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A FIXED POINT THEOREM AND EXISTENCE OF EQUILIBRIUM FOR ABSTRACT ECONOMIES

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In this paper, we shall prove a generalisation of Himmelberg's fixed point theorem and as applications, the existence of equilibrium points for abstract economies given by preference correspondences and utility functions have been established.

1. Introduction

In the last twenty years, the classical Arrow-Debreu result [**1**] on the existence of Walrasian equilibria has been generalised in many directions. Mas-Colell [**13**] has first shown that the existence of an equilibrium can be established without assuming preferences to be total or transitive. Next, by using a maximal element existence theorem, Gale and Mas-Colell [**9**] gave a proof of the existence of a competitive equilibrium without ordered preferences. By using Kakutani's fixed point theorem, Shafer and Sonnenschein [**15**] proved the powerful result on 'the Arrow-Debreu lemma for abstract economies' for the case where preferences may not be total or transitive but has an open graph. On the other hand, Borglin and Keiding [**3**] proved a new existence theorem for a compact abstract economy with KF-majorised preference correspondences. Following their ideas, there have been a number of generalisations of the existence of equilibria for compact abstract economies.

As in [**3, 9, 13, 15**], in most results on the existence of equilibria for abstract economies, the underlying spaces (commodity spaces or choice sets) are always compact and convex. However, in recent papers [**5, 18, 19**], the underlying spaces are not always compact, although paracompactness is needed. Moreover it should be noted that we will encounter many kinds of preferences in various economic situations; so that it is important that we shall consider several types of preferences and obtain some existence results for such correspondences in non-compact (or non-paracompact) settings.

In a recent paper [**17**], Tarafdar proved an extension of Fan's fixed point theorem and using this he proved the existence of an equilibrium point of an abstract economy given by preferences and an economy given by utility functions.

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The purpose of this paper is two-fold. First, we prove a generalisation of Himmelberg's fixed point theorem by relaxing the upper semicontinuity assumption. Second, as applications we prove existence theorems for equilibria of abstract economies given by preference correspondences and by utility functions.

2. Preliminaries

Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A and by $cl\,A$ the closure of A in X. If A is a subset of a vector space, we shall denote by $co\,A$ the convex hull of A. If A is a non-empty subset of a topological vector space X and $S, T : A \to 2^X$ are correspondences, then $co\,T, cl\,T$, $T \cap S : A \to 2^X$ are correspondences defined by $(co\,T)(x) = co\,T(x), (cl\,T)(x) = cl\,T(x)$ and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively. Let B be a non-empty subset of A. Denote the restriction of T on B by $T|_B$.

Let X be a non-empty subset of a topological vector space and $x \in X$. Let $\phi : X \to 2^X$ be a given correspondence. A correspondence $\phi_x : X \to 2^X$ is said to be a Θ-*majorant* of ϕ at x if there exists an open neighbourhood N_x of x in X such that (a) for each $z \in N_x, \phi(z) \subset \phi_x(z)$, (b) for each $z \in N_x, z \notin cl\,co\,\phi_x(z)$ and (c) $\phi_x|_{N_x}$ has open graph in $N_x \times X$. The correspondence ϕ is said to be Θ-*majorised* if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists a Θ-majorant of ϕ at x.

It is clear that every correspondence ϕ having an open graph with $x \notin cl\,co\,\phi(x)$ for each $x \in X$ is a Θ-majorised correspondence. However the following simple correspondence shows a Θ-majorised correspondence which does not have an open graph :

The correspondence $\phi : X = (0,1) \to 2^X$ is defined by

$$\phi(x) = \left(0, x^2\right] \quad \text{for each } x \in X.$$

Then ϕ has no open graph but $\phi_x(z) = (0, z)$ for all $z \in X$ is a Θ-majorant of ϕ at any $x \in X$.

We now state the following definition introduced in [17]. Let X and Y be two topological spaces. Then a correspondence $T : X \to 2^Y$ is said to be *upper semicontinuous (respectively, almost upper semicontinuous)* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighbourhood U of x in X such that $T(y) \subset V$ (respectively, $T(y) \subset cl\,V$) for each $y \in U$. An upper semicontinuous correspondence is clearly almost upper semicontinuous. From the definition, if T is almost upper semicontinuous, then $cl\,T$ is also almost upper semicontinuous. And it should be noted that we do not need the closedness assumption of $T(x)$ for each $x \in X$ in the definitions as in [17].

The following example shows an almost upper semicontinuous correspondence which is not upper semicontinuous.

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EXAMPLE. Let $X = [0,\infty)$ and $\phi : X \to 2^X$ be defined by

$$\phi(x) = \begin{cases} (1,3), & x = 2, \\ [1,3], & x \neq 2. \end{cases}$$

Then ϕ is not upper semicontinuous at 2 since for an open neighbourhood (1,3) of $\phi(2)$ there does not exist any desired neighbourhood U of 2 such that $T(y) \subset (1,3)$ for all $y \in U$; however $T(y) \subset [1,3]$ for all y in any neighbourhood of 2. Therefore T is almost upper semicontinuous.

Now we recall the following general definitions of equilibrium theory in mathematical economics. Let I be a finite or an infinite set of agents. For each $i \in I$, let X_i be a non-empty set of actions. *An abstract economy (or generalised game)* $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ is defined as a family of ordered quadruples (X_i, A_i, B_i, P_i) where X_i is a non-empty topological vector space (a choice set), $A_i, B_i : \Pi_{j\in I} X_j \to 2^{X_i}$ are constraint correspondences and $P_i : \Pi_{j\in I} X_j \to 2^{X_i}$ is a preference correspondence. An *equilibrium* for Γ (Shafer-Sonneinschein type) is a point $\widehat{x} \in X = \Pi_{i\in I} X_i$ such that for each $i \in I, \widehat{x}_i \in clB_i(\widehat{x})$ and $P_i(\widehat{x}) \cap A_i(\widehat{x}) = \emptyset$. When $A_i = B_i$ for each $i \in I$, our definitions of an abstract economy and an equilibrium coincide with the standard definition of Shafer-Sonnenschein [**15**] or in [**3, 18, 19**]. For each $i \in I, P_i' : X \to 2^X$ will denote the correspondence defined by $P_i'(x) = \{y \in X : y_i \in P_i(x)\}$ ($= \pi_i^{-1}(P_i(x))$, where $\pi_i : X \to X_i$ is the i-th projection).

And we shall use the following notation:

$$X^i = \prod_{\substack{j\in I \\ j\neq i}} X_j$$

and let $\pi_i : X \to X_i$, $\pi^i : X \to X^i$ be the projections of X onto X_i and X^i, respectively. For any $x \in X$, we simply denote $\pi^i(x) \in X^i$ by x^i and $x = (x^i, x_i)$.

In [**10**], Greenberg introduced a further generalised concept of equilibrium as follows: Under the same settings as above, let $\Psi = \{\psi_i\}_{i\in I}$ be a family of functions $\psi_i : X \to R^+$ for each $i \in I$. A Ψ- *quasi-equilibrium* for Γ is a point $\widehat{x} \in X$ such that for all $i \in I$,

(1) $\widehat{x}_i \in clA_i(\widehat{x})$,

(2) $P_i(\widehat{x}) \cap A_i(\widehat{x}) = \emptyset$ and/or $\psi_i(\widehat{x}) = 0$.

As remarked in [**10**], quasi-equilibrium can be of special interest for economies with a tax authority and the result of Shafer-Sonnenschein [**15**] cannot be applied in this problem.

Next we give another definition of equilibrium for an abstract economy given by utility functions. By following Debreu [4], an abstract economy $\Gamma = (X_i, A_i, f_i)_{i\in I}$ is

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defined as a family of ordered triples (X_i, A_i, f_i) where X_i is a non-empty topological vector space (a choice set), $A_i : \Pi_{j \in I} X_j = X \to 2^{X_i}$ is a constraint correspondence and $f_i : \Pi_{j \in I} X_j \to R$ is a utility function (payoff function). An *equilibrium* for Γ (Nash type) is a point $\widehat{x} \in X$ such that for each $i \in I, \widehat{x}_i \in cl A_i(\widehat{x})$ and

$$f_i(\widehat{x}) = f_i(\widehat{x}^i, \widehat{x}_i) = \inf\{f_i(\widehat{x}_1, \dots, \widehat{x}_{i-1}, z, \widehat{x}_{i+1}, \dots) | z \in cl\, A_i(\widehat{x})\}.$$

It should be noted that if $A_i(x) = X_i$ for all $x \in X$, then the concept of an equilibrium for Γ coincides with the well-known Nash equilibrium [**14**]. And as is remarked in [**17**], two types of equilibrium points coincide when the preference correspondence P_i can be defined by

$$P_i(x) = \{z_i \in X_i | f_i(x^i, z_i) < f_i(x)\} \text{ for each } x \in X.$$

3. A generalisation of Himmelberg's fixed point theorem

We begin with the following lemma:

Lemma 1. *Let X be a non-empty subset of a topological space and D be a non-empty compact subset of X. Let $T : X \to 2^D$ be an almost upper semicontinuous correspondence such that for each $x \in X, T(x)$ is closed. Then T is upper semicontinuous.*

Proof: For any $x \in X$, let U be an open neighbourhood of $T(x)$ in D. Since $T(x)$ is closed in D, there exists an open neighbourhood V of $T(x)$ such that

$$T(x) \subset V \subset cl\, V \subset U.$$

Since T is almost upper semicontiunuous at x, for such open neighbourhood V of $T(x)$, we can find an open neighbourhood W of x such that $T(y) \subset cl\, V \subset U$ for all $y \in W$. Therefore T is upper semicontinuous at x. □

For any upper semicontinuous correspondence $T : X \to 2^Y$, $co\,T$ and $cl\,co\,T$ are not necessarily upper semicontinuous in general even if $X = Y$ is compact convex in a locally convex Hausdorff topological vector space. However the almost upper semicontinuity can be preserved as follows:

Lemma 2. *Let X be a convex subset of a locally convex Hausdorff topological vector space E and D be a non-empty compact subset of X. Let $T : X \to 2^D$ be an almost upper semicontinuous correspondence such that for each $x \in X, co\,T(x) \subset D$.*

Then $cl\,co\,T$ is almost upper semicontinuous.

Proof: For any $x \in X$, let U be an open set containing $cl\,co\,T(x)$. Since $cl\,co\,T(x)$ is closed in D, we can find an open convex neighbourhood N of 0 such that

$$cl\,co\,T(x) + N \subset cl(cl\,co\,T(x) + N) = cl\,co\,T(x) + cl\,N \subset U.$$

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Clearly $V = cl\,co\,T(x) + N$ is an open convex set containing $cl\,co\,T(x)$ and $V \subset U$. Since T is almost upper semicontinuous, there exists an open neighbourhood W of x in X such that $T(y) \subset cl\,V$ for all $y \in W$. Since V is convex, $cl\,co\,T(y) \subset cl\,V \subset cl\,U$ for all $y \in W$. Therefore $cl\,co\,T$ is almost upper semicontinuous. ∎

REMARK. In the above Lemma 2, we do not know whether the correspondence $co\,T$ is almost upper semicontinuous even when T is upper semicontinuous.

We now prove the following generalisation of Himmelberg's fixed point theorem.

THEOREM 1. *Let X be a convex subset of a locally convex Hausdorff topological vector space E and D be a non-empty compact subset of X. Let $S, T : X \to 2^D$ be almost upper semicontinuous correspondences such that*

(1) *for each $x \in X$, $\emptyset \neq co\,S(x) \subset T(x)$,*
(2) *for each $x \in X$, $T(x)$ is closed.*

Then there exists a point $\hat{x} \in D$ such that $\hat{x} \in T(\hat{x})$.

PROOF: For each $x \in X$, since $co\,S(x) \subset T(x)$ and $T(x)$ is closed, we have $cl\,co\,S(x) \subset T(x)$. By Lemma 2, the correspondence $cl\,co\,S : X \to 2^D$ is also almost upper semicontinuous , so that by Lemma 1, $cl\,co\,S$ is upper semicontinuous and closed convex valued in D. Therefore by Himmelberg's fixed point theorem [**11**], there exists a point $\hat{x} \in D$ such that $\hat{x} \in cl\,co\,S(\hat{x}) \subset T(\hat{x})$, which completes the proof. ∎

REMARK. Theorem 1 is a generalisation of the recent result of Tarafdar [**17**, Theorem 2.1], and also simplifies his proof.

COROLLARY 1. *Let X be a convex subset of a locally convex Hausdorff topological vector space E and D be a non-empty compact subset of X. Let $S : X \to 2^D$ be an almost upper semicontinuous correspondences such that for each $x \in X, co\,S(x)$ is a non-empty subset of D.*

Then there exists a point $\hat{x} \in D$ such that $\hat{x} \in cl\,co\,S(\hat{x})$.

PROOF: We define a correspondence $T : X \to 2^D$ by $T(x) = cl\,co\,S(x)$ for all $x \in X$. Then by Lemma 2, T is almost upper semicontinuous. Clearly the pair (S,T) satisfies all conditions of Theorem 1, so that there exists a point $\hat{x} \in D$ such that $\hat{x} \in T(\hat{x})$. ∎

When $S = T$ in Theorem 1, we obtain Himmelberg's fixed point theorem as a corollary:

COROLLARY 2. [**11**] *Let X be a convex subset of a locally convex Hausdorff topological vector space and D be a non-empty compact subset of X. Let $T : X \to 2^D$ be an upper semicontinuous correspondence such that for each $x \in X, T(x)$ is a non-empty closed convex subset of D. Then there exists a point $\hat{x} \in D$ such that $\hat{x} \in T(\hat{x})$.*

4. Existence of equilibria in abstract economies

In this section, we consider both kinds of economy described in the preliminaries (that is, an abstract economy given by preference correspondences [Shafer-Sonnenschein type] in compact settings and an abstract economy given by utility functions [Nash type] in non-compact settings) and prove the existence of equilibrium points or quasi-equilibrium points for either case by using the fixed point theorems in Section 3.

First, using Θ-majorised correspondences we shall prove an equilibrium existence of a compact abstract economy, which generalises the powerful result of Shafer-Sonnenschein. For simplicity, we may assume that $A_i = B_i$ for each $i \in I$ in an abstract economy.

Theorem 2. *Let $\Gamma = (X_i, A_i, P_i)_{i\in I}$ be an abstract economy where I is a countable set such that for each $i \in I$,*

(1) *X_i is a non-empty compact convex subset of a metrisable locally convex Hausdorff topological vector space,*
(2) *for each $x \in X = \Pi_{i\in I}X_i, A_i(x)$ is non-empty convex,*
(3) *the correspondence $cl\,A_i : X \to 2^{X_i}$ is continuous,*
(4) *the corespondence P_i' is Θ-majorised.*

Then Γ has an equilibrium choice $\widehat{x} \in X$, that is, for each $i \in I$, $\widehat{x}_i \in cl\,A_i(\widehat{x})$ and $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$.

Proof: Let $i \in I$ be fixed. Since P_i' is Θ-majorised, for each $x \in X$, there exists a correspondence $\phi_x : X \to 2^{X_i}$ and an open neighbourhood U_x of x in X such that $P_i(z) \subset \phi_x(z)$ and $z_i \notin cl\,co\,\phi_x(z)$ for each $z \in U_x$, and $\phi_x|_{U_x}$ has an open graph in $U_x \times X_i$. By the compactness of X, the family $\{U_x : x \in X\}$ of an open cover of X contains a finite subcover $\{U_{x_j} : j \in J\}$, where $J = \{1, 2, \dots, n\}$. For each $j \in J$, we now define $\phi_j : X \to 2^{X_i}$ by

$$\phi_j(z) = \begin{cases} \phi_{x_j}(z), & \text{if } z \in U_{x_j}, \\ X_i, & \text{if } z \notin U_{x_j}, \end{cases}$$

and next we define $\Phi_i : X \to 2^{X_i}$ by

$$\Phi_i(z) = \bigcap_{j\in J} \phi_j(z) \quad \text{for each } z \in X.$$

For each $z \in X$, there exists $k \in J$ such that $z \in U_{x_k}$ so that $z_i \notin cl\,co\,\phi_{x_k}(z) = cl\,co\,\phi_k(z)$; thus $z_i \notin cl\,co\,\Phi_i(z)$. We now show that the graph of Φ_i is open in $X \times X_i$. For each $(z, x) \in$ graph of Φ_i, since $X = \cup_{j\in J} U_{x_j}$, there exist $\{i_1, \dots, i_k\} \subset J$ such that $z \in U_{x_{i_1}} \cap \dots \cap U_{x_{i_k}}$. Then we can find an open neighbourhood U of z in X such that $U \subset U_{x_{i_1}} \cap \dots \cap U_{x_{i_k}}$. Since $\phi_{x_{i_1}}(z) \cap \dots \cap \phi_{x_{i_k}}(z)$ is an open subset

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of X_i containing x, there exists an open neighbourhood V of x in X_i such that $x \in V \subset \phi_{x_{i_1}}(z) \cap \cdots \cap \phi_{x_{i_k}}(z)$. Therefore we have an open neighbourhood $U \times V$ of (z, x) such that $U \times V \subset$ graph of Φ_i, so that the graph of Φ_i is open in $X \times X_i$. And it is clear that $P_i(z) \subset \Phi_i(z)$ for each $z \in X$.

Next, since $X \times X_i$ is compact and metrisable, so is perfectly normal. Since the graph of Φ_i is open in $X \times X_i$, by Corollary 4.2 of Dugundji [**6**, p.148] there exists a continuous function $C_i : X \times X_i \to [0,1]$ such that $C_i(x,y) = 0$ for all $(x,y) \notin$ graph of Φ_i and $C_i(x,y) \neq 0$ for all $(x,y) \in$ graph of Φ_i. For each $i \in I$, we define a correspondence $F_i : X \to 2^{X_i}$ by

$$F_i(x) = \{y \in cl\, A_i(x) : C_i(x,y) = \max_{z \in cl\, A_i(x)} C_i(x,z)\}.$$

Then by Proposition 3.1.23 of Aubin and Ekeland [**2**], F_i is upper semicontinuous and for each $x \in X, F_i(x)$ is non-empty closed. Then a correspondence $G : X \to 2^X$ defined by $G(x) = \Pi_{i \in I} F_i(x)$ is also upper semicontinuous by Lemma 3 of Fan [**7**] and $G(x)$ is a non-empty compact subset of X for each $x \in X$. Therefore by Corollary 1, there exists a point $\widehat{x} \in X$ such that $\widehat{x} \in cl\, co\, G(\widehat{x})$; that is, $\widehat{x} \in cl\, co\, G(\widehat{x}) \subset \Pi_{i \in I} cl\, co\, F_i(\widehat{x})$. Since $F_i(\widehat{x}) \subset cl\, A_i(\widehat{x})$ and $A_i(\widehat{x})$ is convex, $cl\, co\, F_i(\widehat{x}) \subset cl\, A_i(\widehat{x})$. Therefore $\widehat{x}_i \in cl\, A_i(\widehat{x})$ for each $i \in I$. It remains to show that $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$. If $z_i \in A_i(\widehat{x}) \cap P_i(\widehat{x}) \neq \emptyset$, then $C_i(\widehat{x}, z_i) > 0$ so that $C_i(\widehat{x}, z_i') > 0$ for all $z_i' \in F_i(\widehat{x})$. This implies that $F_i(\widehat{x}) \subset \Phi_i(\widehat{x})$, which implies $\widehat{x}_i \in cl\, co\, F_i(\widehat{x}) \subset cl\, co\, \Phi_i(\widehat{x})$; this is a contradiction. ∎

REMARKS. (1) In a finite dimensional space, for a compact set A, $co\, A$ is compact and convex . Therefore when X_i is a subset of R^n, we can relax the assumption (b) of the definition of Θ-majorant as follows without affecting the conclusion of Theorem 2:

$$(b') \text{ for each } z \in N_x,\ z \notin co\, \phi_x(z).$$

And in this case, Theorem 2 generalises the Shafer-Sonnenschein theorem [**15**] in two aspects, that is, (i) P_i need not have open graph and (ii) an index set I may not be finite.

(2) Theorem 2 is a generalisation of the recent result of Tarafdar [**17**, Theorem 3.2].

Using the concept of Ψ-quasi-equilibrium described in the preliminaries, we further generalise Theorem 2 as follows:

THEOREM 3. *Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy where I is a countable set such that for each $i \in I$,*

(1) *X_i is a non-empty compact convex subset of a metrisable locally convex Hausdorff topological vector space,*

(2) *$\psi_i : X = \Pi_{i \in I} X_i \to R^+$ is a non-negative real-valued lower semicontinuous function,*

(3) *for each $x \in X, A_i(x)$ is non-empty convex,*

(4) *the correspondence $cl\, A_i : X \to 2^{X_i}$ is continuous for all x with $\psi_i(x) > 0$ and is almost upper semicontinuous for all x with $\psi_i(x) = 0$,*

(5) *the corespondence P_i' is Θ-majorised.*

Then Γ has a Ψ-quasi-equilibrium choice $\widehat{x} \in X$, that is, for each $i \in I$,

(a) $\widehat{x}_i \in cl\, A_i(\widehat{x})$,

(b) $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$ *and/or* $\psi_i(\widehat{x}) = 0$.

Proof: We can repeat the proof of Theorem 2 again. In the proof of Theorem 2, for each $i \in I$ we shall replace the correspondence F_i by a new correspondence $\widetilde{F}_i : X \to 2^{X_i}$ defined by $\widetilde{F}_i(x) = \{y \in cl\, A_i(x) : C_i(x,y)\,\psi_i(x) = \max\limits_{z \in cl\, A_i(x)} C_i(x,z)\,\psi_i(x)\}$ for each $x \in X$.

Since $\{x \in X : \psi_i(x) > 0\}$ is open, $\widetilde{F}_i$ is also upper semicontinuous. In fact, for any open set V containing $\widetilde{F}_i(x)$, if $\psi_i(x) = 0$ then $\widetilde{F}_i(x) = cl\, A_i(x) \subset V$. Since $cl\, A_i$ is upper semicontinuous, there exists an open neighbourhood W of x such that $\widetilde{F}_i(y) \subset cl\, A_i(y) \subset V$ for all $y \in W$; if $\psi_i(x) > 0$, then by Proposition 3.1.23 in [2] $\widetilde{F}_i(x) = F_i(x)$ is also upper semicontinuous at x, so that there exists an open neighbourhood W of x such that $F_i(y) \subset V$ for each $y \in W$. Then $W' = W \cap \{z \in X : \psi_i(z) > 0\}$ is an open neighbourhood of x such that $\widetilde{F}_i(y) \subset V$ for each $y \in W'$. Therefore $\widetilde{F}_i$ is upper semicontinuous.

Then $G = \Pi_{i \in I} F_i : X \to 2^X$ is also upper semicontinuous by Lemma 3 of Fan [7] and $G(x)$ is a non-empty compact subset of X for each $x \in X$. Therefore by the same proof as in Theorem 2, there exists a point $\widehat{x} \in X$ such that $\widehat{x}_i \in cl\, A_i(\widehat{x})$ for each $i \in I$. Finally, if $\psi_i(\widehat{x}) = 0$, then the conclusion (b) holds. In case $\psi_i(\widehat{x}) > 0$, if $z_i \in A_i(\widehat{x}) \cap P_i(\widehat{x}) \neq \emptyset$, then $C_i(\widehat{x}, z_i) > 0$ so that $C_i(\widehat{x}, z_i') > 0$ for all $z_i' \in F_i(\widehat{x})$. This implies that $F_i(\widehat{x}) \subset \Phi_i(\widehat{x})$, which implies $\widehat{x}_i \in cl\, co\, F_i(\widehat{x}) \subset cl\, co\, \Phi_i(\widehat{x})$; this is a contradiction. Therefore we have $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$. ∎

It should be noted that Theorem 3 generalises the theorem of Greenberg [**10**] and Theorem 3.2 in [**17**] in many aspects as in the Remarks of Theorem 2.

As remarked before, in most results on the existence of equilibria for abstract economies, the underlying spaces (commodity spaces or choice sets) are always compact and convex. However, in recent papers [**5, 18, 19**], the underlying spaces are not always compact and it should be noted that we will encounter many kinds of correspondences in various economic situations; so it is important that we shall consider several types of correspondences and obtain some existence results in non-compact settings.

Finally we prove the quasi-equilibrium existence theorem of Nash type non-compact abstract economy.

THEOREM 4. *Let I be any (possibly uncountable) index set and for each $i \in I$, let X_i be a convex subset of a locally convex Hausdorff topological vector space E_i and D_i be a non-empty compact subset of X_i. For each $i \in I$, let $f_i : X = \prod_{i \in I} X_i \to R$ be a continuous function and $\psi_i : X \to R^+$ be a non-negative real-valued lower semicontinuous function. For each $i \in I$, $S_i : X \to 2^{D_i}$ be a continuous correspondence for all $x \in X$ with $\psi_i(x) > 0$ and be almost upper semicontinuous for all $x \in X$ with $\psi_i(x) = 0$ such that*

(1) *$S_i(x)$ is a non-empty closed convex subset of D_i,*
(2) *$x_i \to f_i(x^i, x_i)$ is quasi-convex on $S_i(x)$.*

Then there exists an equlibrium point $\widehat{x} \in D = \prod_{i \in I} D_i$ such that for each $i \in I$,

(a) $\widehat{x}_i \in S_i(\widehat{x})$,
(b) $f_i(\widehat{x}^i, \widehat{x}_i) = \inf_{z \in S_i(\widehat{x})} f_i(\widehat{x}^i, z)$ *and/or* $\psi_i(\widehat{x}) = 0$.

PROOF: For each $i \in I$, we now define a correspondence $V_i : X \to 2^{X_i}$ by

$$V_i(x) = \{y \in S_i(x) \mid f_i(x^i, y)\, \psi_i(x) = \inf_{z \in S_i(x)} f_i(x^i, z)\, \psi_i(x)\}.$$

Since $\{x \in X : \psi_i(x) > 0\}$ is open, for each $x \in X$ with $\psi_i(x) > 0, V_i$ is upper semicontinuous at x by Proposition 3.1.23 in [**2**] and the same argument of the proof of Theorem 3; and for each $x \in X$ with $\psi_i(x) = 0, V_i(x) = S_i(x)$ so that V_i is also upper semicontinuous at x. Therefore for each $x \in X, V_i$ is upper semicontinuous at x and $V_i(x)$ is non-empty compact and convex.

Now we define $V : X \to 2^D$ by

$$V(x) = \prod_{i \in I} V_i(x) \qquad \text{for each} \quad x \in X.$$

Then, by Lemma 3 of Fan [**7**], V is also upper semicontinuous, and $V(x)$ is a non-empty compact convex subset of D for each $x \in X$. Therefore, by Corollary 2 there exists a point $\widehat{x} \in D$ such that $\widehat{x} \in V(\widehat{x})$, that is, for each $i \in I$, we have

(a) $\widehat{x}_i \in V_i(\widehat{x}) \subset S_i(\widehat{x})$ and
(b) $f_i(\widehat{x}^i, \widehat{x}_i) = \inf_{z \in S_i(\widehat{x})} f_i(\widehat{x}^i, z)$ and/or $\psi_i(x) = 0$. □

Theorem 4 is a non-compact generalisation of Theorem 3.1 in [**17**], and the Nash equilibrium existence theorem in [**14**]. And also Theorem 4 is a quasi-equilibrium type generalisation of a recent result of Im and Kim [**12**].

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