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CENTERED BASES, NESTED BASES, AND COMPLETABILITY OF ARONSZAJN SPACES

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Definition. [**1**] A base B for the topology of a space S is *centered* provided every perfectly decreasing filterbase F in B is regular and either F is free or F converges. A centered base which contains no free perfectly decreasing filterbase is said to be *complete.*

Definition. [**2**] A collection B of subsets of a space S is said to be *nested* if it has the property that if $U, V \in B$ and U is a proper subset of V, then $\bar{U} \subseteq V$. Of primary interest is the case when B is a base for the topology of S.

Definition. [**11**] A regular T_0 space having a (monotonically complete) base of countable order is called a (*complete*) *Aronszajn space.* An Aronszajn space S is said to be *completable* if S is homeomorphic to a dense subspace of a complete Aronszajn space X and X is called a *completion* of S.

A T_1 space having a (complete) centered base is a (complete) Aronszajn space and a (complete) Aronszajn space having a nested base has a (complete) centered base. It is not known if either of these statements characterizes spaces having centered bases. The spaces described in [**7**; **8**, Theorem 3; 9, Theorems 1 and 8; and **10**, Theorem 1] are non-completable Aronszajn spaces. The space described in [**4**, Example 1] is an extremely simple example of a separable Aronszajn space which cannot be densely embedded in any Aronszajn space containing a dense complete subspace (see [**5**]).

The primitive concepts of Wicke and Worrell are essential tools in the investigation of Aronszajn spaces and it is assumed that the reader is familiar with the terminology and techniques found in [**12**] and in the references to the works of these authors listed in the bibliography of [**12**]. The definitions found in these references will not be duplicated here. We assume throughout this paper that (S, τ) denotes a regular T_0 space and N is the set of positive integers. If W is a well-ordered collection of sets and M is a subset of some member of W, then $W(M)$ denotes the first member of W containing M. If $M = \{P\}$, then $W(P)$ denotes $W(M)$.

THEOREM 1. *If S is a first-countable Tychonoff space such that* card $S \leqq c$, *then S has a nested base of regular open sets.*

Proof. The following proof is due to Gary Gruenhage. Let S be well-ordered so that the cardinality of each initial segment is less than c. Suppose $x \in S$ and

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for each $y < x$, assume that B_y is a nested collection of regular open sets in S which forms a base for S on $\{z \in S | z \leqq y\}$ and card $B_y < c$. Suppose U is an open set containing x and f is a continuous function from S to $[0, 1]$ such that $f(x) = 0$ and $f(P) = 1$ for each $P \in S - U$. For each $0 < r < 1$, let U_r denote the interior of the closure of $f^{-1}[0, r)$. Then U_r is a regular open set contained in U such that if $r < s < 1$, then $\bar{U}_r \subseteq U_s$. So for each $b \in B = \bigcup_{y<x} B_y$, there is at most one r such that $b \subseteq U_r$ but $\bar{b} \not\subseteq U_r$ and at most one s such that $U_s \subseteq b$ but $\bar{U}_s \not\subseteq b$. Since card $B < c$, there is an r such that $B \cup \{U_r\}$ is nested. This process may be applied to each U in a countable local base at x so as to extend B to a nested collection B_x of regular open sets which forms a base for S on $\{y \in S | y \leqq x\}$ and card $B_x < c$. The result then follows.

THEOREM 2. *If S contains an open set D such that there exist a base B for D, no member of which is closed in S, and a closed discrete subset K of $\bar{D} - D$ having cardinality $2^{|B|}$, then some base for S fails to contain a nested base.*

Proof. Let $C = \{G \subseteq B | G$ is a base for $D\}$. Then $|C| \leqq 2^{|B|} = |K|$. Let $\varphi : C \to K$ be an injection and for each $x \in S - D$, let B_x be a base at x such that each set in B_x contains no point of $K - \{x\}$. Let

$$\mathscr{B} = B \cup \{U_{b,g}(x) | x \in \varphi(G) \text{ for some } G \in C, b \in B_x, g \in G \text{ such that } g \cap b = \emptyset \text{ and } U_{b,g}(x) = b \cup g\} \cup \{U | U \in B_x \text{ for some } x \in S - D - \varphi(C)\}.$$

$\mathscr{B}$ is a base for the topology of S. Let $\mathscr{B}_0 \subseteq \mathscr{B}$ be a base for the topology of S. Then $\mathscr{B}_0 \cap \mathscr{B}$ contains a base G for D. Let $x = \varphi(G)$. The only sets in $\mathscr{B}_0$ containing x are of the form $U_{b,g}(x)$ for $g \in G$. So there exist $g \in G$ and $b \in B_x$ such that $U_{b,g}(x) \in \mathscr{B}_0$. However $g \in \mathscr{B}_0$ and g is properly contained in $U_{b,g}(x)$. But g is not closed so $\bar{g} \not\subseteq g$ and $\bar{g} \cap b = \emptyset$. So $\bar{g} \not\subseteq b \cup g = U_{b,g}(x)$ and $\mathscr{B}_0$ is not nested.

Comment. The tangent disc space Γ is a simple example of a space which has a nested base by Theorem 1 but some base for Γ fails to contain a nested base by Theorem 2. It can be shown that no space as described in the hypothesis of Theorem 2 can be normal nor can it have a point-countable base.

LEMMA. *If S has a primitive base* [**12**] *and B is a base for S, then there exist bases $B_1, B_2, \ldots$ for S such that $B_n \subseteq B$ and $B_n \cap B_m = \{\{x\} | \{x\} \in \tau\}$ if $n \neq m$.*

Proof. Let $C = \{\{x\} | \{x\} \in \tau\}$. There is an open primitive sequence $W = W_1, W_2, \ldots$ of S such that if $P \in S$, $\{W_n(P) | n \in N\}$ is a decreasing local base at P. Let V_1 be a well-ordered subcollection of B covering S such that if $v \in V$, there is a $P \in S$ such that $v = V_1(P)$ and for each $Q \in S$, $V_1(Q) \subseteq W_1(Q)$. Since V_1 contains no perfectly decreasing subcollection, it follows that if $P \in S$ and $B - V_1$ contains no local base at P, then $\{P\} \in C$. So $B' = (B - V_1) \cup C$ is a base for S. There is a well-ordered subcollection V_2 of B' covering S such that if $v \in V_2$, there is a $P \in S$ such that $v = V_2(P)$ and for each $Q \in S$, $V_2(Q) \subseteq V_1(Q) \cap W_2(Q)$.

Continuing this process we construct a sequence $V_1, V_2, \ldots$ of well-ordered subcollections of B covering S such that if $n \neq m$, $V_n \cap V_m \subseteq C$ and if $P \in S$, then $\{V_n(P)|n \in N\}$ is a decreasing local base at P. Let A be a countable disjoint collection of infinite subsets of N and for each $\alpha \in A$, let $B_\alpha = \bigcup_{i \in \alpha} V_i$. The bases required in the lemma are obtained merely by renaming $\{B_\alpha|\alpha \in A\}$.

THEOREM 3. *If $\{S_n|n \in N\}$ is a countable collection of T_1 spaces each of which has a centered base, then $\prod_{n \in N} S_n$ has a centered base.*

Proof. For each n, let B_n be a centered base for S_n and let $B_{n1}, B_{n2}, \ldots$ be a sequence of bases for S_n as in the previous lemma. For each m, let W_{nm}^1, $W_{nm}^2, \ldots$ be an open primitive sequence of S_n such that for each k, $W_{nm}^k \subseteq B_{nm}$. It is easy to show that

$$\{w_1 \times \ldots x w_k \times S_{k+1} \times S_{k+2} \times \ldots |k \geqq 1 \text{ and for some } j \geqq 1, \\ w_i \in W_{ik}{}^j \text{ for } i = 1, \ldots, k\}$$

is a centered base for $\prod_{n \in N} S_n$.

Definition. [**3**] An *ortho-base* for a space X is a base B for the topology of X such that if $F \subseteq B$, then either $\bigcap F$ is open or $\bigcap F = \{x\}$ and F is a local base at x.

Definition. A *uniform base* for a space X is a base B for the topology of X such that if $x \in X$ and F is an infinite subset of B each element of which contains x, then F is a local base at x.

Comment. Every space having an ortho-base is orthocompact and for T_0 developable spaces, the converse is true. However the space of countable ordinals is an orthocompact Aronszajn space which does not have an ortho-base. Clearly a uniform base for S is an ortho-base for S but the tangent disc space has an ortho-base but no uniform base. Although we will not use the result, it is well-known that S has a uniform base if and only if S is a metacompact Moore space. Thus half of the following theorem is essentially a result in [**1**] which shows that every metacompact Moore space has a nested base.

THEOREM 4. *If S is an Aronszajn space having an ortho-base (resp. uniform base), then every ortho-base (resp. uniform base) for S contains a centered base (resp. nested base) for S.*

Proof. First let B be an ortho-base for S. There is an open primitive sequence $W_1, W_2, \ldots$ of S such that $W_n \subseteq B$ for each n, $A = \bigcup_{n=1}^{\infty} W_n$ is a base of countable order for S and if $w \in W_n$ for $n > 1$, then $\bar{w}$ is a subset of each set in $\bigcup_{i<n} W_i$ which contains w. Since no W_n contains a perfectly decreasing collection, it follows that A is a centered base for S. If in addition B is a uniform

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base, then each W_n is point-finite and can be taken to be a minimal cover of S. It then follows that A is nested.

Definition. If Γ is a collection of sequences of subsets of S and $U, V \subseteq S$, then *U is Γ-embedded in V* means if $(g_n) \in \Gamma$ and for each n, $g_n \cap U \neq \emptyset$, then for some n, $g_n \subseteq V$.

Definition. Suppose $G = (G_n)$ is a decreasing sequence of bases for the topology of S and let $D(G)$ denote the collection of all decreasing representatives of G, i.e., $D(G) = \{(g_n) | \text{ for each } n, g_n \in G_n \text{ and } g_{n+1} \subseteq g_n\}$. (We consider the elements of $D(G)$ to be filterbases on S hence the terms coarser, finer, equivalent, converges, intersects, regular, etc. assume their usual meanings in this setting.) If G has the property that for each $g \in D(G)$ and $P \in \bigcap g_n$, g converges to P, then G is called an *Aronszajn sequence for S*. If G is an Aronszajn sequence for S, then G is said to be

(1) *centered* if each element of $D(G)$ is regular;

(2) *complete* if each element of $D(G)$ converges;

(3) *completing* if for each $g \in D(G)$ and $n \in N$, there is a $k \geqq n$ such that g_k is $D(G)$-embedded in g_n;

(4) *strong completing* if for each $U, V \in G_1$ with U a proper subset of V, then U is $D(G)$-embedded in V.

THEOREM 5. *S is a (complete) Aronszajn space if and only if S has a (complete) Aronszajn sequence. S has a (complete) centered base if and only if S has a (complete) centered Aronszajn sequence.*

Definition. Suppose G is an Aronszajn sequence for S. For each $x \in D(G)$, let $[x] = \{y \in D(G) | x \text{ and } y \text{ are equivalent filterbases}\}$. For each $U \in \tau$, let $U^* = \{[x] | U \text{ contains a term of } x\}$. Then $\{U^* | U \in \tau\}$ is a base for a topology τ^* on S^* and S is homeomorphic to a dense subspace of S^*. The space (S^*, τ^*) will be denoted by $S^*(G)$ and will be called the *extension of S by G*.

THEOREM 6. *If G is a (strong) completing Aronszajn sequence for S, then $S^*(G)$ is a completion of S which has a centered base (resp. nested base).*

THEOREM 7. *If S is an Aronszajn space having an ortho-base (resp. uniform base), then S is completable if and only if S has a (strong) completing Aronszajn sequence.*

Proof. One implication follows from Theorem 6. So suppose S is a completable Aronszajn space and B is an ortho-base for S. In [**6**] it is shown that there is an Aronszajn sequence H for S such that if $P \in V \in \tau$, there is a $U \in \tau$ such that $P \in U$ and U is $D(H)$-embedded in V. There is an open primitive sequence W of S such that $W_n \subseteq B$ for each n, each element of $PR(W)$ (the set of all primitive representatives of W) is finer than some element of $D(H)$, and if $n > 1$ and $w \in W_n$, then w is $D(H)$-embedded, hence $PR(W)$-embedded, in each set in $\bigcup_{i<n} W_i$ which contains w.

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If for each n, $G_n = \bigcup_{i \geqq n} W_i$, then every element of $D(G)$ is finer than some element of $PR(W)$. It follows that G is an Aronszajn sequence for S and U is $D(G)$-embedded in V if and only if U is $PR(W)$-embedded in V. Since no W_n contains a perfectly decreasing collection, it follows that G is completing. The modification for the case where B is a uniform base is the same as in Theorem 4.

COROLLARY. *Every completable Aronszajn space having an ortho-base (resp. uniform base) has a completion having a centered base (resp. nested base).*

THEOREM 8. *If X is a completion of S having a centered base (resp. nested base) of regular open sets, then S has a (strong) completing Aronszajn sequence G such that $S^*(G)$ is homeomorphic to X.*

Proof. Let H be a centered complete Aronszajn sequence of X such that each set in H_1 is regular and for the case that X has a nested base, let H_1 be nested. For each n, let $G_n = \{hnS|h \in H_n\}$. The remaining details are straightforward upon realizing that each element of $D(G)$ is the trace on S of some element of $D(H)$ and therefore, each element of $D(G)$ converges in X.

COROLLARY. *Every Tychonoff complete Aronszajn space which has cardinality not greater than c (resp., is separable) is the extension of a complete (and separable) metric space by an Aronszajn sequence.*

Proof. Every complete (and separable) Aronszajn space contains a (separable) dense completely metrizable subspace. The result therefore follows from Theorems 1 and 8.

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