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A Continuous Field of Projectionless C^*-Algebras

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Abstract. We use some results about stable relations to show that some of the simple, stable, projectionless crossed products of O_2 by $\mathbb{R}$ considered by Kishimoto and Kumjian are inductive limits of type I C^*-algebras. The type I C^*-algebras that arise are pullbacks of finite direct sums of matrix algebras over the continuous functions on the unit interval by finite dimensional C^*-algebras.

1 Introduction

Quasi-free automorphisms of O_n were defined by Evans in [Ev] as follows. Let $F(H)$ denote the full Fock space $\bigoplus_{n=0}^{\infty}(\otimes^n H)$ for a separable Hilbert space H, where $\otimes^0 H$ denotes the one-dimensional subspace spanned by the vacuum vector Ω. Define a linear map $O\colon H \to B\big(F(H)\big)$ by $O(f)\Omega = f$ and $O(f)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = f \otimes f_1 \otimes \cdots \otimes f_n$. The map O satisfies the relations $O(f)^*O(g) = \langle g|f\rangle 1$ and $\sum_{i=1}^{n} O(h_i)O(h_i)^* \leq 1$ for any orthonormal basis $\{h_i\}$ with equality mod $K\big(F(H)\big)$ if H is finite dimensional. Let $O(H)$ denote the C^*-algebra generated by the image of this map. If $\dim H = \infty$ then $O(H) \cong O_\infty$, and if $\dim H = n < \infty$ then $O(H)/K\big(F(H)\big) \cong O_n$. If U is a unitary on H, then U gives rise to an automorphism of $O(H)$, denoted $O(U)$, such that $O(U)O(f) = O(Uf)$ for all $f \in H$. These in turn induce automorphisms of the O_n's, and such automorphisms are called quasi-free.

In this paper we shall consider the crossed products of O_n by $\mathbb{R}$ actions induced by one-parameter subgroups of $U(n)$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{C}^n$ and let $S_1 = \pi \circ O(e_1), \ldots, S_n = \pi \circ O(e_n)$, where π is the quotient map from $O(\mathbb{C}^n)$ onto O_n. Then, up to covariant isomorphism, all such automorphism groups are of the form $\alpha_t(S_1) = e^{2\pi i \lambda_1 t} S_1, \ldots, \alpha_t(S_n) = e^{2\pi i \lambda_n t} S_n$ for some real numbers $\lambda_1, \ldots, \lambda_n$. If all of the λ's are zero, then we have just a trivial action and the crossed product is the suspension of O_n. If we assume that $\lambda_1 \neq 0$ then, since a real crossed product is not changed, up to isomorphism, by scaling the real parameter, we may assume that $\lambda_1 = 1$. We then have an $n-1$ parameter family of crossed products $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$, where $\lambda = (\lambda_2, \ldots, \lambda_n)$.

In [Ks] Kishimoto showed that $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$ is simple if and only if one of the following two conditions holds:

1. $\{1, \lambda_2, \ldots, \lambda_n\}$ generates $\mathbb{R}$ as a closed subgroup and $\lambda_2, \ldots, \lambda_n$ are all positive.
2. $\{1, \lambda_2, \ldots, \lambda_n\}$ generates $\mathbb{R}$ as a closed subsemigroup.

In the special case of O_2, this reduces to saying that λ_2 is irrational, case 1 holding when $\lambda_2 > 0$, case 2 holding when $\lambda_2 < 0$. In [KK2] Kishimoto and Kumjian

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showed that in either of the above cases $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$ is stable. Furthermore, in case 1 it is projectionless and in case 2 it is purely infinite. In [KK1] the same authors showed that in case 1 there is a unique trace on $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$.

Combining Cuntz's calculation of the K-theory of the O_n's with Connes' analog of the Thom isomorphism theorem for real crossed products, *cf.* [Cu] and [Co], we see that $K_0(O_n \rtimes_{\alpha^\lambda} \mathbb{R}) \cong 0$ and $K_1(O_n \rtimes_{\alpha^\lambda} \mathbb{R}) \cong \mathbb{Z}/(n-1)\mathbb{Z}$.

Elliott has conjectured that all stable, simple, separable, nuclear C^*-algebras not of type I are characterised up to isomorphism by K-theoretic and tracial invariants. More precisely, the proposed invariant consists of the K_1 group, the ordered K_0 group, the cone of densely defined, lower semi-continuous traces, denoted T^+, and the natural pairing between K_0 and T^+. This conjecture has been borne out in many special cases; *cf.* [E1] for a survey. In [E2] Elliott showed that a wide variety of values of the above invariant occur for C^*-algebras satisfying the classification hypotheses and arising as inductive limits of type I C^*-algebras. Among the values given by Elliott's construction are those of the simple projectionless crossed products of O_n mentioned above.

This suggested that these algebras could be inductive limits of type I C^*-algebras, and our object in the present paper is to show that, at least for λ's in a dense category 2 subset of the positive multi-indices, this is the case. Our methods will involve first obtaining a continuous field of C^*-algebras by varying the $\mathbb{R}$ action, then carefully analysing some of the fibres of the field for which the action results in a crossed product with a more tractable structure, deducing an inductive limit structure for these fibres, and finally, using results from the theory of stable relations, in particular those in [ELP], to deduce the existence of similar inductive limit structure in other fibres. The proofs for general O_n are virtually the same as for the the special case of O_2, with just some additional complication in the book-keeping. Thus for most of the paper we shall concentrate on O_2 and leave the general case to the final section, where we also deal with some other questions raised along the way.

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2 A Continuous Field of C^*-Algebras

It follows from Corollary 3.6 of [Rie] that the algebras $O_2 \rtimes_{\alpha^\lambda} \mathbb{R}$ fit into a continuous field with the images of fixed elements of $C_c(\mathbb{R}, O_2)$ in each crossed product giving continuous sections (*cf.* also [ENN]). In this section we shall analyse the rational fibres of this continuous field, arriving at a description of them as mapping tori over C^*-algebras that we can describe in terms of generators and relations. In the next section these presentations are analysed in more detail.

Suppose $\lambda = p/q$ is a rational number expressed in lowest terms. Then the action α^λ is periodic with period q. We make the change of variables $t \mapsto qt$ to get an action with period 1 for which the crossed product is isomorphic. For the remainder of this section we shall call this new action α, explicitly, $\alpha_t(S_1) = e^{2\pi iqt}S_1$, $\alpha_t(S_2) = e^{2\pi ipt}S_2$.

In [Bl, Prop. 10.3.2] it is shown that the crossed product $O_2 \rtimes_\alpha \mathbb{R}$ is isomorphic to the mapping torus of $O_2 \rtimes_\alpha \mathbb{T}$ by an automorphism generating the dual action of $\mathbb{Z}$. (*Cf.* [OP] for a more general version of this theorem.)

To analyse these mapping-tori it will be convenient to express O_2 in a different way. Write $M_{2^\infty} \otimes K$ as the infinite tensor product, indexed by the integers, of copies of M_2 trailing off as the identity to the right and as e_{11} to the left. (Assume we have fixed systems of matrix units $\{e_{ij}^k\}_{k\in\mathbb{Z}}$ for each factor in the tensor product.) Let β denote the automorphism of $M_{2^\infty} \otimes K$ given by shifting one position to the right in the infinite tensor product. Observe that β scales the unique trace on $M_{2^\infty} \otimes K$ by 1/2, *i.e.*, $\tau \circ \beta = (1/2)\tau$. If e is the unit of the 0-th copy of M_2 (*i.e.*, $e = \cdots e_{11}^{-2} \otimes e_{11}^{-1} \otimes 1 \otimes 1 \otimes \cdots$), then it follows from [Rør] that $e\big((M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}\big)\, e \cong O_2$, and in fact more is true. Define an action $\tilde{\alpha}_t$ of $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ on $(M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}$ as follows. On $M_{2^\infty} \otimes K$ in its tensor product expression $\tilde{\alpha}_t$ is the product type automorphism given by infinitely many copies of $u_t = \gamma_1^t e_{11} + \gamma_2^t e_{22}$, where γ_1^t and γ_2^t denote $e^{2\pi i q t}$ and $e^{2\pi i p t}$ respectively. If V is the adjoined unitary in the multiplier algebra of $(M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}$ implementing β, then $\tilde{\alpha}_t(V) = \gamma_1^t V$. This action fixes e, so we get an action, also denoted $\tilde{\alpha}_t$, of $\mathbb{T}$ on $e\big((M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}\big)\, e$. We then have the following.

Lemma 2.1 *The C^* dynamical systems $(O_2, \alpha_t, \mathbb{T})$ and $\Big(e\big((M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}\big)\, e, \tilde{\alpha}_t, \mathbb{T}\Big)$ are covariantly isomorphic.*

Proof At this point we introduce a simplification in our notation. When we write $e_{ij}^k \otimes \cdots \otimes e_{st}^{k+n}$ for an element of $M_{2^\infty} \otimes K$ it is assumed that all entries to the right of those shown are 1's and all those to the left are e_{11}'s, so for example $e_{11}^{-1} \otimes 1^0 \otimes 1^1$ is the element e. Let $T_1 = Ve$ and $T_2 = (e_{11}^{-1} \otimes e_{21}^0 \otimes 1^1)Ve$. Then straightforward computations show that $T_1^* T_1 = T_2^* T_2 = e$, $T_1 T_1^* + T_2 T_2^* = e$, $\tilde{\alpha}_t(T_1) = \gamma_1^t T_1$ and $\tilde{\alpha}_t(T_2) = \gamma_2^t T_2$. Thus by the universal property of O_2 there is an isomorphism of O_2 onto the sub-C^*-algebra of $(M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}$ generated by T_1 and T_2 intertwining the actions. It remains to show that T_1 and T_2 generate all of $e\big((M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}\big)\, e$. Notice that as $\beta^{-k}(e) \geq e$ in $(M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}$ for $k \geq 0$, we have $eAV^k e = (eAe)(eVe)^k$ for all $A \in M_{2^\infty} \otimes K$. As $eVe = T_1$ we have only to see that $e(M_{2^\infty} \otimes K)e \subseteq C^*(T_1, T_2)$. This follows from the observations that $e(M_{2^\infty} \otimes K)e$ is generated by those tensors with e_{11} in all positions to the left of zero, that if $eAe \in C^*(T_1, T_2)$ for some $A \in M_{2^\infty} \otimes K$ then $\beta^k(eAe) = (Ve)^k A (eV^*)^k = T_1^k A T_1^{*k} \in C^*(T_1, T_2)$ for all $k \geq 0$, and the calculations $T_1 T_2^* = e_{21}^0$, $T_1 T_1^* = e_{11}^0$ and $T_2 T_2^* = e_{22}^0$, which show that the matrix units for the 0-th copy of M_2 are included.

The next step in our analysis is to interchange the order of the two crossed products. On the dense subalgebra $C_c(\mathbb{Z}, M_{2^\infty} \otimes K)$ of $(M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}$ we have $(\tilde{\alpha}_t g)(k) = (t,k)^q \alpha_t\big(g(k)\big)$ for all $g \in C_c(\mathbb{Z}, M_{2^\infty} \otimes K)$, $t \in \mathbb{R}/\mathbb{Z}$, and $k \in \mathbb{Z}$, where (t,k) denotes the pairing of an element of $\mathbb{T}$ with an element of $\mathbb{Z} \cong \hat{\mathbb{T}}$. Consider the action, denoted $\tilde{\beta}$, on $(M_{2^\infty} \otimes K) \rtimes_{\tilde{\alpha}} \mathbb{T}$ defined by $(\tilde{\beta}_k f)(t) = \overline{(t,k)}^q \beta_k\big(f(t)\big)$ for all $f \in C_c(\mathbb{T}, M_{2^\infty} \otimes K)$, $t \in \mathbb{R}/\mathbb{Z}$, and $k \in \mathbb{Z}$. We then have the following lemma, which is easy to check.

Lemma 2.2 *The map $\Phi\colon C\big(\mathbb{T}, C_c(\mathbb{Z}, M_{2^\infty} \otimes K)\big) \to C_c\big(\mathbb{Z}, C(\mathbb{T}, M_{2^\infty} \otimes K)\big)$ given*

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by $(\Phi f)(k,t) = \overline{(t,k)}^q f(t,k)$ *for all* $t \in \mathbb{R}/\mathbb{Z}$ *and* $k \in \mathbb{Z}$ *extends to an isomorphism of* $\big((M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}\big) \rtimes_{\tilde\alpha} \mathbb{T}$ *with* $\big((M_{2^\infty} \otimes K) \rtimes_{\tilde\alpha} \mathbb{T}\big) \rtimes_{\tilde\beta} \mathbb{Z}$.

Now we dissect the crossed product $(M_{2^\infty} \otimes K) \rtimes_{\tilde\alpha} \mathbb{T}$ while keeping track of β. If we view $M_{2^\infty} \otimes K$ as the limit of the inductive system $M_2 \to M_2 \otimes M_2 \otimes M_2 \to \otimes^5 M_2 \to \cdots$, where the maps are the inclusions given by $x \mapsto e_{11} \otimes x \otimes 1$, we see that the action $\tilde\alpha$ is of inductive limit type, *i.e.*, it leaves invariant each of the sub-C^*-algebras in this increasing sequence. The crossed product of the limit, $M_{2^\infty} \otimes K$, is then the inductive limit of the crossed products $(M_{2^{2n+1}}) \rtimes_{\tilde\alpha} \mathbb{T}$ with the obvious inclusions. The action $\tilde\alpha$ of $\mathbb{T}$ on $M_{2^{2n+1}}$ is exterior equivalent to the trivial action via the unitary cocycle $t \mapsto \otimes^{2n+1} u_t$. Thus we get an isomorphism $\psi_n \colon (M_{2^{2n+1}}) \rtimes_{\tilde\alpha} \mathbb{T} \to (M_{2^{2n+1}}) \rtimes_{\mathrm{id}} \mathbb{T}$ given by $(\psi_n x)(t) = x(t)(\otimes^{2n+1} u_t)$ for all $x \in C(\mathbb{T}, M_{2^{2n+1}})$. The Fourier transform defines an isomorphism $F \colon (M_{2^{2n+1}}) \rtimes_{\mathrm{id}} \mathbb{T} \to (M_{2^{2n+1}}) \otimes C_0(\mathbb{Z})$ by $(Ff)(k) = \int_{\mathbb{R}/\mathbb{Z}} f(t)(t,k)\,dt$ for all $k \in \mathbb{Z}$, so we have that $(M_{2^\infty} \otimes K) \rtimes_{\tilde\alpha} \mathbb{T}$ is AF. To determine its structure completely we have only check how the inclusions of the $(M_{2^{2n+1}}) \rtimes_{\tilde\alpha} \mathbb{T}$'s one into the next are transformed by the exterior equivalence and the Fourier transform. We get a diagram:

$$\begin{tikzcd}
(M_{2^{2n+1}}) \rtimes_{\tilde\alpha} \mathbb{T} \arrow[r, "\psi_n"] \arrow[d, "x\mapsto e_{11}\otimes x\otimes 1"'] & (M_{2^{2n+1}}) \rtimes_{\mathrm{id}} \mathbb{T} \arrow[r, "F"] \arrow[d, "\Gamma_n^1"'] & (M_{2^{2n+1}}) \otimes C_0(\mathbb{Z}) \arrow[d, "\Gamma_n^2"'] \\
(M_{2^{2n+3}}) \rtimes_{\tilde\alpha} \mathbb{T} \arrow[r, "\psi_{n+1}"] & (M_{2^{2n+3}}) \rtimes_{\mathrm{id}} \mathbb{T} \arrow[r, "F"] & (M_{2^{2n+3}}) \otimes C_0(\mathbb{Z}).
\end{tikzcd}$$

For the first square we see that $(\Gamma_n^1 y)(t) = \gamma_1^t\big(e_{11} \otimes y(t) \otimes u_t\big)$ for all $t \in \mathbb{T}$, $y \in C(\mathbb{T}, M_{2^{2n+1}})$, and for the second $(\Gamma_n^2 z)(k) = e_{11} \otimes \big(z(2q+k) \otimes e_{11} + z(q+p+k) \otimes e_{22}\big)$, for all $k \in \mathbb{Z}$, $z \in C_c(\mathbb{Z}, M_{2^{2n+1}})$. We thus arrive at the following description:

Lemma 2.3 $(M_{2^\infty} \otimes K) \rtimes_{\tilde\alpha} \mathbb{T}$ *is isomorphic to the limit of the inductive system* $\{M_{2^{2n+1}} \otimes C_0(\mathbb{Z}), \Gamma_n^2\}_{n=0}^\infty$.

Next we see what becomes of the automorphism $\tilde\beta$. We have a similar diagram:

$$\begin{tikzcd}
(M_{2^{2n+1}}) \rtimes_{\tilde\alpha} \mathbb{T} \arrow[r, "\psi_n"] \arrow[d, "x\mapsto \gamma_1^{-t} e_{11}\otimes e_{11}\otimes x(t)"'] & (M_{2^{2n+1}}) \rtimes_{\mathrm{id}} \mathbb{T} \arrow[r, "F"] \arrow[d, "\tilde\beta'"'] & (M_{2^{2n+1}}) \otimes C_0(\mathbb{Z}) \arrow[d, "\tilde\beta''"'] \\
(M_{2^{2n+3}}) \rtimes_{\tilde\alpha} \mathbb{T} \arrow[r, "\psi_{n+1}"] & (M_{2^{2n+3}}) \rtimes_{\mathrm{id}} \mathbb{T} \arrow[r, "F"] & (M_{2^{2n+3}}) \otimes C_0(\mathbb{Z}).
\end{tikzcd}$$

For the first square we get $(\tilde\beta' y)(t) = \gamma_1^t e_{11} \otimes e_{11} \otimes y(t)$, for all $t \in \mathbb{T}$ and $y \in C(\mathbb{T}, M_{2^{2n+1}})$, and for the second $\big(\tilde\beta''(z)\big)(k) = e_{11} \otimes e_{11} \otimes z(k+q)$ for all $z \in C_c(\mathbb{Z}, M_{2^{2n+1}})$.

The dual action $\hat\alpha$ of $\mathbb{Z}$ on $e\big((M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}\big) e \rtimes_{\tilde\alpha} \mathbb{T}$ is just the restriction of the dual action of $\mathbb{Z}$, also denoted $\hat\alpha$, on $\big((M_{2^\infty} \otimes K) \rtimes_\beta \mathbb{Z}\big) \rtimes_{\tilde\alpha} \mathbb{T} \cong \big((M_{2^\infty} \otimes K) \rtimes_{\tilde\alpha} \mathbb{T}\big) \rtimes_{\tilde\beta} \mathbb{Z}$. Since $\hat\alpha$ commutes with $\tilde\beta$ to describe it we need only see what it does on

$(M_{2^\infty} \otimes K) \rtimes_{\tilde{\alpha}} \mathbb{T}$. We again get a diagram:

$$\begin{tikzcd}
(M_{2^{2n+1}}) \rtimes_{\tilde{\alpha}} \mathbb{T} \arrow[r, "\psi_n"] \arrow[d, "(\hat{\alpha} f)(t)=(t,1)f(t)"'] & (M_{2^{2n+1}}) \rtimes_{\mathrm{id}} \mathbb{T} \arrow[r, "F"] \arrow[d, "\hat{\alpha}'"'] & (M_{2^{2n+1}}) \otimes C_0(\mathbb{Z}) \arrow[d, "\hat{\alpha}''"'] \\
(M_{2^{2n+1}}) \rtimes_{\tilde{\alpha}} \mathbb{T} \arrow[r, "\psi_{n+1}"] & (M_{2^{2n+1}}) \rtimes_{\mathrm{id}} \mathbb{T} \arrow[r, "F"] & (M_{2^{2n+1}}) \otimes C_0(\mathbb{Z}),
\end{tikzcd}$$

and it is easy to check that $(\hat{\alpha}' f)(t) = (t,1)f(t)$ for all $t \in \mathbb{T}$ and $f \in C(\mathbb{T}, M_{2^{2n+1}})$, $(\hat{\alpha}'' f)(k) = f(k+1)$ for all $k \in \mathbb{Z}$ and $f \in C(\mathbb{T}, M_{2^{2n+1}})$.

Finally, we cut down by the projection e. Cutting down the inductive system by e at each stage gives us $M_2 \to M_{2^2} \to M_{2^3} \to \cdots$, where the inclusions are $x \mapsto x \otimes 1$ at each stage. Using the same symbols for the cutdown versions of the maps found above we get:

$$\begin{aligned}
\tilde{\beta}''(z)(k) &= e_{11} \otimes z(k+q) \\
(\hat{\alpha}'' f)(k) &= f(k+1) \\
(\Gamma_n^2 z)(k) &= z(2q+k) \otimes e_{11} + z(p+q+k) \otimes e_{22},
\end{aligned}$$

and $e\big((M_{2^\infty} \otimes K) \rtimes_{\tilde{\alpha}} \mathbb{T}\big)\, e \cong \varinjlim\{M_{2^n} \otimes C_0(\mathbb{Z}), \Gamma_n^2\}$.

We may summarise this situation in the following two diagrams:

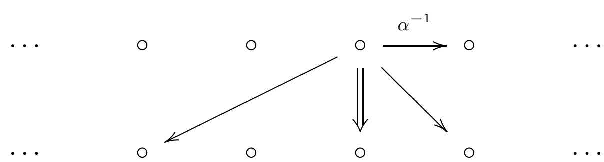

Diagram 1: Infinite case ($\lambda < 0$)

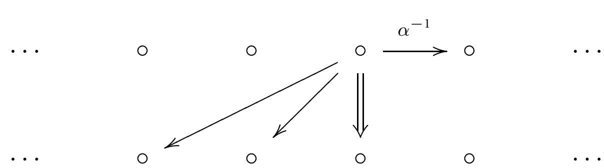

Diagram 2: Finite case ($\lambda > 0$)

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In these diagrams the dots and downward thin arrows represent the Bratteli diagram of the inductive system we have found for $e\big((M_{2^\infty} \otimes K) \rtimes_{\tilde{\alpha}} \mathbb{T}\big) e$. (The arrows shown are repeated at every dot in the diagram, which extends infinitely to either side and downward.) The thick downward arrow represents an adjoined partial isometry implementing $\tilde{\beta}''$ on that summand, *i.e.*, setting a minimal projection in the upper dot equivalent to a minimal projection in the lower dot, and the arrow labelled α^{-1} is supposed to indicate that the automorphism $\hat{\alpha}''$ moves everything one step over. It is also to be understood that there is a compatibility between the fat arrows and the Bratteli diagram, the partial isometries with the same range and base spaces being identified. (All this will be precisely stated in the next section.) From this point on we shall only consider the finite case, leaving the infinite case to the closing remarks in Section 5. In the next section we shall undertake an analysis of the universal C^*-algebras given by generators and relations described by such a diagram, and conclude that these diagrams in fact give a complete description of the positive rational fibres of our continuous field.

3 The Rational Fibres

Theorem 3.1 *Let W^n denote the set of words of length n in the letters a, b. Let p and q be two distinct positive integers with $(p, q) = 1$. Let $A(p, q)$ denote the universal C^*-algebra with generators and relations as described below. Then $A(p, q)$ is a simple AF algebra. Furthermore, the map α given below on the generators defines an automorphism, to be denoted $\hat{\alpha}$, of $A(p, q)$, and for $\lambda = p/q$ we have $O_2 \rtimes_{\alpha^\lambda} \mathbb{R} \cong M_{\hat{\alpha}}\big(A(p, q)\big)$, the mapping torus of $A(p, q)$ by $\hat{\alpha}$.*

Generators:

$$E^{k,n}_{w,v} \quad \text{for } k \in \mathbb{Z}, n \in \mathbb{N}, w, v \in W^n$$

$$V^{k,n} \quad \text{for } k \in \mathbb{Z}, n \in \mathbb{N}$$

Relations:

$$E^{k,n}_{w,v} E^{l,n}_{s,t} = 0 \quad \text{if } l \neq k \tag{1}$$

$$E^{k,n}_{w,v} E^{k,n}_{s,t} = \delta_{vs} E^{k,n}_{w,t} \tag{2}$$

$$E^{k,n*}_{w,v} = E^{k,n}_{v,w} \tag{3}$$

$$E^{k,n}_{w,v} = E^{k-p,n+1}_{wb,vb} + E^{k-q,n+1}_{wa,va} \tag{4}$$

$$V^{k,n*} V^{k,n} = \sum_{w \in W^n} E^{k,n}_{w,w} \tag{5}$$

$$V^{k,n} E^{k,n}_{w,v} V^{k,n*} = E^{k,n+1}_{aw,av} \tag{6}$$

$$V^{k,n} E^{k,n}_{w,w} = V^{k-p,n+1} E^{k-p,n+1}_{wb,wb} + V^{k-q,n+1} E^{k-q,n+1}_{wa,wa} \tag{7}$$

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Automorphism:

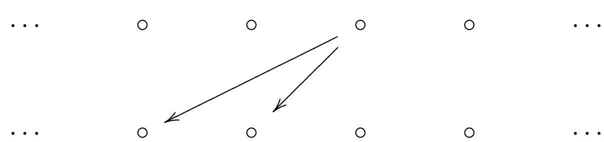
$$\alpha(E^{k,n}_{w,v}) = E^{k-1,n}_{w,v} \quad \alpha(V^{k,n}) = V^{k-1,n}$$

We shall undertake some preliminary analysis before proving this theorem. For the sake of simplicity we shall in the following assume $q > p$, though obviously analogous statements hold with similar proofs if $p > q$. The K-theory of $A(p,q)$ will be discussed in Section 5.

In order to keep track of the generators and relations it will be helpful to use diagrams like Diagram 2 above as a sort of short-hand to represent them. We do this as follows. For a fixed pair (n,k) relations (2) and (3) show that $\{E^{k,n}_{w,v} \mid w, v \in W^n\}$ generates a quotient of a full matrix algebra isomorphic to $M_{2^n}(\mathbb{C})$. (We shall later see that it's not a proper quotient, the natural map is injective.) We represent one of these subalgebras with a dot, $\circ$.

For fixed n, relation (1) shows that the dots corresponding to different values of k represent orthogonal subalgebras. $\{E^{k,n}_{w,v} \mid w, v \in W^n, k \in \mathbb{Z}\}$ with relations (1), (2), and (3) thus gives a presentation of a copy of $c_0(\mathbb{Z}) \otimes M_{2^n}(\mathbb{C})$. We represent these subalgebras by rows of dots, one row for each n, the dots being the orthogonal summands as above.

Relation (4) gives an inclusion of the subalgebra corresponding to the n-th row into that of the $n+1$-st row. We represent this inclusion in the fashion of a Bratteli diagram, using thin arrows to indicate which minimal direct summands are mapped into which.



(In this picture, the longer downward arrow moves q dots over, the short one moves p dots over.)

Relations (5) and (6), in combination with (2) and (3), show that $V^{k,n}$ is a partial isometry with support projection in the k-th dot in the n-th row subalgebra and range projection in the k-th dot in the $n+1$-st row subalgebra that sets a minimal projection in its support dot Murray-von Neumann equivalent to a minimal projection in its range dot. We shall represent one of these partial isometries by a fat downward arrow from its support dot to its range dot. Putting all of the above together we arrive at a picture like Diagram 2 above.

Below we shall deduce rules for reading off the structure of $A(p,q)$ from finite subsets of the full sets of generators and relations, using these pictures as an aid.

Let $S(n,L,K)$ denote the set of generators $\{E^{k,n+1}_{w,v} \mid w, v \in W^{n+1}, L-K-q \le k \le L+K\} \cup \{E^{k,n}_{w,v} \mid w, v \in W^n, |L-k| \le K\} \cup \{V^{k,n} \mid |L-k| \le K\}$, where L is any integer and K is any positive integer. Let $A\bigl(S(n,L,K)\bigr)$ denote the sub-C^*-algebra of $A(p,q)$ generated by $S(n,L,K)$. From relations (2), (3) and (5) it follows that each $V^{n,k}$ is a sum of terms $(V^{n,k}E^{k,n}_{w,w})$. Now repeated applications of relations (4) and (7) show that for any finite set of generators F, the sub-C^*-algebra of $A(p,q)$ generated by the

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elements of F is contained in some $A\bigl(S(n,L,K)\bigr)$. This shows that, in a sense made precise below, to see that $A(p,q)$ is AF we need only look at pieces of the diagram that look like

$$\begin{tikzcd}
 & & \circ^{k-q} \arrow[dll] \arrow[dl] \arrow[d, Rightarrow] & & \circ^{k-p} \arrow[dlll] \arrow[dl] \arrow[d, Rightarrow] & \circ^{k} \arrow[dlll] \arrow[dl] \arrow[d, Rightarrow] & n \\
\circ_{k-2q} & \circ_{k-p-q} & \circ_{k-q} & \circ_{k-2p} & \circ_{k-p} & \circ_{k} & n+1
\end{tikzcd}$$

Diagram 3

i.e., finite, two row diagrams whose lower rows contain the ends of the arrows emanating from the dots in their upper rows. (In this diagram we have assumed $p/q < 1/2$.)

Let $C^*\bigl(S(n,L,K)\bigr)$ denote the universal C^*-algebra given by the following presentation. The generators of $C^*\bigl(S(n,L,K)\bigr)$ are identified with the the elements of $S(n,L,K)$ and the relations are those in the statement of Theorem 3.1 that only involve elements of $S(n,L,K)$, in other words, $C^*\bigl(S(n,L,K)\bigr)$ is a universal C^*-algebra for a diagram like Diagram 3 above. Clearly the identification of the generators gives a surjective map from $C^*\bigl(S(n,L,K)\bigr)$ to $A\bigl(S(n,L,K)\bigr)$. Our analysis below will show that $C^*\bigl(S(n,L,K)\bigr)$, and hence $A\bigl(S(n,L,K)\bigr)$, is finite dimensional.

In the following we shall refer to a diagram of the form

$$\begin{tikzcd}
 & & \circ \arrow[dll] \arrow[dl] \arrow[d, Rightarrow] & n \\
\circ_{k-q} & \circ_{k-p} & \circ_{k} & n+1
\end{tikzcd}$$

Diagram 4

as a mer. The C^*-algebra generated by the elements shown in this diagram (with only the relations shown) is $M_{2^{n+2}} \oplus M_{2^{n+2}}$, where dot $k-q$ includes into the first summand with multiplicity one, dot $k-p$ goes into the second summand with multiplicity one, and dot k includes into each summand with multiplicity one. We shall use these mers to define a relation between the dots in the lower row of a two row diagram such as Diagram 3 above. We shall say that dot A is related by a mer to dot B if there is a sub-diagram of the form of Diagram 4 with A as the target of the fat arrow and B as the target of a thin arrow. We shall also say that A is related to B by mers if there is a

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sequence of dots starting with A and ending with B with each dot related to the next (what could also be called being related by a polymer). Finally, we say that any dot is related (by mers) to itself. (In terms of the pictures, dot A is related to dot B if there is a zig-zag path going up fat arrows and down thin ones from dot A to dot B.)

Lemma 3.2 *Consider the two row diagram corresponding to some set of generators $S(n, L, K)$, and consider a dot, $\#(l, n+1)$ in the lower row that is not related by mers in this diagram to any dot lying to the left of it. Then the elements of the set S described below form the first row of matrix units for a full matrix algebra contained in $C^*\big(S(n, L, K)\big)$. Furthermore, this matrix algebra is a direct summand of $C^*\big(S(n, L, K)\big)$.*

Let $w \in W^{n+1}$ be distinguished as the first word in W^{n+1}, and fix an element $s \in W^n$. Then S is the union of those A_νs described below, where ν is a word in the letters p and q, for which the elements of A_ν are words in the elements of $S(n, L, K)$; explicitly, moving the sum of the letters of ν to the right of dot $\#(l, n+1)$ in the diagram for $S(n, L, K)$ brings one to a dot still in the diagram, and if ν begins with a p, then A_ν is included only if $V^{l+p,n*} \in S(n, L, K)$.

$$A_0 = \{E^{l,n+1}_{w,v} \mid v \in W^{n+1}\}$$

$$A_p = \{E^{l,n+1}_{w,sb} V^{l+p,n*} E^{l+p,n+1}_{as,v} \mid v \in W^{n+1}\}$$

$$A_q = \{E^{l,n+1}_{w,sa} V^{l+q,n*} E^{l+q,n+1}_{as,v} \mid v \in W^{n+1}\}$$

$$A_{pq} = \{E^{l,n+1}_{w,sb} V^{l+p,n*} E^{l+p,n+1}_{as,sa} V^{l+p+q,n*} E^{l+p+q,n+1}_{as,v} \mid v \in W^{n+1}\}$$

$$A_{qp} = \{E^{l,n+1}_{w,sa} V^{l+q,n*} E^{l+q,n+1}_{as,sb} V^{l+p+q,n*} E^{l+p+q,n+1}_{as,v} \mid v \in W^{n+1}\}$$

and in general, if m is a word in p's and q's, the sum of whose letters is $|m|$, then an element in A_{mq} consists of the word in A_m ending in $E^{l+|m|,n+1}_{as,sa}$ followed by $V^{l+|m|+q,n*} E^{l+|m|+q,n+1}_{as,v}$ for some $v \in W^{n+1}$, and an element of A_{mp} consists of the word in A_m ending in $E^{l+|m|,n+1}_{as,sb}$ followed by $V^{l+|m|+p,n*} E^{l+|m|+p,n+1}_{as,v}$ for some $v \in W^{n+1}$.

Proof It is straightforward to check that for $v, u \in S$, $v^* v$ is a projection, $vv^* = uu^*$, and $uv^* = 0$ if $u \neq v$, so the elements of S do form the first row for a system of matrix units for a full matrix algebra. Call this matrix algebra F_l and let 1_{F_l} denote its unit. (Strictly speaking we still have to show that F_l is not zero, but this will be obvious from the sequel.)

To show that F_l is a direct summand of $C^*\big(S(n, L, K)\big)$ it will suffice to show that 1_{F_l} is in the centre of $C^*\big(S(n, L, K)\big)$ and that $1_{F_l} C^*\big(S(n, L, K)\big) 1_{F_l} = F_l$.

To see that 1_{F_l} is in the centre of $C^*\big(S(n, L, K)\big)$ it suffices to check that it commutes with the elements of $S(n, L, K)$. We check this first for the E's. $1_{F_l} = \sum_{m | A_m \subseteq S} (\sum_{v \in A_m} v^* v)$. Consider an element $E^{k,n+1}_{f,g}$. As any $v \in A_m$ ends in a matrix unit of dot $\#(l + |m|, n+1)$, we have that $E^{k,n+1}_{f,g} \perp v^* v$ if $v \in A_m$ and $l + |m| \neq k$. Suppose $l + |m| = k$ and consider $\sum_{v \in A_m} v^* v$. If $|m| = 0$ then this is the unit of the $(k, n+1)$-th dot and it commutes with $E^{k,n+1}_{f,g}$. If $|m| \neq 0$ then $\sum_{v \in A_m} v^* v = \sum_{v \in A_m} (\sum_{t \in W^{n+1}} E^{k,n+1}_{t,as} v^* v E^{k,n+1}_{as,t})$, and we get $E^{k,n+1}_{f,g} (\sum_{v \in A_m} v^* v) = \sum_{v \in A_m} (E^{k,n+1}_{f,as} v^* v E^{k,n+1}_{as,g}) = (\sum_{v \in A_m} v^* v) E^{k,n+1}_{f,g}$. Thus, we have that 1_{F_l} commutes with $E^{k,n+1}_{f,g}$.

Checking that 1_{F_l} commutes with the V's in $S(n, L, K)$ is a bit more tedious. We first introduce some notation. Fix a $V^{n,k}$ in $S(n, L, K)$. Define

$$\Lambda = \{m \mid A_m \subseteq S \text{ and } |m| = k - l\}$$
$$\Lambda_p = \{m \mid A_m \subseteq S \text{ and } |m| = k - l - p\}$$
$$\Lambda_q = \{m \mid A_m \subseteq S \text{ and } |m| = k - l - q\}.$$

Then we have that

$$1_{F_l} V^{k,n} = \sum_{m \in \Lambda} \Big(\sum_{v \in A_m} v^* v \Big) V^{k,n}$$

and

$$V^{k,n} 1_{F_l} = \overbrace{\sum_{d \in \Lambda_p} V^{k,n} \Big(\sum_{v \in A_d} v^* v \Big)}^{\text{blob 1}} + \overbrace{\sum_{t \in \Lambda_q} V^{k,n} \Big(\sum_{v \in A_t} v^* v \Big)}^{\text{blob 2}},$$

where we use the convention that a sum over an empty index set gives 0. Notice that $\Lambda = \{fp \mid f \in \Lambda_p\} \cup \{gq \mid g \in \Lambda_q\}$. (This uses that $k \neq l$. That $k = l$ is not a possibility follows from our definition of $S(n, L, K)$ and the assumption that dot $(l, n+1)$ is not related by mers to any dots further left in the diagram for $S(n, L, K)$.) We then have

$$\sum_{m \in \Lambda} \Big(\sum_{v \in A_m} v^* v \Big) V^{k,n} = \overbrace{\sum_{d \in \Lambda_p} \Big(\sum_{v \in A_{dp}} v^* v \Big) V^{k,n}}^{\text{blob 3}} + \overbrace{\sum_{t \in \Lambda_q} \Big(\sum_{v \in A_{tq}} v^* v \Big) V^{k,n}}^{\text{blob 4}}.$$

We shall show that blob 1 is equal to blob 3 (a similar argument shows that blob 2 is equal to blob 4). Look first at blob 3.

$$\sum_{d \in \Lambda_p} \Big(\sum_{v \in A_{dp}} v^* v \Big) V^{k,n} = \sum_{d \in \Lambda_p} \Big\{ \sum_{t \in W^{n+1}} E^{k,n+1}_{t,as} V^{k,n} w_d^* w_d V^{k,n*} E^{k,n+1}_{as,t} V^{k,n} \Big\},$$

where w_d is the word in A_d that ends in $E^{k-p,n+1}_{as,sb}$. In the above $E^{k,n+1}_{t,as} V^{k,n} w_d^* w_d V^{k,n*} \cdot E^{k,n+1}_{as,t} V^{k,n}$ is equal to 0 if t starts with a b, and if $t = ar$ for some r in W^n it is equal to $E^{k,n+1}_{ar,as} V^{k,n} w_d^* w_d E^{k,n}_{s,r} = E^{k,n+1}_{ar,as} V^{k,n} w_d^* w_d E^{k-p,n+1}_{sb,rb}$ by relations (4), (5) and (6). Thus we get:

$$(\dagger) \qquad \begin{aligned} \sum_{d \in \Lambda_p} \Big(\sum_{v \in A_{dp}} v^* v \Big) V^{k,n} &= \sum_{d \in \Lambda_p} \Big\{ \sum_{r \in W^n} E^{k,n+1}_{ar,as} V^{k,n} w_d^* w_d E^{k-p,n+1}_{sb,rb} \Big\} \\ &= \sum_{d \in \Lambda_p} \Big\{ \sum_{r \in W^n} V^{k,n} E^{k-p,n+1}_{rb,sb} w_d^* w_d E^{k-p,n+1}_{sb,rb} \Big\} \end{aligned}$$

where we have used for the second equality the fact that $E^{k,n+1}_{ar,as}V^{k,n} = V^{k,n}E^{k,n}_{r,s} = V^{k,n}(E^{k-p,n+1}_{rb,sb} + E^{k-q,n+1}_{ra,sa})$ and that $E^{k-q,n+1}_{ra,sa}w^*_d = 0$. Now we look at blob 1.

$$
\begin{aligned}
\sum_{d\in\Lambda_p} V^{k,n}\Bigl(\sum_{v\in A_d} v^*v\Bigr) &= \sum_{d\in\Lambda_p} V^{k,n}\Bigl(\sum_{u\in W^n} E^{k-p,n+1}_{ub,ub} + E^{k-q,n+1}_{ua,ua}\Bigr)\Bigl(\sum_{v\in A_d} v^*v\Bigr)\\
&= \sum_{d\in\Lambda_p} V^{k,n}\Bigl(\sum_{u\in W^n} E^{k-p,n+1}_{ub,ub}\Bigr)\Bigl(\sum_{v\in A_d} v^*v\Bigr)\\
&= \sum_{d\in\Lambda_p} V^{k,n}\Bigl\{\sum_{u\in W^n} E^{k-p,n+1}_{ub,sb}w^*_d w_d E^{k-p,n+1}_{sb,ub}\Bigr\}
\end{aligned}
$$

where the first equality follows from relations (2), (3), (5) and (6), and the second comes from the fact that elements of A_d for $d \in \Lambda_p$ end in a matrix unit for dot $\#(k-p, n+1)$. The last expression in the above is clearly equal to the expression for blob 3 given in equation (†) above. Thus we have that 1_{F_l} is in the centre of $C^*\bigl(S(n,L,K)\bigr)$.

To show that $1_{F_l}C^*\bigl(S(n,L,K)\bigr)1_{F_l} = F_l$ it will suffice to show that $1_{F_l}x1_{F_l} \in F_l$ for each element $x \in S(n,L,K)$. For any $E^{k,n+1}_{f,g}$ and $Y \in S$ either $YE^{k,n+1}_{f,g}$ is zero or it is also in S. Thus $1_{F_l}E^{k,n+1}_{f,g}1_{F_l} = (\sum_{Y\in S} Y^*Y)E^{k,n+1}_{f,g}(\sum_{Y\in S} Y^*Y) \in C^*(S) = F_l$. Suppose $Y \in A_m$, so Y ends in a matrix unit of dot $\#(l+|m|, n+1)$, and suppose $V^{k,n} \in S(n,L,K)$. If $l+|m| \neq k$ then $YV^{k,n} = 0$. If $l+|m| = k$, then $|m| \neq 0$ as we have assumed that dot $\#(l, n+1)$ is not related by mers to any dots further left in the diagram for $S(n,L,K)$. We have $Y = ZV^{k,n*}E^{k,n+1}_{as,v}$ for some $v \in W^{n+1}$ and some element Z of S ending in either a matrix unit for dot $\#(k-p, n+1)$ or one for dot $\#(k-q, n+1)$. $YV^{k,n} = Z(V^{k,n*}E^{k,n+1}_{as,v}V^{k,n})$. If v begins with a b, then $V^{k,n*}E^{k,n+1}_{as,v}V^{k,n} = 0$, and if $v = ar$ for some $r \in W^n$ then $V^{k,n*}E^{k,n+1}_{as,ar}V^{k,n} = E^{k,n+1}_{s,r} = E^{k-p,n+1}_{sb,rb} + E^{k-q,n+1}_{sa,ra}$. Now one of $ZE^{k-p,n+1}_{sb,rb}$, $ZE^{k-q,n+1}_{sa,ra}$ is zero and the other is an element of S. Thus $1_{F_l}V^{k,n}1_{F_l} \in C^*(S) = F_l$.

Lemma 3.3 *Consider the diagram describing a set of generators $S(n,L,K)$ and the C^*-algebra $C^*\bigl(S(n,L,K)\bigr)$. Suppose $l_1,\dots,l_m$ are the numbers of those dots in the lower row of the diagram that are not related by mers to dots further left. Then by Lemma 3.2 there is a full matrix algebra F_{l_i}, a direct summand of $C^*\bigl(S(n,L,K)\bigr)$ containing the matrix units of the l_i-th dot. We have in fact $C^*\bigl(S(n,L,K)\bigr) \cong F_{l_1}\oplus\cdots\oplus F_{l_m}$. Furthermore, the multiplicity with which a dot is contained in F_{l_i} is given by the number of ways in which it is related by polymers to dot l_i. Finally, the inclusion of the sub-C^*-algebra generated by just the dots into $C^*\bigl(S(n,L,K)\bigr)$ is unital.*

Thus if we look back at Diagram 3 (even though this isn't of the form $C^*\bigl(S(n,L,K)\bigr)$) we see that the algebra generated by the elements shown has three direct summands, of size $3\cdot 2^{(n+1)}$, $5\cdot 2^{(n+1)}$ and $3\cdot 2^{(n+1)}$. Dots $k-2q$, $k-p-q$ and $k-2p$ include into these, one dot into each summand in the order shown, each with multiplicity one. Also dot $k-q$ includes with multiplicities one, one and zero respectively into these summands, dot $k-p$ with multiplicities zero, one and one, and dot k with multiplicities one, two and one.

Proof It is convenient at this point to generalise our notation from Lemma 3.2. We add a superscript to the symbols S and A_m defined there to indicate the dot in question, so the sets in Lemma 3.2 will now be denoted S^l and A^l_m.

The final comment about the inclusion being unital is obvious.

We show first that the various F_{l_i} are orthogonal. Suppose dots l_1 and l_2 are not related by polymers to dots further left and $l_1 < l_2$. From our assumption if $v \in S^{l_1}$ and $u \in S^{l_2}$ we have that $vu = uv = 0$, since any element of S^l begins with a matrix unit of dot $\#(l_1, n+1)$ and ends with a matrix unit of a dot related by polymers to dot $\#(l_1, n+1)$, and similarly for dot $\#(l_2, n+1)$. We also have that $v^*u = u^*v = 0$ for any $v \in S^{l_1}$, $u \in S^{l_2}$, so it remains only to show that $uv^* = vu^* = 0$. We need only consider a $u \in S^{l_2}$ and a $v \in S^{l_1}$ that end in matrix units for the same dot, in fact the same matrix unit, so suppose $u \in A^{l_2}_{\mu}$ and $v \in A^{l_1}_{\nu}$, where $l_2 + |\mu| = l_1 + |\nu|$. Then since l_1 and l_2 are assumed to be unrelated μ is not a tail of ν, so reading from right to left we must come to a point where μ and ν have different letters, $\mu = rqt$, $\nu = xpt$ say, for some words r, t, x. Looking at the partial isometries u and v as words in the generators we see that there are partial isometries U_r, U_x and Y such that $U_r \in A^{l_2}_r$, U_r ends in a matrix unit $E^{l_2+|r|,n+1}_{as,sa}$, $U_x \in A^{l_1}_x$, U_x ends the matrix unit $E^{l_1+|x|,n+1}_{as,sb}$, $u = U_rY$ and $v = U_xY$. That $uv^* = 0$ now follows from observing that U_r and U_x have orthogonal supports, both contained in the range of Y.

To show that $C^*\big(S(n,L,K)\big) \cong F_{l_1} \oplus \cdots \oplus F_{l_m}$ it will suffice to show that the unit of $C^*\big(S(n,L,K)\big)$ is in $F_{l_1} + \cdots + F_{l_m}$. We shall show that if dot $\#(k, n+1)$ is the fat foot of a mer, then $C^*(S^k) \subseteq C^*(S^{k-p}) + C^*(S^{k-q})$. Since the unit of dot $\#(k, n+1)$ is in $C^*(S^k)$, this will allow us to conclude, by moving right to left through the diagram, that the unit of $C^*\big(S(n,L,K)\big)$ is contained in the sum of the $C^*(S^l)$ for dots $\#(l, n+1)$ that are not at the fat feet of mers. These however are exactly the F_{l_i}'s. So consider a dot $\#(k, n+1)$ that is the fat foot of a mer, and let $v \in S^k$. Then $v = E^{k,n+1}_{w,sa}Z$ or $v = E^{k,n+1}_{w,sb}X$ for some Z or X in $C^*\big(S(n,L,K)\big)$ (assume the former, a similar argument works for the other case). Observe that

$$\begin{aligned} E^{k,n+1}_{w,w} &= (E^{k,n+1}_{w,as}V^{k,n}E^{k-q,n+1}_{sa,w})(E^{k-q,n+1}_{w,sa}V^{k,n*}E^{k,n+1}_{as,w}) \\ &\quad + (E^{k,n+1}_{w,as}V^{k,n}E^{k-p,n+1}_{sb,w})(E^{k-p,n+1}_{w,sb}V^{k,n*}E^{k,n+1}_{as,w}), \end{aligned}$$

so we have

$$\begin{aligned} v &= (E^{k,n+1}_{w,as}V^{k,n}E^{k-q,n+1}_{sa,w})(E^{k-q,n+1}_{w,sa}V^{k,n*}E^{k,n+1}_{as,sa}Z) \\ &\quad + (E^{k,n+1}_{w,as}V^{k,n}E^{k-p,n+1}_{sb,w})(E^{k-p,n+1}_{w,sb}V^{k,n*}E^{k,n+1}_{as,sa}Z). \end{aligned}$$

The first expression on the right above is in $C^*(S^{k-q})$ and the second is in $C^*(S^{k-p})$.

Now finally we determine the multiplicities of the inclusions. Suppose as above we have a set of generators $S(n,L,K)$ and that the corresponding diagram contains dots $\#(l, n+1)$ and $\#(k, n+1)$ where the former is not related by polymers to any dots further left. Consider a minimal projection $E^{k,n+1}_{w,w}$ in dot $\#(k, n+1)$. The multiplicity of the inclusion of this dot into F_l is the rank of the projection $1_{F_l}(E^{k,n+1}_{w,w})1_{F_l}$ in F_l. If $l = k$, then A^l_0 is a set of generators for dot $\#(k, n+1)$, while v^*v is orthogonal to dot $\#(k, n+1)$ for any $v \in S^l \setminus A^l_0$, so the statement is clear in this case. If $l+|\mu| \neq k$, then $A^l_\mu \perp E^{k,n+1}_{w,w}$. Thus if there are no words μ for which $l + |\mu| = k$, in particular if $k < l$, then the multiplicity is zero. Notice that $C^*(A^l_\mu)$ is a full matrix algebra whose unit, $1_{A^l_\mu}$, commutes with the matrix units of any dot on the $n+1$st row, so $1_{F_l}(E^{k,n+1}_{w,w})1_{F_l}$ is the sum of the projections $1_{A^l_\mu}(E^{k,n+1}_{w,w})1_{A^l_\mu}$ where μ runs over those words such that $l + |\mu| = k$ (we use here the definition of $S(n, L, K)$). Given such a μ, looking at the formula for the elements of A^l_μ we see that there is exactly one v in A^l_μ for which $v^*v(E^{k,n+1}_{w,w})v^*v = v^*v$, while for all other $u \in A^l_\mu$ we have $u^*uE^{k,n+1}_{w,w} = 0$. Thus the rank of $1_{F_l}(E^{k,n+1}_{w,w})1_{F_l}$ in F_l is the number of words μ for which $l + |\mu| = k$. This is exactly the number of ways by which dot $\#(k, n+1)$ is related by polymers to dot $\#(l, n+1)$.

Proof of Theorem 3.1 It follows easily from Lemmas 3.2, 3.3 and the discussion preceding Lemma 3.2 that $A(p, q)$ is AF. To show that $A(p, q)$ is simple we use the following simple observation: If p and q are coprime natural numbers, then for some natural number $N(p, q)$ all numbers greater than $N(p, q)$ may be written as sums of positive multiples of p and q. Notice that in the diagram for $S(n, L, K)$ the direct summands of $C^*\big(S(n, L, K)\big)$ are labeled by the last q dots to the left on the lower row, as these dots are unrelated by mers, and every other dot is related to at least one of these by a polymer. Consider the following increasing sequence of sub-C^*-algebras of $A(p, q)$.

$$A_n = A\big(S(n, -K_n, K_n)\big)$$

where $K_1 = 1$ and $K_{n+1} = K_n + N(p, q) + 1 + q$. Repeated applications of relations (4) and (7) show that $A(p, q)$ is the inductive limit of the sequence $A_1 \subseteq A_2 \subseteq \cdots$. Relations (4) and (7) define a map from $C^*\big(S(n, -K_n, K_n)\big)$ to $C^*\big(S(n+1, -K_{n+1}, K_{n+1})\big)$ by sending the terms on the left, read in the first C^*-algebra to the terms on the right, read in the second C^*-algebra. Call the limit of this sequence C, and write C_n for $C^*\big(S(n, -K_n, K_n)\big)$. The analysis carried out above implies that the map $C_n \to C_{n+1}$ sends each minimal direct summand of C_n with non-zero multiplicities into every minimal direct summand of C_{n+1}. It follows from this that all of these maps are injective and that C is a simple C^*-algebra. We get a commutative diagram:

$$\begin{tikzcd}
C_1 \arrow[r, hook] \arrow[d] & C_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r] & C \arrow[d] \\
A_1 \arrow[r, hook] & A_2 \arrow[r, hook] & \cdots \arrow[r] & A(p,q).
\end{tikzcd}$$

Since C is simple the last downward map is an isomorphism (with inverse given by the universal property of $A(p, q)$), and hence so are all the downward maps at the finite stages. In fact it is now easy to see that $A\big(S(n, L, K)\big) \cong C^*\big(S(n, L, K)\big)$ by the obvious map for any $S(n, L, K)$, and we henceforth identify them in this way.

It is easy to see that the map α is a bijection of the set of generators with itself that preserves the relations and so extends to an automorphism $\hat{\alpha}$ of $A(p,q)$.

The proof is concluded by showing that there exists a covariant homomorphism of $\big(A(p,q),\hat{\alpha}\big)$ onto $(O_2 \rtimes_{\alpha^\lambda} \mathbb{T}, \hat{\alpha}^\lambda)$ and referring to [Bl] or [OP] (that such a map must be an isomorphism follows from simplicity of $A(p,q)$). We have already observed that the E's along with relations (1)–(4) give a presentation of $e\big((M_{2^\infty} \otimes K) \rtimes_{\tilde{\alpha}} \mathbb{T}\big)\, e$. The explicit correspondence is given as follows. If we identify the letter a with 1 and b with 2, then the matrix units $E_{w,w}$ for $w, v \in W^n$ are in a natural correspondence with the matrix units of M_{2^n} coming from its expression as an n-fold tensor product of copies of M_2. With this identification $E^{k,n}_{v,w}$ gets mapped to the corresponding matrix unit in the k-th direct summand of $M_{2^n} \otimes C_0(\mathbb{Z})$ in our direct limit decomposition of $e\big((M_{2^\infty} \otimes K) \rtimes_{\tilde{\alpha}} \mathbb{T}\big)\, e$. Let V denote the adjoined unitary in the multiplier algebra of $(M_{2^\infty} \otimes K) \rtimes_{\tilde{\alpha}} \mathbb{T} \rtimes_{\tilde{\beta}} \mathbb{Z}$ implementing the action $\tilde{\beta}''$. Then if we assume the identification of the matrix units with the E's already made, we may send $V^{k,n}$ to $\sum_{w \in W^{n+1}} (E^{k,n+1}_{w,w} V e)$ and check that relations (1)–(7) are satisfied. Straightforward computations then show that the $*$-homomorphism into $O_2 \rtimes_{\alpha^\lambda} \mathbb{T}$ given by the universal property of $A(p,q)$ has a dense image and carries $\hat{\alpha}$ onto $\hat{\alpha}''$.

Definition 3.4 In [ELP] the following definitions are given. A *zero dimensional NCCW-complex* is just a finite dimensional C^*-algebra. A *one-dimensional NCCW-complex* is a pull-back of the following form. Let F_1 and F_2 be two finite dimensional C^*-algebras with unital maps $\alpha_1, \alpha_2 \colon F_1 \to F_2$. Let ev(0), ev(1) denote the maps from $F_2 \otimes C[0,1]$ to F_2 given by evaluation at zero and one respectively. We then form the pull-back of the following diagram:

$$\begin{array}{ccc} & & F_1 \\ & & \Big\downarrow{\scriptstyle \alpha_1 \oplus \alpha_2} \\ F_2 \otimes C[0,1] & \xrightarrow[\mathrm{ev}(0)\oplus\mathrm{ev}(1)]{} & F_2 \oplus F_2. \end{array}$$

We shall weaken these requirements slightly. Define a *non-unital one-dimensional NCCW-complex* as a pull-back as above, but without the requirement that the maps α_1 and α_2 be unital. Given a description of a non-unital one-dimensional NCCW-complex as above we call F_1 the *zero skeleton*, $F_2 \otimes C[0,1]$ the *one-cell* and the map $\alpha_1 \oplus \alpha_2$ the *attaching map*.

We then have the following corollary.

Corollary 3.5 *For any rational number $\lambda \in \mathbb{Q} \cap (0,\infty) \setminus \{1\}$, $O_2 \rtimes_{\alpha^\lambda} \mathbb{R}$ is an inductive limit of non-unital one-dimensional NCCW-complexes with the property that every direct summand of the zero skeleton is included with non-zero multiplicity into each direct summand of the fibre of the one-cell at both endpoints by the attaching map.*

Proof Observe that both A_n and $\hat{\alpha}(A_n)$ are included into A_{n+1} in the fashion described. If we let B_n denote the non-unital NCCW-complex consisting of continuous functions from $[0,1]$ into A_{n+1} that are in A_n at zero and whose value at one is $\hat{\alpha}$ of their value at zero we get an increasing sequence $B_1 \subseteq B_2 \subseteq \cdots$, of sub-$C^*$-algebras of $M_{\hat{\alpha}}\big(A(p,q)\big)$ whose union is dense.

4 Stable Relations and the Main Result for O_2

In this section we shall require some results about stable relations. We refer the reader to [L] and [ELP] for details.

We use the notation $C^*\langle G|R\rangle$ to denote the universal C^*-algebra with generators G subject to the relations R, if it exists. $C^*\langle\{x_1,\dots,x_n\} \mid \|x_i\| \leq 2\rangle$ does exist and we shall denote it by $F_n^{(2)}$. In the following we shall regard relations among n variables to be just elements of this C^*-algebra. We say that a C^*-algebra $C^*\langle G|R\rangle$ is *finitely presented* if R and G are both finite sets. As observed in [ELP, 2.2.5] this is no real constraint on R.

Definition 4.1 **(*cf.* [L, 13.2.2])** The n-tuple $(a_1,\dots,a_n)$ of elements in a C^*-algebra A is called a *representation* of a set of relations $R \subseteq F_n^{(2)}$ if $\|a_j\| \leq 2$ and $\Phi_{\mathbf{a}}(p) = 0$ for all $p \in R$, where $\Phi_a \colon F_n^{(2)} \to A$ is induced by $x_i \mapsto a_i$. If for some $\delta < 1$ only $\|\Phi_{\mathbf{a}}(p)\| \leq \delta$ is required to hold for all $p \in R$, then $(a_1,\dots,a_n)$ is called a *δ-representation.*

Definition 4.2 **(*cf.* [L, 13.2.1])** By a *finite, bounded* set of relations in n variables we mean a finite subset R of $F_n^{(2)}$ such that $\|x_j + I_R\| \leq 1$, where I_R is the ideal generated by R.

Definition 4.3 A finite, bounded set of relations R in $F_n^{(2)}$ is said to be *weakly stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $(a_1,\dots,a_n)$ is a δ-representation of R in a C^*-algebra A there exists a representation $(b_1,\dots,b_n)$ of R in A such that $\|a_i - b_i\| < \varepsilon$ for $i = 1,\dots,n$.

It follows from the results of [ELP] and [L] that any non-unital one-dimensional NCCW-complex can be finitely presented with a weakly stable set of relations (in fact we may assume just one relation).

Lemma 4.4 *A projectionless quotient of a non-unital one-dimensional NCCW- complex having the property that every direct summand of the zero skeleton is mapped non-trivially into every direct summand of the fibre of the one-cell at both endpoints by the attaching map is a non-unital one-dimensional NCCW-complex.*

Proof Suppose we are given a non-unital one-dimensional NCCW-complex satisfying our hypotheses: A is isomorphic to the pullback of F_1 and $F_2 \otimes C[0,1]$ with $\alpha_1 \oplus \alpha_2 \colon F_1 \to F_2 \oplus F_2$ and $\mathrm{ev}(0) \oplus \mathrm{ev}(1) \colon F_2 \otimes C[0,1] \to F_2 \oplus F_2$ where F_1 and F_2 are finite dimensional algebras and both α_1 and α_2 map every direct summand of F_1 into every direct summand of F_2 with non-zero multiplicity. Then the primitive spectrum of A, denoted $\sigma(A)$, is the union of a finite set, identified with the spectrum of F_1, and a finite number of copies of $\mathbb{R}$, identified with $\sigma\big(F_2 \otimes C_0(0,1)\big)$, with the topology specified as follows (*cf.* [El2]). The relative topology on $\sigma(F_1)$ is the discrete topology, each line in $\sigma\big(F_2 \otimes C_0(0,1)\big)$ is open in $\sigma(A)$ and homeomorphic to a line in the relative topology, taking the union of any line in $\sigma\big(F_2 \otimes C_0(0,1)\big)$ with any point of $\sigma(F_1)$ with the relative topology gives the one point compactification of the line, and every point in $\sigma(A)$ is closed.

Let I be a proper ideal in A. The spectrum of I is then identified in a natural way with an open subset of $\sigma(A)$, and the spectrum of A/I is homeomorphic to the compliment $\sigma(A) \setminus \sigma(I)$. We shall first see what the set $\sigma(I)$ can be.

Consider $\sigma(I) \cap \sigma(F_1)$. Suppose this set is not empty and is not all of $\sigma(F_1)$. Since it is not empty, there is some neighbourhood of a point in $\sigma(F_1)$ contained in it, and this neighbourhood must contain the ends of all of the lines. Thus any point in $\sigma(F_1) \setminus \sigma(I)$ is an isolated point of $\sigma(A/I)$, which therefore has an elementary C^*-algebra as a direct summand, and in particular has projections.

Suppose that $\sigma(I) \supseteq \sigma(F_1)$. Then we have that $\sigma(A/I)$ is contained in the union of finitely many closed intervals, and A/I is a quotient of a finite direct sum of matrix algebras over $C[0,1]$. As these are unital, so is A/I, in particular, it has a projection.

We have reduced the proof of our lemma to the case in which $\sigma(I)$ is contained in $\sigma\big(F_2 \otimes C(0,1)\big)$. If we write $\sigma\big(F_2 \otimes C(0,1)\big) \cong L_1 \coprod \cdots \coprod L_k$ where each L is homeomorphic to $\mathbb{R}$, it is easy to see that, if A/I is projectionless, then for each i, $L_i \cap \sigma(I)$ must be connected (otherwise A/I would contain a quotient of a matrix algebra over $C[0,1]$). Furthermore, $\sigma(I)$ can not intersect all of the L_i's non-trivially or we would again have projections in the quotient. Thus we have only to consider the case in which $\sigma(I)$ is a disjoint union of open intervals, at most one in each L_i, and not in every L_i. We show that in this case the quotient is a non-unital one-dimensional NCCW-complex. Suppose $F_2 = A_1 \oplus \cdots \oplus A_k$, where each A is a full matrix algebra, and let $\varphi_j^{(1)}, \varphi_j^{(2)}$ denote the maps from F_1 to the j-th minimal direct summand of F_2 at the left and right endpoints respectively. Suppose further that the ideal I cuts the lines L_l to L_k. Define F_1' and F_2' as follows. $F_1' = F_1 \oplus (A_l^1 \oplus \cdots \oplus A_k^1) \oplus (A_l^2 \oplus \cdots \oplus A_k^2)$ and $F_2' = (A_1 \oplus \cdots \oplus A_{l-1}) \oplus (A_l^1 \oplus \cdots \oplus A_k^1) \oplus (A_l^2 \oplus \cdots \oplus A_k^2)$, where the superscripts are just to distinguish the two copies. Define maps α_1', α_2' from F_1' to F_2' as follows. α_1' maps F_1 into the 1-st $l-1$ summands of F_2' by $\varphi_1^{(1)}, \ldots, \varphi_{l-1}^{(1)}$ respectively. It maps F_1 into $A_l^1, \ldots, A_k^1$ by $\varphi_l^{(1)}, \ldots, \varphi_k^{(1)}$ respectively. α_1' maps $A_l^1, \ldots, A_k^1$ to $A_l^2, \ldots, A_k^2$ respectively each with the identity map. All the other partial maps for α_1' are zero. α_2' maps F_1 into the first $l-1$ summands of F_2' by $\varphi_1^{(2)}, \ldots, \varphi_{l-1}^{(2)}$ respectively. It maps F_1 into $A_l^2, \ldots, A_k^2$ by $\varphi_l^{(2)}, \ldots, \varphi_k^{(2)}$ respectively, it maps $A_l^2, \ldots, A_k^2$ to $A_l^1, \ldots, A_k^1$ respectively, each with the identity map, and all of the other partial maps for α_2' are zero. It is then easy to see that A/I is isomorphic to the non-unital one-dimensional NCCW-complex with zero-skeleton F_1', one-cell $F_2' \otimes C[0,1]$ and attaching map $\alpha_1' \oplus \alpha_2'$.

We shall refer to the additional condition on a non-unital one-dimensional NCCW-complex mentioned in Lemma 4.4 as the endpoint property.

Definition 4.5 We shall say that a separable C^*-algebra A has the local approximation property with respect to a class of C^*-algebras $\mathfrak{C}$ if for every finite set F of elements of A and every $\varepsilon > 0$ there is a $C \in \mathfrak{C}$ and a $*$-homomorphism $\varphi\colon C \to A$ such that each element of F lies within ε of the image of φ.

Lemma 4.6 *If a separable projectionless C^*-algebra has the local approximation property with respect to the class of non-unital one-dimensional NCCW-complexes having* *the endpoint property, then it is an inductive limit of a sequence of non-unital one-dimensional NCCW-complexes.*

Proof From Lemma 4.4 we may assume that the algebra has the local approximation property with injective maps for the class of non-unital one-dimensional NCCW-complexes. The statement now follows from [L, 15.2.2].

We refer the reader to [Dix] for facts about continuous fields of C^*-algebras.

Lemma 4.7 *Let δ and ε such that $1 > \delta > 0$ and $1 - \delta > \varepsilon > 0$ be given. Let $\{A(t) \mid t \in X\}$ be a continuous field of C^*-algebras over a topological space X with continuous sections Γ, $t_0 \in X$, and suppose that $y_1, \dots, y_n \in \Gamma$ are such that $\|y_i(t_0)\| < 2$ and $\big(y_1(t_0), \dots, y_n(t_0)\big)$ is a δ-representation of a relation $r \in F_n^{(2)}$. Then there is a neighbourhood V of t_0 such that for all $s \in V$, $\big(y_1(s), \dots, y_n(s)\big)$ is a $(\delta + \varepsilon)$-representation of r.*

Proof Let P be a $*$-polynomial in the generators $x_1, \dots, x_n$ of $F_n^{(2)}$ such that $\|P - r\| < \varepsilon/3$ in $F_n^{(2)}$. Since $s \mapsto \big\|P\big(y_1(s), \dots, y_n(s)\big)\big\|$ is continuous on X there is some neighbourhood V of t_0 on which it varies by less than $\varepsilon/3$ and on which $\|y_i(\cdot)\| < 2$. If $\psi_s\colon F_n^{(2)} \to A(s)$ denotes the map induced by $x_i \mapsto y_i(s)$, which is well defined for $s \in V$, then $\|\psi_s(r)\| < \delta + \varepsilon$ for all s in V.

Lemma 4.8 *Let $\{A(t) \mid t \in X\}$ be a continuous field of C^*-algebras over a topological space X, and suppose that $B \cong C^*\langle x_1, \dots, x_n \mid r\rangle$ is a finitely generated C^*-algebra described by a weakly stable relation r. Let $\varepsilon > 0$, $t_0 \in X$ and continuous sections $y_1, \dots, y_m$ be given. Suppose that there is a $*$-homomorphism $\psi_{t_0}\colon B \to A(t_0)$ such that $\{y_1(t_0), \dots, y_m(t_0)\} \subseteq_{\varepsilon/2} \psi_{t_0}(B)$. Then there is a neighbourhood V of t_0 such that for every $s \in V$ there is a $*$-homomorphism $\psi_s\colon B \to A(s)$ with $\{y_1(s), \dots, y_m(s)\} \subseteq_\varepsilon \psi_s(B)$.*

Proof Find $*$-polynomials $p_1, \dots, p_m$ in n variables such that

$$\big\| p_j\big(\psi_{t_0}(x_1), \dots, \psi_{t_0}(x_n)\big) - y_j(t_0)\big\| < 3\varepsilon/4$$

for $j = 1, \dots, m$. Find continuous sections $z_1, \dots, z_n$ such that $z_i(t_0) = \psi_{t_0}(x_i)$ for $i = 1, \dots, n$. There exists a neighbourhood U of t_0 such that for all $s \in U$ and for each j, $\big\| p_j\big(z_1(s), \dots, z_n(s)\big) - y_j(s)\big\| < 3\varepsilon/4$. Choose ε' such that if $a_1, \dots, a_n$, $a_1', \dots, a_n'$ are elements in a C^*-algebra and $\|a_i - a_i'\| < \varepsilon'$ for $i = 1, \dots, n$, then for each j, $\|p_j(a_1, \dots, a_n) - p_j(a_1', \dots, a_n')\| < \varepsilon/4$. There exists a δ such that if $(w_1, \dots, w_n)$ is a δ-representation of r in a C^*-algebra, then there exists a representation $(w_1', \dots, w_n')$ of r with $\|w_i - w_i'\| < \varepsilon'$ for each i. By Lemma 4.7 above there exists a neighbourhood W of t_0 such that if $s \in W$ then $\big(z_1(s), \dots, z_n(s)\big)$ is a δ-representation of r. Thus for $s \in W$ there exists a representation $(z_1^s, \dots, z_n^s)$ of r in $A(s)$ with $\|z_i^s - z_i(s)\| < \varepsilon'$ for each i. Define $\psi_s\colon B \to A(s)$ for each $s \in U \cap W$ by $x_i \mapsto z_i^s$ for $i = 1, \dots, n$. Now it is easy to see that for $s \in U \cap W$ we have $\{y_1(s), \dots, y_m(s)\} \subseteq_\varepsilon \psi_s(B)$.

Corollary 4.9 *Suppose $t \in \mathbb{Q} \cap (0, \infty) \setminus \{1\}$, $\varepsilon > 0$, and $f_1, \ldots, f_n \in C_c(\mathbb{R}, O_2)$ are given. Then there exists a neighbourhood U of t, a non-unital one-dimensional NCCW-complex A with the endpoint property, and, for every $s \in U$, a $*$-homomorphism $\psi_s \colon A \to O_2 \rtimes_{\alpha^s} \mathbb{R}$ such that $\{\varphi_s(f_1), \ldots, \varphi_s(f_n)\} \subseteq_\varepsilon \psi_s(A)$, where φ_s denotes the canonical inclusion of $C_c(\mathbb{R}, O_2)$ into $O_2 \rtimes_{\alpha^s} \mathbb{R}$.*

Proof The corollary is immediate from Lemma 4.8 above and Corollary 3.5.

Theorem 4.10 **(The main result for O_2)** *The set of irrational numbers λ for which $O_2 \rtimes_{\alpha^\lambda} \mathbb{R}$ is an inductive limit of non-unital one-dimensional NCCW-complexes is a dense set of Baire category 2 in $(0, \infty)$.*

Proof From Lemma 4.6 we only have to show that the local approximation property with respect to the class of non-unital one-dimensional NCCW-complexes with the endpoint property holds for such a set. Pick a countable dense subset of $C_c(\mathbb{R}, O_2)$ and call it $\mathfrak{G}$. To conclude that a given $O_2 \rtimes_{\alpha^s} \mathbb{R}$ has the local approximation property it will suffice to show that the elements in $\varphi_s(\mathfrak{G})$ may be approximated. From Corollary 4.9 above, for each finite subset $F \subseteq \mathfrak{G}$, $\varepsilon > 0$, and each $t \in \mathbb{Q} \cap (0, \infty) \setminus \{1\}$ there is a neighbourhood $V(t, F, \varepsilon)$ of t, a non-unital one-dimensional NCCW-complex $B(t, F, \varepsilon)$ with the endpoint property and for every $s \in V(t, F, \varepsilon)$ a $*$-homomorphism $\psi(t, s, F, \varepsilon) \colon B(t, F, \varepsilon) \to O_2 \rtimes_{\alpha^s} \mathbb{R}$ such that $\varphi_s(F) \subseteq_\varepsilon \psi(t, s, F, \varepsilon) B(t, F, \varepsilon)$. Let $G(\varepsilon, F) = \bigcup_{t \in \mathbb{Q} \cap (0,\infty) \setminus \{1\}} V(t, F, \varepsilon)$. Then for every $s \in G(\varepsilon, F)$, $\varphi_s(F)$ is approximately contained to within ε by the image of a non-unital one-dimensional NCCW-complex with the end point property. Observe that $G(\varepsilon, F)$ contains a dense open set. Let ε_n be a sequence of positive numbers converging to zero and let $\mathfrak{F}(\mathfrak{G})$ denote the set of finite subsets of $\mathfrak{G}$. Then the set $G = \bigcap_{F \in \mathfrak{F}(\mathfrak{G})} \bigcap_{\varepsilon_n} G(\varepsilon_n, F)$ is contained in the set of points s in $(0, \infty)$ for which $O_2 \rtimes_{\alpha^s} \mathbb{R}$ has the local approximation property, and G clearly contains a dense G_δ set. It follows that $G \setminus \mathbb{Q}$ is a dense set of Baire category 2.

5 O_n and Closing Remarks

In this section we shall describe how the results of Sections 2 to 4 generalise to O_n for $n > 2$. As the statements and their proofs closely resemble those for O_2, we shall be brief.

The actions of $\mathbb{R}$ on O_n we are considering depend on n real parameters $\lambda_1, \ldots, \lambda_n$. Omitting the trivial case and rescaling we may assume that $\lambda_1 = 1$. By [Rie] we get a continuous field of C^*-algebras over $\mathbb{R}^{n-1}$ with fibres $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$ where $\lambda = (1, \lambda_2, \ldots, \lambda_n)$ is a multi-index. As in the case of O_2 we shall analyse the fibres for which α^λ is periodic, that is, when $\lambda_1, \ldots, \lambda_n$ are all rational numbers. If $\lambda = (1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_2, \ldots, \lambda_n \in \mathbb{Q}$, we may by rescaling the real parameter get a new action, also to be called α^λ, with parameters $p_1, \ldots, p_n \in \mathbb{Z}$ such that $\gcd(p_1, \ldots, p_n) = 1$. We may restrict attention to the case in which the p's are all distinct. [Bl, Prop. 10.3.2] may be applied to conclude that $O_n \rtimes_{\alpha^\lambda} \mathbb{R} \cong M_{\hat{\alpha}^\lambda}(O_n \rtimes_{\alpha^\lambda} \mathbb{T})$. We proceed to analyse $O_n \rtimes_{\alpha^\lambda} \mathbb{T}$ in the same way as for the special case of O_2, writing

O_n as $e(M_{n^\infty} \otimes K \rtimes_\beta \mathbb{Z})e$ *etc.* We may again display the results of this analysis in diagrams.

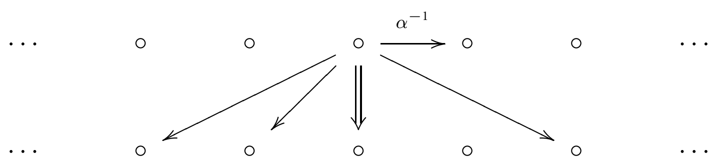

Diagram 5

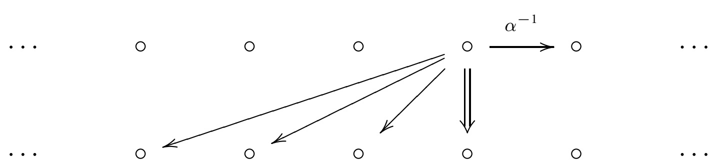

Diagram 6

In these diagrams a dot in the j-th row represents a copy of M_{n^j} and the dots and thin arrows give a Bratteli diagram for an inductive limit of algebras $M_{n^j} \otimes C_0(\mathbb{Z})$. The number of thin downward arrows emanating from each dot is n, the fat arrows, as before denote partial isometries setting minimal projections in the upper dots equivalent to minimal projections in the lower dots. The finite case is distinguished here by having all of the thin arrows on the same side of the fat arrow (*i.e.*, all the parameters of the same sign). By an analysis similar to that carried out in Section 3 for the special case of O_2, one can show that the universal C^*-algebra described by a diagram such as Diagram 6 above is an AF algebra. Using the observation that if $p_1, \ldots, p_n$ are natural numbers with greatest common divisor 1 then there some natural number N such that any $k \geq N$ may be written as a sum of positive multiples of $p_1, \ldots, p_n$, we may again get an inductive system for this algebra in which each minimal direct summand of the n-th algebra is mapped with non-zero multiplicity into each minimal direct summand of the $n + 1$-st algebra. From this analogues of Theorem 3.1 and Corollary 3.5 follow. The arguments of Section 4 apply unchanged to prove the following.

Theorem 5.1 **(The main result for O_n)** *The set of points $(\lambda_2, \ldots, \lambda_n)$ in $(0, \infty)^{n-1}$ for which $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$ is an inductive limit of non-unital one-dimensional NCCW-complexes contains a dense G_δ set.*

Remark The diagrammatic representation of the $A(p,q)$'s allows one to read off fairly simple expressions for their K-theory. In Section 3 we found that the minimal direct summands of $C^*\big(S(n,L,K)\big)$ are labelled by the q leftmost dots in the lower row of the diagram for $S(n,L,K)$, $(q > p)$, and that the minimal projections for these dots are minimal in there respective summands of $C^*\big(S(n,L,K)\big)$. Thus the ordered K_0 group of $A(p,q)$ is a direct limit of copies of the simplicial group $\mathbb{Z}^q$.

It is easy to see that for any projection in one of the $C^*\big(S(n,L,K)\big)$'s, $A(p,q)$ contains an equivalent orthogonal projection. Since $A(p,q)$ is a simple AF algebra its K_0 group is a simple ordered group, so every element is an order unit. We thus conclude that the scale of $K_0\big(A(p,q)\big)$ is the whole positive cone, and that $A(p,q)$ is stable.

It remains only to determine the multiplicity matrices for the maps $\mathbb{Z}^q \to \mathbb{Z}^q$ for some suitably chosen nest of finite dimensional sub-C^*-algebras. We shall describe recursively a sequence of sub-diagrams of the diagram for $A(p,q)$, and the sub-C^*-algebras will be those given by the generators represented in these sub-diagrams. The first sub-diagram, D_0, is just $S(0,0,0)$. D_n is a sub-diagram of the diagram for $A(p,q)$ that is contained between the n-th and $n+1$-st rows of dots. D_{n+1} will be the sub-diagram contained between the $n+1$-st and $n+2$-nd rows given as follows. In the $n+1$-st row D_{n+1} includes all of the dots included in the bottom row of D_n, and one more on the left. D_{n+1} contains all of the arrows emanating from these dots. In the $n+2$-nd row D_{n+1} contains the dots at the feet of any of the arrows, fat or thin, in D_{n+1}, and any dot between two such. (Note that in this context it would only make sense to talk of omitting fat arrows from such a sub-diagram, as the thin ones represent relations among the generators, not additional generators.) In Diagram 7 the leftmost portion of two successive sub-diagrams for $A(1,2)$ is shown.

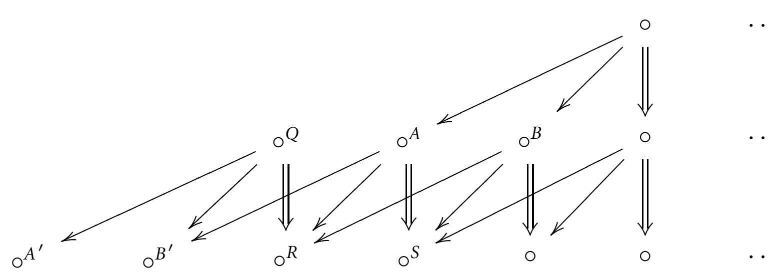

Diagram 7

If we denote by A_n the sub-C^*-algebra of $A(p,q)$ described by D_n, applications of the relations described in Section 3 imply that $A_n \subseteq A_{n+1}$ and $\overline{\bigcup A_n} = A(p,q)$.

To determine the multiplicity with which a minimal direct summand of A_n is included into a minimal direct summand of A_{n+1}, we need only check the rank of a

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minimal projection of the summand of A_n when cut down by the unit of the summand of A_{n+1} in question. Even though the A_n's are not strictly speaking of the form $C^*\big(S(n,L,K)\big)$ (they differ only in that they may not have an odd number of dots in the top row of their corresponding diagram), the analysis of Section 3 may still be applied. A minimal projection in a given minimal direct summand of A_n is given by taking a minimal projection in one of the q leftmost dots in the lower row of D_n, as we saw in Section 3. For a particular minimal direct summand of A_{n+1} we find the desired multiplicity by writing one of these projections as a sum of two projections in dots on the $(n+2)$-nd row and using the rules given in Lemma 3.3 for determining the multiplicities with which these dots are included in the minimal direct summands of A_{n+1}. Notice that this implies that the matrix describing the multiplicities depends on only the leftmost portion of D_n and D_{n+1}, as illustrated in Diagram 7, and therefore does not depend on n.

We illustrate this procedure for $A(1,2)$. The minimal direct summands for an algebra A_n are labelled by the dots marked A, B in the diagram, and those for A_{n+1} by the dots marked A', B'. To get from A to A' by going down a thin arrow and along a polymer there is only one choice: One goes down to dot R, up to dot Q and down to A'. To get from dot B to dot A' one may go down to R, up to Q and down to A', or down to S, up to A, down to R, up to Q and down to A'. Thus the summand of A_n labelled A includes into the summand of A_{n+1} labelled A' with multiplicity one, the summand of A_n labelled B includes into the same summand with multiplicity two. Doing the other two combinations we conclude that

$$K_0\big(A(1,2)\big) \cong \varinjlim \left\{\mathbb{Z}^2, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\right\}.$$

As this matrix has determinant -1, $K_0\big(A(1,2)\big) \cong \mathbb{Z}^2$ as a group. Its positive cone is the half space lying above the line through the origin with slope $(1-\sqrt{5})/2$.

Remark Finally, a remark about the diagrams for the infinite case. These are really not important for the object of this paper, but they do perhaps provide some additional insight about the conditions on the projectionless/purely infinite dichotomy found by Kishimoto and Kumjian. Consider Diagram 5. Starting at a given dot in this diagram, one may, by going first down to the right several times and then down to the left several times, return by thin arrows to the same column. Thus the fat arrows going straight down set projections in the dot we started with equivalent to sub-projections of themselves, when they are written in terms of elements of a lower row. So in these algebras one can "see" some of the infinite projections.

References

[Bl] B. Blackadar, *K-theory for operator algebras.* Math. Sci. Res. Inst. Publ. **5**, Springer-Verlag, Berlin-New York, 1986.

[Co] A. Connes, *An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of $\mathbb{R}$.* Adv. Math. **39**(1981), 31–55.

[Cu] J. Cuntz, *K-theory for certain C^*-algebras.* Ann. of Math. (2) **113**(1981), 181–197.

[Dix] J. Dixmier, *C^*-algebras.* North Holland, Amsterdam-New York-Oxford, 1977.

[ELP] S. Eilers, T. Loring and G. Pedersen, *Stability of anti-commutation relations: An application of non-commutative CW-complexes.* J. Reine Angew. Math, to appear.

[E1] G. A. Elliott, *The classification problem for amenable C^*-algebras.* Proc. ICM (Zurich, 1994), Vol. 1, 2, Birkhauser, Basel, 1995, 922–932.

[E2] ———, *An invariant for simple C^*-algebras.* Invited Papers/Articles Sollicités (eds. J. B. Carrell and R. Murty), Canadian Mathematical Society, Ottawa, 1996, 61–90.

[ENN] G. A. Elliott, T. Natsume and R. Nest, *The Heisenberg group and K-Theory.* K-Theory **7**(1993), 409–428.

[Ev] D. E. Evans, *On O_n.* Publ. Res. Inst. Math. Sci. **16**(1980), 915–927.

[Ks] A. Kishimoto, *Simple crossed products by locally compact abelian groups.* Yokohama Math. J. **28**(1980), 69–85.

[KK1] A. Kishimoto and A. Kumjian, *Simple, stably projectionless C^*-algebras arising as crossed products.* Canad. J. Math. (5) **48**(1996), 980–996.

[KK2] ———, *Crossed products of Cuntz algebras by quasi-free automorphisms.* Fields Inst. Commun. **13**(1997), 173–192.

[L] T. Loring, *Lifting solutions to perturbing problems in C^*-algebras.* Fields Institute Monographs **8**, Amer. Math. Soc., Providence, RI, 1997.

[OP] D. Olesen and G. Pedersen, *Partially inner C^*-dynamical systems.* J. Funct. Anal. **66**(1986), 262–281.

[Ped] G. K. Pedersen, *C^*-algebras and their automorphism groups.* Academic Press, London-New York-San Francisco, 1979.

[Rie] M. Rieffel, *Continuous fields of C^*-algebras coming from group cocycles and actions.* Math. Ann. **283**(1989), 631–643.

[Rør] M. Rørdam, *Classification of certain infinite simple C^*-algebras.* J. Funct. Anal. **131**(1995), 415–458.

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