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MODULAR ANNIHILATOR A^*-ALGEBRAS

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1. **Introduction.** We find several equivalent conditions for an A^*-algebra with dense socle to be completely continuous. Such an A^*-algebra is modular annihilator [10]. We also study modular annihilator A^*-algebras with the weak (β_k)-property and obtain a necessary and sufficient condition for such algebras to be dual.

Let A be a modular annihilator A^*-algebra and $\mathfrak{A}$ the completion of A in the auxiliary norm; $\mathfrak{A}$ is a dual B^*-algebra. Let $\mathrm{cl}_{\mathfrak{A}}(S)$ denote the closure of the set S in $\mathfrak{A}$. In §3 we show that the mapping $M \to \mathrm{cl}_{\mathfrak{A}}(M) = \mathfrak{M}$ sets up a one-to-one correspondence between the maximal modular two-sided ideals M of A and the maximal modular two-sided ideals $\mathfrak{M}$ of $\mathfrak{A}$, and $M = \mathfrak{M} \cap A$. Using this fact we obtain several necessary and sufficient conditions for an A^*-algebra with dense socle to be completely continuous. We thus show that [7, p. 28, Theorem 15], which was stated for a dual A^*-algebra, also holds for an A^*-algebra with dense socle. In §4 we investigate modular annihilator A^*-algebras with the weak (β_k)-property. Several equivalent formulations of this property are included.

2. **Preliminaries.** All algebras under consideration are over the complex field C. An A^*-algebra A is a Banach*-algebra on which there is defined a second norm $|\cdot|$ under which A is a normed algebra with the property that $|x^*x| = |x|^2$, $x \in A$. The norm $|\cdot|$ is called an auxiliary norm on A. The completion $\mathfrak{A}$ of A in an auxiliary norm is a B^*-algebra.

For any set S in an algebra A, let $l_A(S)$ and $r_A(S)$ be respectively the left and right annihilators of S in A. An algebra A is modular annihilator if every maximal modular left (right) ideal of A has a non-zero right (left) annihilator. A semi-simple Q-algebra with dense socle is modular annihilator [10, p. 41, Lemma 3.11]; in particular a semi-simple Banach algebra with dense socle is modular annihilator. A Banach algebra A is an annihilator algebra if $r_A(J) \neq (0)$ for every proper closed left ideal J and $l_A(R) \neq (0)$ for every proper closed right ideal R of A. A Banach algebra A is dual if $l_A(r_A(J)) = J$ for every closed left ideal J and $r_A(l_A(R)) = R$ for every closed right ideal R of A.

Let A be a dual B^*-algebra. Then every closed *-subalgebra of A is a dual algebra and every minimal closed two-sided ideal of A is *-isomorphic to $LC(H)$, the algebra of all compact linear operators on a Hilbert space H. Thus if a minimal closed two-sided ideal is finite dimensional then it is a full matrix algebra with

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identity. Every closed two-sided of A which has an identity is finite dimensional [6, pp. 221–223].

A normed algebra is completely continuous (c.c.) if every $a \in A$ is c.c., i.e., the mappings $x \to ax$ and $x \to xa$ are completely continuous operators on A. Let A be a semi-simple Banach algebra and let e be a minimal idempotent of A. Then the closed two-sided ideal generated by e is a minimal closed two-sided ideal of A. (See the proof of [5, p. 158, Theorem 5].)

The socle of an algebra A will be denoted by S_A. If S is a subset of a normed algebra A, $\mathrm{cl}_A(S)$ will denote the closure of S in A. For all other concepts used see [8].

3. **Completely continuous modular annihilator A^*-algebras.** From [1, p. 6, (1.3)] it follows that a modular annihilator A^*-algebra A has a unique auxiliary norm. Thus A can be embedded in a unique B^*-algebra $\mathfrak{A}$. From now on we shall refer to $\mathfrak{A}$ as the completion of A. By [3, p. 287, Lemma 2.6], $\mathfrak{A}$ has dense socle and hence, by [6, p. 222, Theorem 2.1], $\mathfrak{A}$ is dual.

LEMMA 3.1. *Let A be a modular annihilator A^*-algebra and $\mathfrak{A}$ the completion of A. Then:*

(i) (0) *is the only two-sided ideal I of $\mathfrak{A}$ such that $A \cap I = (0)$.*

(ii) *Every nonzero two-sided ideal I of $\mathfrak{A}$ contains a minimal idempotent belonging to A.*

Proof. (i) Suppose I is a two-sided ideal of $\mathfrak{A}$ such that $A \cap I = (0)$. We claim that $AI = (0)$. In fact, assume that $AI \neq (0)$. Since, by [3, p. 287, Lemma 2.6], S_A is dense in A relative to the auxiliary norm $|\cdot|$, there exists a self-adjoint minimal idempotent e in A such that $eI \neq (0)$. Let $z \in I$ be such that $ez \neq 0$. Then $ezz^*e = \lambda e$, $\lambda \neq 0$ in C. This shows that $e \in I$ which clearly contradicts the fact that $A \cap I = (0)$. Hence $AI = (0)$ and consequently $\mathfrak{A}I = (0)$. Since $\mathfrak{A}$ is semi-simple, we must have $I = (0)$.

(ii) Suppose I is a nonzero two-sided ideal of $\mathfrak{A}$. Then, by (i), $A \cap I \neq (0)$. Thus $J = A \cap I$ is a nonzero two-sided ideal of A and, by the semi-simplicity of A, $AJ \neq (0)$. Hence $AI \neq (0)$. The argument in (i) now shows that I contains a minimal idempotent of A.

THEOREM 3.2. *Let A be a modular annihilator A^*-algebra and $\mathfrak{A}$ the completion of A. If $\mathfrak{M}$ is a maximal modular two-sided ideal of $\mathfrak{A}$, then $M = \mathfrak{M} \cap A$ is a maximal modular two-sided ideal of A and $\mathfrak{M} = \mathrm{cl}_{\mathfrak{A}}(M)$. Conversely, if M is a maximal modular two-sided ideal of A, then $\mathfrak{M} = \mathrm{cl}_{\mathfrak{A}}(M)$ is a maximal modular two-sided ideal of $\mathfrak{A}$ and $M = \mathfrak{M} \cap A$. Thus $M \to \mathrm{cl}_{\mathfrak{A}}(M) = \mathfrak{M}$ sets up a one-to-one correspondence between the maximal modular two-sided ideals M of A and the maximal modular two-sided ideals $\mathfrak{M}$ of $\mathfrak{A}$, and $M = \mathfrak{M} \cap A$.*

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Proof. Let $\mathfrak{M}$ be a maximal modular two-sided ideal of $\mathfrak{A}$. Since $\mathfrak{A}$ is modular annihilator, by [2, p. 574, Theorem 6.4], $\mathfrak{A}/\mathfrak{M}$ is finite dimensional. Let $\mathfrak{I}=r_{\mathfrak{A}}(\mathfrak{M})$. Since $\mathfrak{I} \cap \mathfrak{M}=(0)$ and $\mathfrak{I}+\mathfrak{M}=\mathfrak{A}$, $\mathfrak{I}$ is a finite dimensional minimal closed two-sided ideal of $\mathfrak{A}$. Since $\mathfrak{A}$ is dual, $\mathfrak{I}$ is a full matrix algebra with identity e; $\mathfrak{I}=e\mathfrak{A}=\mathfrak{A}e$. It follows now that $\mathfrak{M}=(1-e)\mathfrak{A}=\mathfrak{A}(1-e)$. Moreover, by Lemma 3.1 (ii), $\mathfrak{I}$ contains a minimal idempotent $f \in A$. As $\mathfrak{I}=\mathrm{cl}_{\mathfrak{A}}(AfA)$ and $J=\mathrm{cl}_A(AfA)$ is dense in $\mathfrak{I}$, we have $\mathfrak{I}=J$ and consequently $e \in A$. Hence $\mathfrak{I}$ is also a minimal closed two-sided ideal of A and so $M=(1-e)A=A(1-e)$ is a maximal modular two-sided ideal of A. Clearly $M=\mathfrak{M} \cap A$ and $\mathfrak{M}=\mathrm{cl}_{\mathfrak{A}}(M)$.

Conversely, suppose M is a maximal modular two-sided ideal of A. Then, by the argument above, $I=r_A(M)$ is a finite dimensional minimal closed two-sided ideal of A. Since $\mathrm{cl}_{\mathfrak{A}}(I)=I$, I is also a minimal closed two-sided ideal of $\mathfrak{A}$ and so $I=eA=Ae$, where e is an idempotent in A. Hence $M=(1-e)A=A(1-e)$ and $\mathfrak{M}=(1-e)\mathfrak{A}=\mathfrak{A}(1-e)$ is a maximal modular two-sided ideal of $\mathfrak{A}$. Clearly $\mathfrak{M}=\mathrm{cl}_{\mathfrak{A}}(M)$ and $M=\mathfrak{M} \cap A$.

COROLLARY 3.3. *Let A be a modular annihilator A^*-algebra and $\mathfrak{A}$ the completion of A. Then A and $\mathfrak{A}$ have the same minimal closed two-sided ideals of finite dimension.*

COROLLARY 3.4. *Let A be a modular annihilator A^*-algebra and $\mathfrak{A}$ the completion of A. Then A is strongly semi-simple if and only if $\mathfrak{A}$ is strongly semi-simple.*

Proof. Let $\{\mathfrak{M}_\alpha\}$ be the family of all maximal modular two-sided ideals of A. Then, by Theorem 3.2, $\{M_\alpha=\mathfrak{M}_\alpha \cap A\}$ is the family of all maximal modular two-sided ideals of A. Thus $(\bigcap_\alpha \mathfrak{M}_\alpha) \cap A=\bigcap_\alpha (\mathfrak{M}_\alpha \cap A)=\bigcap_\alpha M_\alpha$. Therefore $\bigcap_\alpha \mathfrak{M}_\alpha=(0)$ implies that $\bigcap_\alpha M_\alpha=(0)$ and conversely, by Lemma 3.1 (i), $\bigcap_\alpha M_\alpha=(0)$ implies $\bigcap_\alpha \mathfrak{M}_\alpha=(0)$. This completes the proof.

We recall that an A^*-algebra with dense socle is modular annihilator. With the help of the observations above we obtain the following theorem which was proved by Ogasawara and Yoshinaga for a dual A^*-algebra [7, p. 28, Theorem 15]:

THEOREM 3.5. *Let A be an A^*-algebra with dense socle and $\mathfrak{A}$ the completion of A. Then the following statements are equivalent*:

(i) A is a c.c. algebra.

(ii) $\mathfrak{A}$ is a c.c. algebra.

(iii) A is strongly semi-simple.

(iv) There exists an orthogonal family $[e_\alpha\}$ of self-adjoint idempotents in the center of A such that $ae_\alpha=0$, for all α, implies $a=0$.

Proof. (i)→(ii). Suppose A is c.c. Since every minimal left (right) ideal of A is of the form $Ae(eA)$, where e is an idempotent, every such ideal is finite dimensional. As S_A is dense in $\mathfrak{A}$, it follows that $\mathfrak{A}$ is c.c.

(ii)→(iii). Suppose $\mathfrak{A}$ is c.c. Then every minimal closed two-sided ideal of $\mathfrak{A}$ is finite dimensional. In fact, let I be a minimal closed two-sided ideal of $\mathfrak{A}$ and J a minimal left ideal of the algebra I. Since $J=\mathfrak{A}e$, where e is a minimal idempotent of $\mathfrak{A}$, J is finite dimensional. As I has a faithful (left regular) representation on J, I is also finite dimensional. Thus $l_{\mathfrak{A}}(I)$ is a maximal modular two-sided ideal. But $\mathfrak{A}$ is the direct topological sum of its minimal closed two-sided ideals. Hence $\mathfrak{A}$ is strongly semi-simple and so, by Corollary 3.4, A is strongly semi-simple.

(iii)→(iv). Suppose (iii) holds. Then, by Corollary 3.4, $\mathfrak{A}$ is strongly semi-simple and therefore is the direct topological sum of its finite dimensional minimal closed two-sided ideals I_α each of which belongs to A. Let e_α be the identity of I_α. Then clearly $\{e_\alpha\}$ has all the properties stated in (iv).

(iv)→(i). Suppose $\{e_\alpha\}$ is a family of self-adjoint idempotents in A with the properties stated in (iv). Then clearly $\{e_\alpha\}$ belongs to the center of $\mathfrak{A}$ and so $I_\alpha=e_\alpha\mathfrak{A}$ is a closed two-sided ideal of $\mathfrak{A}$ with identity and hence finite dimensional. Now $I=\{x\in\mathfrak{A}:xe_\alpha=0 \text{ for all } \alpha\}$ is the (0) ideal of $\mathfrak{A}$ by Lemma 3.1 (i), since $A\cap I=(0)$. Hence $\mathfrak{A}$ is the direct topological sum of the I_α and so is c.c. Thus every minimal left ideal of A is finite dimensional and since S_A is dense in A, it follows that A is also c.c.

REMARK. It is easy to see from the proof above that the statements (ii), (iii) and (iv) of Theorem 3.5 are equivalent for any modular annihilator A^*-algebra.

4. **Modular annihilator A^*-algebras with the weak (β_k)-property.** Let A be a Banach*-algebra. A is said to have the weak (β_k)-property if, for every minimal left ideal I of A, there exists a constant $k>0$ such that $\|x\|^2\le k\,\|x*x\|$ for all $x\in I$. Now suppose that A has the property that $x^*x=0$ implies $x=0$. Then every minimal left ideal I of A is of the form $I=Ae$, where e is a self-adjoint minimal idempotent [8, p. 261, Lemma (4.10.1)], and the scalar-valued function (x,y) given by $(x,y)e=y^*x$ on I is an inner product on I. Let $|x|_0=(x,x)^{1/2}$, $x\in I$. It can be shown that A has the weak (β_k)-property if and only if every minimal left ideal I is complete in the norm $|\cdot|_0$, or what is the same, the norms $|\cdot|_0$ and $\|\cdot\|$ are equivalent on every minimal left ideal I of A. (See [8, p. 263, Theorem (4.10.6)] and its proof.) In particular if a minimal left ideal is finite dimensional then the two norms $|\cdot|_0$ and $\|\cdot\|$ are equivalent on I. Thus a c.c. A^*-algebra has the weak (β_k)-property.

Let A be an A^*-algebra. It follows from [9] that if A is a dense two-sided ideal of a dual B^*-algebra then A has the weak (β_k)-property. Also A is an annihilator algebra if and only if A has dense socle and the weak (β_k)-property. Thus having the weak (β_k)-property is weaker than being a dense two-sided ideal of a dual B^*-algebra. Also a modular annihilator A^*-algebra with the weak (β_k)-property need not be an annihilator algebra unless it has dense socle.

The following lemma shows that A^*-algebras which are modular annihilator and have the weak (β_k)-property are the most general semi-simple Banach*-algebras with these properties; more precisely:

LEMMA 4.1. *Let A be a semi-simple modular annihilator Banach*-algebra with the weak (β_k)-property. Then A is a modular annihilator A^*-algebra.*

Proof. Since $l_A(S_A)=(0)$ [4, p. 516], it follows from [8, p. 73, Corollary (2.5.8.)] that A has the unique norm topology and consequently the involution in A is continuous. Now suppose $x \in A$ is such that $x^*x=0$ and let I be a minimal left ideal of A. Then, for each $a \in I$, $(xa)^*(xa)=a^*x^*xa=(0)$. Hence, by the weak (β_k)-property, $\|xa\|^2=0$ which implies that $xa=0$ and consequently that $xI=(0)$. It follows now that $xS_A=(0)$ and hence that $x=0$ since $l_A(S_A)=(0)$. Applying [8, p. 262, Corollary (4.10.4)] completes the proof.

LEMMA 4.2. *Let A be a modular annihilator A^*-algebra and $\mathfrak{A}$ the completion of A. Then following statements are equivalent:*

(i) *A has the weak (β_k)-property.*

(ii) *A and $\mathfrak{A}$ have the same socle.*

(iii) *For every closed right ideal I of A, $\mathrm{cl}_{\mathfrak{A}}(I)S_A \subset I$.*

(iv) $\mathfrak{A}S_A \subset A$.

Proof. (i)→(ii). If (i) holds, then every minimal left (right) ideal of $\mathfrak{A}$ is also a minimal left (right) ideal of $\mathfrak{A}$ and hence S_A is a dense two-sided ideal of $\mathfrak{A}$. Thus if I is a minimal left ideal of $\mathfrak{A}$, then $I \cap S_A$ is a nonzero left ideal of $\mathfrak{A}$ and therefore, by the minimality of I, we must have $I=I \cap S_A$. Hence $S_{\mathfrak{A}} \subset S_A$ and, as $S_A \subset S_{\mathfrak{A}}$, we obtain $S_{\mathfrak{A}}=S_A$.

(ii)→(iii). Suppose (ii) holds, and let J be a minimal left ideal of A. Since J is also a minimal left ideal of $\mathfrak{A}$, the norms $|\cdot|$ and $\|\cdot\|$ are equivalent on J. Thus if $x \in \mathrm{cl}_{\mathfrak{A}}(I)$ and $y \in J$ then, since $xy \in J$, we have $\|xy\| \le k\,|xy| \le k\,|x|\,|y|$, for some constant $k>0$. But $|y| \le \beta\,\|y\|$ [8, p. 187, Corollary (4.1.10)]. Hence $\|xy\| \le c\,|x|\,\|y\|$ for some constant $c>0$. Now let $\{x_n\}$ be a sequence in I such that $|x_n-x|\to 0$. Then $\|x_ny-xy\|\to 0$ since $\|x_ny-xy\| \le c\,|x_n-x|\,\|y\|$. Thus $xy \in I$ and therefore $\mathrm{cl}_{\mathfrak{A}}(I)S_A \subset I$.

(iii)→(iv). Since S_A is dense in $\mathfrak{A}$, $\mathrm{cl}_{\mathfrak{A}}(\mathrm{cl}_A(S_A))S_A=\mathfrak{A}S_A$. Hence if (iii) holds, then $\mathfrak{A}S_A \subset \mathrm{cl}_A(S_A) \subset A$.

(iv)→(i). If $\mathfrak{A}S_A \subset A$, then clearly every minimal left ideal I of A is also a minimal left ideal of $\mathfrak{A}$. Thus I is complete in the norm $|\cdot|$, whence (i).

THEOREM 4.3. *Let A be a modular annihilator A^*-algebra with the weak (β_k)-property and let $\mathfrak{A}$ be the completion of A. Then, for every closed right ideal I of A, we have:*

(i) $l_{\mathfrak{A}}(\mathrm{cl}_{\mathfrak{A}}(I))=\mathrm{cl}_{\mathfrak{A}}(l_A(I))$;

(ii) $\mathrm{cl}_{\mathfrak{A}}(I) \cap A=r_A(l_A(I))$;

(iii) $\mathrm{cl}_{\mathfrak{A}}(I)=\mathfrak{A}$ *if and only if* $S_A \subset I$.

Proof. Since A has the weak (β_k)-property and $\mathfrak{A}$ is dual, the proofs of (i) and (ii) are exactly the same as those given for similar statements in [9, p. 54, Lemma 5.5] for an annihilator A^*-algebra. Statement (iii) is an immediate consequence of the facts that $\mathrm{cl}_{\mathfrak{A}}(I)S_A \subset I$, S_A is dense in $\mathfrak{A}$ and $\mathfrak{A}S_A = S_A$.

COROLLARY 4.4. *Let A be a modular annihilator A^*-algebra with the weak (β_k)-property and let $\mathfrak{A}$ be the completion of A. Then A is dual if and only if, for every closed right ideal I of A,* $\mathrm{cl}_{\mathfrak{A}}(I) \cap A = I$.

REFERENCES

1. B. A. Barnes, *Subalgebras of modular annihilator algebras*, Proc. Cambridge Philos. Soc. **66** (1969), 5–12.

2. ——, *Modular annihilator algebras*, Canad. J. Math. **18** (1966), 566–578.

3. ——, *Algebras with spectral expansion property*, Illinois J. Math. **11** (1967), 284–290.

4. ——, *On the existence of minimal ideals in a Banach algebra*, Trans. Amer. Math. Soc. **133** (1968), 511–517.

5. F. F. Bonsall and A. W. Goldie, *Annihilator algebras*, Proc. London Math. Soc. **14** (1954), 154–167.

6. I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. **70** (1951), 219–255.

7. T. Ogasawara and K. Yoshinaga, *Weakly completely continuous Banach*-algebras*, J. Sci. Hiroshima Univ. Ser. A18 (1954), 15–36.

8. C. E. Rickart, *General Theory of Banach algebras*, Van Nostrand, Princeton, N.J. 1960.

9. B. J. Tomiuk and P. K. Wong, *Annihilator and complemented Banach*-algebras*, J. Austral. Math. Soc. **13** (1971), 47–66.

10. B. Yood, *Ideals in topological rings*, Canad. J. Math. **16** (1964), 28–45.

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