Glasgow Math. J. **60** (2018) 527–537. © Glasgow Mathematical Journal Trust 2017.
doi:10.1017/S001708951700026X.

LOWER BOUNDS ALONG STABLE MANIFOLDS

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(Received 21 February 2017; revised 28 March 2017; accepted 1 September 2017; first published online 2 October 2017)

Abstract. It is well known that along any stable manifold the dynamics travels with an exponential rate. Moreover, this rate is close to the slowest exponential rate along the stable direction of the linearization, provided that the nonlinear part is sufficiently small. In this note, we show that whenever there is also a *fastest* finite exponential rate along the stable direction of the linearization, similarly we can establish a lower bound for the speed of the nonlinear dynamics along the stable manifold. We consider both cases of discrete and continuous time, as well as a nonuniform exponential behaviour.

2000 *Mathematics Subject Classification.* 34D99, 37C75

1. Introduction. In this note, we consider stable invariant manifolds obtained from a sufficiently small nonlinear perturbation of an exponential dichotomy. It is well known that invariant manifolds play a central role in a large part of the theory of differential equations and dynamical systems, both in discrete and continuous time, and in finite and infinite dimension. Not surprisingly, the theory of exponential dichotomies, invariant manifolds and their applications is widely developed. We refer the reader to the books [**4–6**, **10**] for details and references.

We consider both cases of discrete and continuous time. More precisely, we consider stable invariant manifolds respectively for the dynamics

$$v_{n+1} = A_n v_n + f_n(v_n), \quad n \in \mathbb{N}, \tag{1}$$

and

$$v' = A(t)v + f(t, v), \quad t \geq 0, \tag{2}$$

on some Banach space (in fact we consider the more general case of mild differential equations obtained from perturbation a linear evolution family). The linear dynamics $v_{n+1} = A_n v_n$ and $v' = A(t)v$ are assumed to have an exponential dichotomy and the nonlinear terms are assumed to be sufficiently small (see Sections 2 and 3).

In fact, we consider the more general case of a linear dynamics having a nonuniform exponential dichotomy. This type of exponential behaviour is much more typical than the uniform exponential behaviour and plays a central role in a large part of the theory of dynamical systems, most notably in smooth ergodic theory (see, for example, [**1**]). Besides the existence of stable invariant manifold for any sufficiently small perturbation of a nonuniform exponential dichotomy (with a small nonuniformity, which is controlled by the Lyapunov regularity theory), one can study the ergodic properties of the dynamics, obtain a formula for the entropy of an invariant measure, and study the dimension of a hyperbolic invariant measure, among many other nontrivial consequences. Invariant manifolds were first obtained for nonuniformly hyperbolic trajectories by Pesin [**8**]. The first related results in Hilbert spaces were established by Ruelle [**9**]. The case of transformations in Banach spaces under some compactness assumptions was considered by Mañé [**7**].

It is well know that along any stable manifold the dynamics travels at least with an exponential rate. Moreover, this rate is close to the slowest exponential rate along the stable direction of the linearizations $v_{n+1} = A_n v_n$ and $v' = A(t)v$, provided that the nonlinear part is sufficiently small. This means that if we have an upper exponential bound for the stable part of the linear dynamics, we have also a close upper exponential bound along the stable manifold of the corresponding nonlinear dynamics in (1) and (2).

Our main aim in this note is to show that whenever there is also a *fastest* exponential rate along the stable part of the linearization, which means that the linear part has in fact a *strong* nonuniform exponential dichotomy, we can also establish a lower bound for the speed of the nonlinear dynamics along the stable direction. We consider both cases of discrete and continuous time, as well as the general case of a nonuniform exponential behaviour.

We emphasize that in the context of ergodic theory most nonuniform exponential dichotomies are in fact strong nonuniform exponential dichotomies. More precisely, for a flow preserving a finite invariant measure, the linear variational equation of almost all trajectories with a nonuniform exponential dichotomy has a strong nonuniform exponential dichotomy (see for example [**1**]).

2. Stable manifolds for discrete time. Let X be a Banach space. We denote by $B(X)$ the set of all bounded linear operators acting on X. Given a sequence $(A_n)_{n\in\mathbb{N}} \subset B(X)$ of invertible linear operators, we define

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n, & m > n, \\ \mathrm{Id}, & m = n, \\ A_{n-1}^{-1}\cdots A_m^{-1}, & m < n \end{cases}$$

for each $m, n \in \mathbb{N}$. Note that

$$\mathcal{A}(m,n) = \mathcal{A}(m,k)\mathcal{A}(k,n) \quad \text{and} \quad \mathcal{A}(m,n)^{-1} = \mathcal{A}(n,m) \tag{3}$$

for $m, n, k \in \mathbb{N}$. We say that $(A_n)_{n\in\mathbb{N}}$ has a *nonuniform exponential dichotomy* if there exist projections P_n for $n \in \mathbb{N}$ satisfying

$$P_m\mathcal{A}(m,n) = \mathcal{A}(m,n)P_n \quad \text{for } m, n \in \mathbb{N},$$

and there exist constants $a < 0 < b$ and $D > 0$ such that for each $m, n \in \mathbb{N}$ with $m \geq n$ we have

$$\|\mathcal{A}(m,n)P_n\| \leq De^{a(m-n)+\varepsilon n} \quad \text{and} \quad \|\mathcal{A}(n,m)Q_m\| \leq De^{-b(m-n)+\varepsilon m}, \tag{4}$$

https://doi.org/10.1017/S001708951700026X Published online by Cambridge University Press

where $Q_m = \mathrm{Id} - P_m$. For each $n \in \mathbb{N}$, we define the stable and unstable spaces, respectively, by

$$S_n = P_n(X) \quad \text{and} \quad U_n = Q_n(X).$$

Moreover, we say that the sequence $(A_n)_{n\in\mathbb{N}}$ has a *strong nonuniform exponential dichotomy* if it has a nonuniform exponential dichotomy and there exist constants $c \geq -a$ and $d \leq -b$ such that for each $m, n \in \mathbb{N}$ with $m \leq n$ we have

$$\|\mathcal{A}(m,n)P_n\| \leq De^{c(n-m)+\varepsilon n} \quad \text{and} \quad \|\mathcal{A}(n,m)Q_m\| \leq De^{-d(n-m)+\varepsilon m}. \tag{5}$$

Note that given an invertible linear operator $A \in B(X)$, the constant sequence $A_n = A$ has a strong nonuniform exponential dichotomy if and only if the spectrum of A does not intersect the unit circle and A^{-1} is bounded. Before proceeding, following [**3**] we describe a few examples of nonconstant sequences.

EXAMPLE 1. Given $\omega < 0$ and $\rho \geq 0$, let

$$a_n = e^{\omega+\rho[(-1)^n n-1/2]}$$

for $n \in \mathbb{N}$. Then, the sequence

$$A_n = \begin{pmatrix} a_n & 0 \\ 0 & 1/a_n \end{pmatrix} \tag{6}$$

has a strong nonuniform exponential dichotomy provided that ω is sufficiently small.

EXAMPLE 2. Given $\omega < 0$ and $\rho \geq 0$, the sequence A_n in (6) with a_n replaced by

$$b_n = e^{\omega+\rho(n+1)\cos(n+1)-\rho n\cos n-\rho\sin(n+1)+\rho\sin n}$$

for all $n \in \mathbb{N}$ or by

$$c_n = e^{\omega+\rho(n+1)\sin\log(n+1)-\rho n\sin\log n+\rho\sin\log(n+1)-\rho\sin\log n}$$

for all $n \in \mathbb{N}$, also has a strong nonuniform exponential dichotomy, again provided that ω is sufficiently small.

We also consider a sequence of C^1 functions $f_n \colon X \to X$ with $f_n(0) = 0$ and $d_0f_n = 0$ for $n \in \mathbb{N}$. We always assume that there exists $\delta > 0$ such that

$$\|d_vf_n\| \leq \delta e^{-3\varepsilon n} \quad \text{and} \quad \|d_vf_n - d_wf_n\| \leq \delta e^{-3\varepsilon n}\|v - w\| \tag{7}$$

for $n \in \mathbb{N}$ and $v, w \in X$, with the same ε as above. We are interested in the dynamics given by

$$v_{n+1} = F_n(v_n), \quad \text{where } F_n(v) = A_nv + f_n(v).$$

Clearly, letting

$$\mathcal{F}(m,n) = \begin{cases} F_{m-1} \circ \cdots \circ F_n, & m > n, \\ \mathrm{Id}, & m = n, \\ F_{n-1}^{-1} \cdots F_m^{-1}, & m < n, \end{cases}$$

we have $v_m = \mathcal{F}(m,n)(v_n)$ for all $m, n \in \mathbb{N}$.

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Now let $\mathcal{X}$ be the set of all sequences $\phi = (\phi_n)_{n\in\mathbb{N}}$ of C^1 functions $\phi_n : S_n \to U_n$ with $\phi_n(0) = 0$ and $d_0\phi_n = 0$ for $n \in \mathbb{N}$ such that

$$\gamma := \sup\{\|d_\xi \phi_n\| : n \in \mathbb{N}, \xi \in S_n\} < 1$$

and

$$\|d_\xi \phi_n - d_{\bar\xi} \phi_n\| \le \gamma \|\xi - \bar\xi\| \quad \text{for } n \in \mathbb{N} \text{ and } \xi, \bar\xi \in S_n.$$

Given $\phi \in \mathcal{X}$, we consider the C^1 manifolds

$$\mathcal{V}_n = \{(\xi, \phi_n(\xi)) : \xi \in S_n\} \quad \text{for } n \in \mathbb{N}.$$

The following stable manifold theorem can be obtained as in [**2**] (even though the paper takes $\gamma = 1$, the same argument applies with simple changes).

THEOREM 1. *Assume that the sequence $(A_n)_{n\in\mathbb{N}}$ has a nonuniform exponential dichotomy and that the maps f_n satisfy property* (7)*. If $a + \varepsilon < b$ and δ is sufficiently small, then there exists a unique sequence of functions $\phi \in \mathcal{X}$ such that*

$$\mathcal{F}(m, n)(\mathcal{V}_n) = \mathcal{V}_m \quad \textit{for } m \ge n. \tag{8}$$

Moreover, there exists $K > 0$ such that for each $m, n \in \mathbb{N}$ with $m \ge n$ and $\xi, \bar\xi \in S_n$ we have

$$\|\mathcal{F}(m, n)(v) - \mathcal{F}(m, n)(\bar v)\| \le K e^{a(m-n)+\varepsilon n} \|\xi - \bar\xi\|,$$

where $v = (\xi, \phi_n(\xi))$ and $\bar v = (\bar\xi, \phi_n(\bar\xi))$, and so

$$\|d_v \mathcal{F}(m, n)\| \le (1 - \gamma)^{-1} K e^{a(m-n)+\varepsilon n}.$$

Our main aim is to show that when $(A_n)_{n\in\mathbb{N}}$ has a *strong* nonuniform exponential dichotomy, one can in fact also establish lower bounds along the stable manifolds $\mathcal{V}_n$ in Theorem 1 that imitate the first bound in (5). In other words, along the stable manifolds the nonlinear dynamics has the same lower and upper bounds as the linear dynamics along the stable space.

THEOREM 2. *Assume that the sequence $(A_n)_{n\in\mathbb{N}}$ has a strong nonuniform exponential dichotomy and that the maps f_n satisfy property* (7)*. If $a + \varepsilon < b$ and δ is sufficiently small, then for the unique sequence of functions $\phi \in \mathcal{X}$ in Theorem 1 there exists $L > 0$ such that for each $m, n \in \mathbb{N}$ with $m \ge n$ and $\xi, \bar\xi \in S_n$ we have*

$$\|\mathcal{F}(m, n)(v) - \mathcal{F}(m, n)(\bar v)\| \ge L e^{-c(m-n)-\varepsilon m} \|\xi - \bar\xi\|, \tag{9}$$

where $v = (\xi, \phi_n(\xi))$ and $\bar v = (\bar\xi, \phi_n(\bar\xi))$, and

$$\|d_v \mathcal{F}(m, n)\| \ge L e^{-c(m-n)-\varepsilon m}. \tag{10}$$

Proof. Write $v_m = (x_m, y_m)$, with $x_m \in S_m$ and $y_m \in U_m$. It follows from (8) that for the unique sequence $\phi \in \mathcal{X}$ in Theorem 1, given $n \in \mathbb{N}$ and $v_n = (\xi, \phi_n(\xi)) \in S_n \oplus U_n$, for each $m \geq n$ we have

$$\begin{aligned} x_m &= \mathcal{A}(m,n)\xi + \sum_{l=n}^{m-1} \mathcal{A}(m,l+1)P_{l+1}f_l(x_l,\phi_l(x_l)), \\ \phi_m(x_m) &= \mathcal{A}(m,n)\phi_n(\xi) + \sum_{l=n}^{m-1} \mathcal{A}(m,l+1)Q_{l+1}f_l(x_l,\phi_l(x_l)). \end{aligned} \tag{11}$$

For simplicity of the notation, we shall also write

$$x_m = x_m(\xi) \quad \text{and} \quad \phi_m^*(\xi) = \phi_m(x_m(\xi)).$$

We first establish the lower bound in (9). Let

$$v_m(\xi) = \mathcal{F}(m,n)(x_m(\xi), \phi_m^*(\xi)). \tag{12}$$

Then,

$$\begin{aligned} \|v_m(\xi) - v_m(\bar{\xi})\| &\geq \|x_m(\xi) - x_m(\bar{\xi})\| - \|\phi_m^*(\xi) - \phi_m^*(\bar{\xi})\| \\ &\geq (1-\gamma)\|x_m(\xi) - x_m(\bar{\xi})\|. \end{aligned}$$

Hence, to prove the theorem, it suffices to show that

$$\|x_m(\xi) - x_m(\bar{\xi})\| \geq Ce^{-c(m-n)-\varepsilon m}\|\xi - \bar{\xi}\| \quad \text{for } m \geq n$$

for some constant $C > 0$. By (3), one can rewrite the first identity in (11) in the form

$$\xi = \mathcal{A}(n,m)x_m - \sum_{l=n}^{m-1} \mathcal{A}(n,l+1)P_{l+1}f_l(x_l,\phi_l(x_l)). \tag{13}$$

We will show that

$$\|\xi - \bar{\xi}\| \leq 2De^{c(m-n)+\varepsilon m}\|x_m(\xi) - x_m(\bar{\xi})\| \quad \text{for } n \leq m.$$

Writing $x_m = x_m(\xi)$ and $\bar{x}_m = x_m(\bar{\xi})$, it follows from (13) that

$$\begin{aligned} x_n - \bar{x}_n &= \mathcal{A}(n,m)(x_m - \bar{x}_m) \\ &\quad - \sum_{l=n}^{m-1} \mathcal{A}(n,l+1)P_{l+1}\big(f_l(x_l,\phi_l(x_l)) - f_l(\bar{x}_l,\phi_l(\bar{x}_l))\big). \end{aligned}$$

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Using (5), we obtain

$$\begin{aligned}\|x_n - \bar{x}_n\| &\le \|\mathcal{A}(n,m)\| \cdot \|x_m - \bar{x}_m\| \\ &\quad + \sum_{l=n}^{m-1} \|\mathcal{A}(n,l+1)P_{l+1}\| \cdot \big\|f_l(x_l, \phi_l(x_l)) - f_l(\bar{x}_l, \phi_l(\bar{x}_l))\big\| \\ &\le De^{c(m-n)+\varepsilon m}\|x_m - \bar{x}_m\| \\ &\quad + \delta D \sum_{l=n}^{m-1} e^{c(l+1-n)+\varepsilon(l+1)} e^{-3l\varepsilon}(1+\gamma)\|x_l - \bar{x}_l\| \\ &\le De^{c(m-n)+\varepsilon m}\|x_m - \bar{x}_m\| \\ &\quad + \delta D(1+\gamma)e^{c+\varepsilon} \sum_{l=n}^{m-1} e^{c(m-n)} e^{c(l-m)} e^{-2l\varepsilon}\|x_l - \bar{x}_l\|.\end{aligned}$$

Letting $\Gamma_l = e^{c(l-m)}\|x_l - \bar{x}_l\|$ yields the inequality

$$\Gamma_n \le De^{\varepsilon m}\|x_m - \bar{x}_m\| + \delta D(1+\gamma)e^{c+\varepsilon} \sum_{l=n}^{m-1} e^{-2l\varepsilon}\Gamma_l.$$

Now let $\Gamma = \max_{n \le l \le m} \Gamma_l$. Then,

$$\Gamma \le De^{\varepsilon m}\|x_m - \bar{x}_m\| + \frac{\delta D(1+\gamma)e^{c+\varepsilon}}{1 - e^{-2\varepsilon}}\Gamma.$$

Taking δ sufficiently small so that

$$\frac{\delta D(1+\gamma)e^{c+\varepsilon}}{1 - e^{-2\varepsilon}} \le \frac{1}{2} \tag{14}$$

we obtain

$$\Gamma \le 2De^{\varepsilon m}\|x_m - \bar{x}_m\|,$$

which is equivalent to

$$\|x_n - \bar{x}_n\| \le 2De^{c(m-n)+\varepsilon m}\|x_m - \bar{x}_m\|.$$

This concludes the proof of inequality (9).

Now we establish the lower bound in (10). With the notation introduced in (12), we have

$$\|d_\xi v_m\| \ge \|d_\xi x_m\| - \|d_\xi \phi_m^*\|.$$

Moreover, since $\phi \in \mathfrak{X}$, we obtain

$$\|d_\xi \phi_m^*\| \le \|d_{x_m(\xi)}\phi_m(\xi)\| \cdot \|d_\xi x_m\| \le \gamma\|d_\xi x_m\|,$$

and so

$$\|d_\xi v_m\| \ge (1-\gamma)\|d_\xi x_m\|.$$

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In order to bound $\|d_\xi x_m\|$, we take derivatives in (13), thus yielding the identity

$$\mathrm{Id} = \mathcal{A}(n, m) d_\xi x_m - \sum_{l=n}^{m-1} \mathcal{A}(n, l+1) P_{l+1} d_\xi [f_l(x_l, \phi_l(x_l))].$$

Using (5) and (7), we obtain

$$\begin{aligned} 1 &\le \|\mathcal{A}(n, m)\| \cdot \|d_\xi x_m\| + \sum_{l=n}^{m-1} \|\mathcal{A}(n, l+1) P_{l+1}\| \cdot \|d_{v_l(\xi)} f_l\| \cdot \|d_\xi v_l(\xi)\| \\ &\le D e^{c(m-n)+\varepsilon m} \|d_\xi x_m\| + \delta D \sum_{l=n}^{m-1} e^{c(l+1-n)+\varepsilon(l+1)} e^{-2l\varepsilon} (1+\gamma) \|d_\xi x_l\| \\ &\le D e^{c(m-n)+\varepsilon m} \|d_\xi x_m\| + \delta D (1+\gamma) e^{c+\varepsilon} e^{c(m-n)} \sum_{l=n}^{m-1} e^{-2l\varepsilon} e^{c(l-m)} \|d_\xi x_l\|. \end{aligned}$$

Finally, letting $\Upsilon_l = e^{c(l-m)} \|d_\xi x_l\|$ and taking into account that $x_n = \xi$ and so $d_\xi x_n =$ Id, we have

$$\Upsilon_n \le D e^{\varepsilon m} \|d_\xi x_m\| + \delta D (1+\gamma) e^{c+\varepsilon} \sum_{l=n}^{m-1} e^{-2l\varepsilon} \Upsilon_l.$$

Now let $\Upsilon = \max_{n \le l \le m} \Upsilon_l$. Then,

$$\Upsilon \le D e^{\varepsilon m} \|d_\xi x_m\| + \frac{\delta D (1+\gamma) e^{c+\varepsilon}}{1 - e^{-2\varepsilon}} \Upsilon.$$

For δ as in (14), we obtain

$$\Upsilon \le 2 D e^{\varepsilon m} \|d_\xi x_m\|,$$

which yields the inequality

$$1 \le 2 D e^{c(m-n)+\varepsilon m} \|d_\xi x_m\|.$$

This concludes the proof of the theorem. □

As noted in Section 1, along any stable manifold, the dynamics travels at least with an exponential rate. We end this section with an example illustrating the role of Theorem 2: it shows that if the linear dynamics has a strong exponential dichotomy (and not all exponential dichotomies are strong), then there is also a lower bound for the speed of the nonlinear dynamics along the stable manifold.

EXAMPLE 3. Given $\delta \ge 0$, consider the sequence $f_n \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f_n(v) = \delta e^{-3\varepsilon n} v \quad \text{for } n \in \mathbb{N}.$$

Moreover, let A_n be any of the sequences of 2×2 matrices in Examples 1 and 2. By Theorem 1, for each such choice, there exist unique stable manifolds $\mathcal{V}_n$ that are obtained from a sequence of functions $\phi \in \mathfrak{X}$. On the other hand, it follows from Theorem 2 that there is also a lower bound for the speed of the dynamics along the

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stable manifolds. In fact, this bound is given by the constant c in the notion of a strong exponential dichotomy (see (10)), which shows that under the assumptions of Theorem 2 the minimal and maximal speeds along the stable manifolds are precisely those of the linear dynamics along the stable spaces (see (4) and (5)), determined by the constants a and c.

3. Stable manifolds for continuous time. In this section, we obtain a version of Theorem 2 for continuous time. Let X be a Banach space and let $T(t, s)$, for $t, s \geq 0$, be an evolution family on X. This means that $T(t, s) \in B(X)$ for $t, s \geq 0$ and that

$$T(t, t) = \mathrm{Id} \quad \text{and} \quad T(t, s) = T(t, \tau)T(\tau, s) \quad \text{for } t, s, \tau \geq 0.$$

We shall always assume in the paper that the map $(t, s) \mapsto T(t, s)$ is of class C^1. We say that $T(t, s)$ has a *nonuniform exponential dichotomy* if there exist projections $P(t)$ for $t \geq 0$ satisfying

$$P(t)T(t, s) = T(t, s)P(s) \quad \text{for } t, s \geq 0,$$

and there exist constants $a < 0 < b$ and $D > 0$ such that for each $t \geq s \geq 0$ we have

$$\|T(t, s)P(s)\| \leq De^{a(t-s)+\varepsilon s} \quad \text{and} \quad \|T(s, t)Q(t)\| \leq De^{-b(t-s)+\varepsilon t},$$

where $Q(t) = \mathrm{Id} - P(t)$. For each $s \geq 0$, we define the stable and unstable spaces by

$$S(s) = P(s)(X) \quad \text{and} \quad U(s) = Q(s)(X).$$

Moreover, we say that $T(t, s)$ has a *strong nonuniform exponential dichotomy* if it has a nonuniform exponential dichotomy and there exist $c \geq -a$ and $d \leq -b$ such that for each $s \geq t \geq 0$ we have

$$\|T(t, s)P(s)\| \leq De^{c(s-t)+\varepsilon s} \quad \text{and} \quad \|T(s, t)Q(t)\| \leq De^{-d(s-t)+\varepsilon t}. \tag{15}$$

We also consider a C^1 function $f\colon \mathbb{R}_0^+ \times X \to X$ with $f(t, 0) = 0$ and $\partial f(t, 0) = 0$ for $t \geq 0$, where ∂f denotes the partial derivative with respect to v. We always assume that there exists $\delta > 0$ such that

$$\|\partial f(t, v)\| \leq \delta e^{-3\varepsilon t} \quad \text{and} \quad \|\partial f(t, v) - \partial f(t, w)\| \leq \delta e^{-3\varepsilon t}\|v - w\| \tag{16}$$

for $t \geq 0$ and $v, w \in X$. In this section, we consider the problem

$$v(t) = T(t, s)v_s + \int_s^t T(t, \tau)f(\tau, v(\tau))\, d\tau, \tag{17}$$

for some $s \geq 0$ and $v_s \in X$.

Now let $\mathcal{X}$ be the set of continuous functions $\phi\colon F \to X$ of class C^1 in ξ, where

$$F = \{(s, \xi) : s \geq 0, \xi \in S(s)\},$$

with $\phi(s, 0) = 0$, $\partial\phi(s, 0) = 0$ (where $\partial\phi = \partial\phi/\partial\xi$) and

$$\phi(s, \xi) \in U(s) \quad \text{for } s \geq 0 \text{ and } \xi \in S(s),$$

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such that

$$\gamma := \sup\{\|\partial\phi(s,\xi)\| : s \geq 0,\ \xi \in S(s)\} < 1$$

and

$$\|\partial\phi(s,\xi) - \partial\phi(s,\bar{\xi})\| \leq \gamma\|\xi - \bar{\xi}\| \quad \text{for } n \in \mathbb{N} \text{ and } \xi, \bar{\xi} \in S_n.$$

We also consider the graph

$$\mathcal{V}_\phi = \{(s, \xi, \phi(s,\xi)) : s \geq 0,\ \xi \in S(s)\}.$$

Given $s \geq 0$ and $v_s = (\xi, \eta) \in S(s) \times U(s)$, we denote by

$$v(\cdot, s, v_s) = (x(\cdot, s, v_s), y(\cdot, s, v_s))$$

the unique solution of problem (17) or, equivalently, of the system

$$\begin{aligned} x(t) &= T(t,s)P(s)\xi + \int_s^t T(t,\tau)P(\tau)f(\tau, x(\tau), \phi(\tau, x(\tau)))\,d\tau, \\ \phi(t, x(t)) &= T(t,s)Q(s)\eta + \int_s^t T(t,\tau)Q(\tau)f(\tau, x(\tau), \phi(\tau, x(\tau)))\,d\tau \end{aligned} \tag{18}$$

for $t \geq s$. For each $\tau \geq 0$, we consider the semiflow

$$\Psi_\tau(s, v_s) = (s + \tau, v(s + \tau, s, v_s)).$$

The following stable manifold theorem was established in [**2**] (even though the paper takes $\gamma = 1$ the same argument applies with simple changes).

THEOREM 3. *Assume that the evolution family $T(t,s)$ has a nonuniform exponential dichotomy and that the function f satisfies property* (16). *If $a + \varepsilon < b$ and δ is sufficiently small, then there exists a unique function $\phi \in \mathcal{X}$ such that*

$$\Psi_\tau(\mathcal{V}) = \mathcal{V} \quad \textit{for } \tau \geq 0.$$

Moreover, there exists $K > 0$ such that for each $t \geq s \geq 0$ and $\xi, \bar{\xi} \in S(s)$ we have

$$\|\Psi_\tau(s, \xi, \phi(s,\xi)) - \Psi_\tau(s, \bar{\xi}, \phi(s, \bar{\xi}))\| \leq K e^{a(t-s)+\varepsilon s}\|\xi - \bar{\xi}\|$$

and so

$$\|\partial_v \Psi_\tau(s, \xi, \phi(s,\xi))\| \leq (1-\gamma)^{-1} K e^{a(t-s)+\varepsilon s}.$$

We also establish a version of Theorem 2 for continuous time.

THEOREM 4. *Assume that $T(t,s)$ has a strong nonuniform exponential dichotomy and that the function f satisfies property* (16). *If $a + \varepsilon < b$ and δ is sufficiently small, then for the unique function $\phi \in \mathcal{X}$ in Theorem 3 there exists $L > 0$ such that for each $t \geq s \geq 0$ and $\xi, \bar{\xi} \in S(s)$ we have*

$$\|\Psi_\tau(s, \xi, \phi(s,\xi)) - \Psi_\tau(s, \bar{\xi}, \phi(s, \bar{\xi}))\| \geq L e^{-(c+\delta D(1+\gamma))(t-s)-\varepsilon t}\|\xi - \bar{\xi}\|$$

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and

$$\|\partial_v \Psi_\tau(s, \xi, \phi(s, \xi))\| \geq L e^{-(c+\delta D(1+\gamma))(t-s)-\varepsilon t}.$$

Proof. It follows from the first inequality in (18) that

$$x(t-r) = T(t-r, t)P(t)x(t) - \int_{t-r}^{t} T(t-r, \tau)P(\tau)f(\tau, x(\tau), \phi(\tau, x(\tau)))\, d\tau \tag{19}$$

for $r \in [0, t-s]$. Using (15), we obtain

$$\begin{aligned}
&\|x(t-r) - \bar{x}(t-r)\| \\
&\leq De^{cr+\varepsilon t}\|x(t) - \bar{x}(t)\| + \delta D(1+\gamma)\int_{t-r}^{t} e^{c(\tau-t+r)+\varepsilon\tau} e^{-3\varepsilon\tau}\|x(\tau) - \bar{x}(\tau)\|\, d\tau \\
&\leq De^{cr+\varepsilon t}\|x(t) - \bar{x}(t)\| + \delta D(1+\gamma)e^{cr}\int_{s}^{t} e^{c(\tau-t)}\|x(\tau) - \bar{x}(\tau)\|\, d\tau.
\end{aligned}$$

Now let $\Gamma(s) = e^{-cr}\|x(t-r) - \bar{x}(t-r)\|$. Then,

$$\Gamma(r) \leq De^{\varepsilon t}\Gamma(0) + \delta D(1+\gamma)\int_0^r \Gamma(\tau)\, d\tau$$

for $r \in [0, t-s]$. It follows from Gronwall's lemma that

$$\Gamma(r) \leq De^{\varepsilon t}e^{\delta D(1+\gamma)r}\|x(t) - \bar{x}(t)\|,$$

also for $r \in [0, t-s]$, and so

$$\|x(s) - \bar{x}(s)\| \leq De^{\varepsilon t}e^{(c+\delta D(1+\gamma))(t-s)}\|x(t) - \bar{x}(t)\|. \tag{20}$$

Finally, note that if $t = \tau + s$, then

$$\begin{aligned}
&\|\Psi_\tau(s, \xi, \phi(s, \xi)) - \Psi_\tau(s, \bar{\xi}, \phi(s, \bar{\xi}))\| \\
&= \big\|(t, x(t), \phi(t, x(t))) - (t, \bar{x}(t), \phi(t, \bar{x}(t)))\big\| \\
&\geq \|x(t) - \bar{x}(t)\| - \|\phi(t, x(t)) - \phi(t, \bar{x}(t))\| \\
&\geq (1-\gamma)\|x(t) - \bar{x}(t)\|.
\end{aligned}$$

Together with (20) this yields the first inequality in the theorem.

Now, we establish the bound for the derivative. Clearly,

$$\begin{aligned}
\|\partial_v \Psi_\tau(s, \xi, \phi(s, \xi))\| &\geq \|\partial_\xi x(t)\| - \|\partial\phi(t, x(t))\| \cdot \|\partial_\xi x(t)\| \\
&\geq (1-\gamma)\|\partial_\xi x(t)\|.
\end{aligned} \tag{21}$$

In order to bound $\|\partial_\xi x(t)\|$, we proceed as before. Taking derivatives in (19), we get

$$\mathrm{Id} = T(s, t)P(t)\partial_\xi x(t) - \int_s^t T(s, \tau)P(\tau)d_\xi[f(\tau, v(\tau))]\, d\tau.$$

https://doi.org/10.1017/S001708951700026X Published online by Cambridge University Press

Using (15) and (16), we obtain

$$
\begin{aligned}
1 &\leq De^{c(t-s)+\varepsilon t}\|\partial_\xi x(t)\| + \delta D(1+\gamma)\int_s^t e^{c(\tau-s)+\varepsilon\tau}e^{-3\varepsilon\tau}\|\partial_\xi x(\tau)\|\,d\tau \\
&\leq De^{c(t-s)+\varepsilon t}\|\partial_\xi x(t)\| + \delta D(1+\gamma)e^{c(t-s)}\int_s^t e^{-c(t-\tau)}e^{-2\varepsilon\tau}\|\partial_\xi x(\tau)\|\,d\tau.
\end{aligned}
$$

Since $\partial_\xi x(s) = \mathrm{Id}$, it follows again from Gronwall's lemma that

$$1 \leq De^{\varepsilon t}e^{(c+\delta D(1+\gamma))(t-s)}\|\partial_\xi x(t)\|$$

for $t \geq s$. Together with (21) this concludes the proof for the theorem. □

ACKNOWLEDGEMENTS. This work is supported by FCT/Portugal through Grant UID/MAT/04459/2013.

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