

## A CHARACTERISATION OF EUCLIDEAN NORMED PLANES VIA BISECTORS

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(Received 21 May 2018; accepted 9 June 2018; first published online 20 August 2018)

### Abstract

Our main result states that whenever we have a non-Euclidean norm  $\|\cdot\|$  on a two-dimensional vector space  $X$ , there exists some  $x \neq 0$  such that for every  $\lambda \neq 1$ ,  $\lambda > 0$ , there exist  $y, z \in X$  satisfying  $\|y\| = \lambda\|x\|$ ,  $z \neq 0$  and  $z$  belongs to the bisectors  $B(-x, x)$  and  $B(-y, y)$ . We also give several results about the geometry of the unit sphere of strictly convex planes.

2010 *Mathematics subject classification*: primary 46B20; secondary 52A10, 52A21.

*Keywords and phrases*: isosceles orthogonality, strictly convex normed spaces, Euclidean planes, bisectors.

### 1. Introduction

In a normed linear space  $(X, \|\cdot\|)$ , a vector  $x$  is said to be isosceles orthogonal to a vector  $y$  (denoted by  $x \perp_I y$ ) if  $\|x - y\| = \|x + y\|$ . Isosceles orthogonality was introduced by James in [6]. Since then, several papers and surveys have studied properties related to the geometric structure of the space in the light of that notion of orthogonality, and various characterisations (for example, for strict convexity) have been obtained. Two interesting surveys on this topic are [2] and [9], and the monograph [10] gives further background.

In this paper,  $(X, \|\cdot\|)$  will denote a two-dimensional normed space (usually referred to as a *Minkowski plane*), and  $S_X$  and  $B_X$  will stand for the *unit sphere* and the *closed unit ball*, respectively. Since we are dealing with normed spaces,  $B_X$  is always a planar convex body centred at the origin and  $S_X$  coincides with its boundary. The segment joining two points  $x, y$  will be denoted by  $[x, y]$ . As we will deal with segments, intervals and two-dimensional vectors, we need to determine the meaning of  $(x, y)$ . Throughout the paper, this will denote a vector in  $X$ . Of course, we will need to have a basis  $\{e, v\}$  fixed previously, so that  $(x, y)$  means  $xe + yv$ . For open intervals (or segments) we will use the notation  $]x, y[$ , and for semiopen intervals (segments)

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The first author was supported in part by DGICYT project MTM2016-76958-C2-1-P (Spain) and Junta de Extremadura programs GR-15152 and IB-16056; the second author was partially supported by Junta de Extremadura and FEDER funds.

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we will write  $[x, y[$  and  $]x, y]$ . For the linear span of a pair of vectors  $x, y \in X$  we will use  $\langle x, y \rangle$ .

We will utilise the concept of the bisector of the segment joining two points. For  $x, y \in X$ , the bisector of  $[x, y]$  is defined as follows (see, for example, [2–5, 8]):

$$B(x, y) = \{z \in X : \|x - z\| = \|y - z\|\}.$$

In Section 2 we prove Proposition 2.1, stated here in a slightly different way.

**PROPOSITION 1.1.** *A norm  $\|\cdot\|$  on  $X$  is strictly convex if and only if for every nonzero  $z \in X$  there exists, up to  $\pm 1$ , exactly one vector which is isosceles orthogonal to  $z$  in  $S_X$ .*

This solves in the negative the following conjecture, proposed by Alonso, Martini and Wu [2, Conjecture 5.3.], with a different approach to the one used to give the solution that can be found in [1, Proposition 5] combined with [7, Corollary 2.5].

**CONJECTURE 1.2.** *In any non-Euclidean Minkowski plane  $X$ , there exist  $x, y \in S_X$ , with  $x \neq \pm y$ , such that  $B(-x, x) \cap B(-y, y) \neq \{0\}$ .*

As far as we know, the if-and-only-if statement of our Proposition 2.1 cannot be found in the literature.

Section 3 is devoted to the main result (Theorem 3.2) in our paper. We propose the following characterisation of Euclidean normed planes.

**THEOREM 1.3.** *The norm  $\|\cdot\|$  is not Euclidean if and only if, for some  $x \neq 0$  and for each  $\lambda \in (0, +\infty) \setminus \{1\}$ , there is a  $y$  such that  $\|y\| = \lambda\|x\|$ ,  $\langle x, y \rangle = X$  and  $B(-x, x) \cap B(-y, y) \neq 0$ .*

**REMARK 1.4.** The definitions of isosceles orthogonality and bisectors given above imply the equivalences  $z \in B(-x, x)$  if and only if  $x \perp_I z$  if and only if  $x \in B(-z, z)$ .

It is easily checked that bisectors enjoy a certain property of linearity:

$$B(\lambda x + z, \lambda y + z) = z + \lambda B(x, y) \quad \text{for all } x, y, z \in X \text{ and for all } \lambda \in \mathbb{R}.$$

Therefore, for any  $a, b \in X$ ,

$$B(a, b) = \frac{a + b}{2} + \frac{\|a - b\|}{2} B\left(\frac{a - b}{\|a - b\|}, \frac{b - a}{\|b - a\|}\right),$$

so the geometric properties of bisectors can be determined by careful analysis of properties of bisectors of the type  $B(-x, x)$ , with  $x \in S_X$ .

## 2. Side results

We will prove that [2, Conjecture 5.3] is false by proving Proposition 2.1. One implication can be seen in [1, Proposition 5], while the other is proven in [7, Corollary 2.5]. However, our proof of Proposition 2.1 is different and more geometric than the earlier proofs.

**PROPOSITION 2.1.** *Let  $(X, \|\cdot\|)$  be a normed plane. Then it is strictly convex if and only if  $B(-x, x) \cap B(-y, y) = 0$  for every linearly independent pair  $x, y \in S_X$ .*

**REMARK 2.2.** As we have noted in Remark 1.4, every bisector is an affine transformation of a  $B(-x, x)$  for some  $x \in S_X$ . This readily implies that the following statements are equivalent.

- There exists  $\lambda > 0$  such that  $B(-x, x) \cap B(-y, y) = 0$  for linearly independent  $x, y \in \lambda S_X$ .
- $B(-x, x) \cap B(-y, y) = 0$  for every linearly independent pair  $x, y \in S_X$ .
- $B(-x, x) \cap B(-y, y) = 0$  for every  $\lambda > 0$  and linearly independent  $x, y \in \lambda S_X$ .
- $B(z - x, z + x) \cap B(z - y, z + y) = z$  for every  $\lambda > 0, z \in X$  and every linearly independent pair  $x, y \in \lambda S_X$ .
- $B(x, x') \cap B(y, y') = (x + x')/2$ , whenever  $\|x - x'\| = \|y - y'\|$  and  $x + x' = y + y'$ .

**REMARK 2.3.** Our problem is to determine what happens when  $0 \neq z \in X, x, y \in S_X$  are such that  $z \in B(-x, x) \cap B(-y, y)$  and  $x$  and  $y$  are linearly independent. This is equivalent to  $\|z - x\| = \|z + x\|$  and  $\|z - y\| = \|z + y\|$  or, with  $\lambda_x = \|z - x\|^{-1}, \lambda_y = \|z - y\|^{-1}$ , to

$$\lambda_x(z + x), \lambda_x(z - x), \lambda_y(z + y), \lambda_y(z - y) \in S_X.$$

We have, then, two pairs of points in the sphere, say

$$a = \lambda_x(z + x), \quad a' = \lambda_x(z - x); \quad b = \lambda_y(z + y), \quad b' = \lambda_y(z - y),$$

and two positive values (not necessarily different)  $\alpha, \beta$ , such that

$$\alpha z = a + a' = \lambda_x(z + x + z - x) = 2\lambda_x z, \quad \beta z = b + b' = \lambda_y(z + y + z - y) = 2\lambda_y z.$$

In particular,  $\alpha = 2\lambda_x = \|a + a'\|/\|z\|$  and  $\beta = 2\lambda_y = \|b + b'\|/\|z\|$ . In the statement we ask for  $x$  and  $y$  to be linearly independent, and we also have

$$a - a' = \lambda_x(z + x - (z - x)) = 2\lambda_x x, \quad b - b' = \lambda_y(z + y - (z - y)) = 2\lambda_y y,$$

so  $a - a'$  and  $b - b'$  must also be independent. From these last equalities, we obtain  $2\lambda_x = \|a - a'\|$  and  $2\lambda_y = \|b - b'\|$ , so combining with the previous statements, we obtain  $\|a - a'\| = \|a + a'\|/\|z\|$  and  $\|b - b'\| = \|b + b'\|/\|z\|$ , or

$$\frac{\|a + a'\|}{\|a - a'\|} = \|z\| = \frac{\|b + b'\|}{\|b - b'\|}.$$

**PROOF OF THE EASY IMPLICATION OF PROPOSITION 2.1.** Suppose  $\|\cdot\|$  is not strictly convex. Then there is some segment  $[c, c'] \subset S_X$ . Take

$$a = \frac{1}{4}(3c + c'), \quad a' = \frac{1}{4}(-3c' - c), \quad b = c, \quad b' = -\frac{1}{2}(c + c').$$

It is straightforward that  $a, a', b, b' \in S_X$ , that  $0 \neq z = \frac{1}{2}(b + b') = \frac{1}{2}(a + a')$ , and also that  $a - a'$  and  $b - b'$  are independent. Also,  $\|a - a'\| = \|b - b'\| = 2$ , and this implies that  $\|a + a'\|/\|a - a'\| = \|b + b'\|/\|b - b'\|$ . Observe that  $0 \neq z \in B(-x, x) \cap B(-y, y)$ , with  $x = a - z$  and  $y = b - z$ . □

**PROOF OF THE OTHER IMPLICATION OF PROPOSITION 2.1.** For the remainder of this section, let us suppose that  $\|\cdot\|$  is strictly convex.

We will also assume that  $0 \neq z \in X, a, a', b, b' \in S_X, \beta \geq \alpha > 0$  are such that

$$a + a' = \alpha z, \quad b + b' = \beta z, \quad \frac{\|a + a'\|}{\|a - a'\|} = \frac{\|b + b'\|}{\|b - b'\|},$$

and  $a - a'$  and  $b - b'$  are linearly independent.

We will split the proof into several elementary results that may be useful for other purposes.

Let us fix some notation. We consider  $X$  endowed with the basis  $\{z, (a - a')/2\}$ , so that  $a = (\alpha, 1), a' = (\alpha, -1), \delta = 2/\|a - a'\|$  and  $d = (0, \delta), d' = (0, -\delta)$ , with  $a, a', d, d' \in S_X$ . Of course,  $\delta > 1$ . The lines  $r^+$  and  $r^-$  defined, respectively, by  $(0, \delta), (\alpha, 1)$  and  $(0, -\delta), (\alpha, -1)$  are given by  $r^+(x) = \delta + (1 - \delta)x/\alpha, r^-(x) = -\delta + (\delta - 1)x/\alpha$ , and the only point that their graphs have in common is  $c = (\delta\alpha/(\delta - 1), 0)$ . So

$$\begin{aligned} d &= (0, \delta) = (0, r^+(0)), & d' &= (0, -\delta) = (0, r^-(0)), \\ a &= (\alpha, 1) = (\alpha, r^+(\alpha)), & a' &= (\alpha, -1) = (\alpha, r^-(\alpha)), \\ c &= (\delta\alpha/(\delta - 1), 0) = (\delta\alpha/(\delta - 1), r^+(\delta\alpha/(\delta - 1))). \end{aligned}$$

**LEMMA 2.4.** Consider the convex hull  $\text{conv}\{d, d', c\}$  and the vertical line  $\{\alpha\} \times \mathbb{R}$ . The following symmetric inclusions hold:

$$\begin{aligned} \text{conv}\{d, d', c\} \cap ]0, \alpha[ \times \mathbb{R} &\subset \text{int}(B_X), \\ B_X \cap ]\alpha, \infty[ \times \mathbb{R} &\subset \text{int}(\text{conv}\{d, d', c\}). \end{aligned}$$

**PROOF.** Observe that

$$\begin{aligned} \text{conv}\{d, d', c\} \cap ]0, \alpha[ \times \mathbb{R} &= \text{conv}\{d, d', a, a'\} \cap ]0, \alpha[ \times \mathbb{R}, \\ \text{conv}\{d, d', c\} \cap ]\alpha, \infty[ \times \mathbb{R} &= \text{conv}\{a, a', c\} \cap ]\alpha, \infty[ \times \mathbb{R}. \end{aligned}$$

For the first part, take  $x = (x_1, x_2) \in \text{conv}\{a, a', d, d'\}$ , with  $x_1 \in ]0, \alpha[$ . We will show that  $x \in \text{int}(B_X)$ . Since

$$x \in \text{conv}\{(0, r^+(0)), (0, r^-(0)), (\alpha, r^+(\alpha)), (\alpha, r^-(\alpha))\},$$

we have  $r^-(x_1) \leq x_2 \leq r^+(x_1)$ . As  $\|\cdot\|$  is strictly convex, both  $(x_1, r^+(x_1))$  and  $(x_1, r^-(x_1))$  belong to the interior of  $B_X$ . As  $(x_1, x_2)$  is a convex combination of  $(x_1, r^+(x_1))$  and  $(x_1, r^-(x_1))$ , we also have  $x \in \text{int}(B_X)$ .

For the second part, let  $x = (x_1, x_2) \in B_X$ , with  $x_1 > \alpha$ . Suppose that  $x_2 \geq r^+(x_1) = \delta + (1 - \delta)x_1/\alpha$ . Then the strict convexity of  $\|\cdot\|, \|x\| \leq 1$  and  $\|d\| = 1$  imply that

$$\begin{aligned} 1 > \left\| \frac{\alpha}{x_1}x + \frac{x_1 - \alpha}{x_1}d \right\| &= \left\| \frac{\alpha}{x_1}(x_1, x_2) + \frac{x_1 - \alpha}{x_1}(0, \delta) \right\| \\ &= \left\| \left( \alpha, \frac{\alpha x_2}{x_1} \right) + \left( 0, \delta - \frac{\alpha \delta}{x_1} \right) \right\| = \left\| \left( \alpha, \delta + \frac{x_2 - \delta}{x_1} \alpha \right) \right\|. \end{aligned}$$

Now

$$\delta + \frac{\alpha(x_2 - \delta)}{x_1} \geq \delta + \frac{\alpha}{x_1} \left( \delta + (1 - \delta) \frac{x_1}{\alpha} - \delta \right) = 1,$$

which implies that  $(\alpha, 1)$  is a convex combination of  $(\alpha, -1)$  and  $(\alpha, \delta + (x_2 - \delta)\alpha/x_1)$ . But  $\|(\alpha, 1)\| = \|(\alpha, -1)\| = 1$  and  $\|(\alpha, \delta + (x_2 - \delta)\alpha/x_1)\| < 1$ , a contradiction. The case  $x_2 \leq r^-(x_1)$  is analogous.  $\square$

**PROPOSITION 2.5.** *For the ball  $(\alpha, 0) + \delta^{-1}B_X$  we have essentially the same symmetric inclusions:*

$$\begin{aligned} B_X \cap (]\alpha, \infty[ \times \mathbb{R}) &\subset (\alpha, 0) + \delta^{-1} \text{int}(B_X), \\ ((\alpha, 0) + \delta^{-1}B_X) \cap (]-\infty, \alpha[ \times \mathbb{R}) &\subset \text{int}(B_X). \end{aligned}$$

**PROOF.** Let  $x = (x_1, x_2) \in B_X$  be such that  $x_1 > \alpha$ . We may suppose  $x_2 \geq 0$ . Instead of showing that  $x$  belongs to the interior of  $(\alpha, 0) + \delta^{-1}B_X$ , we shall see that

$$(\delta x_1 - \delta \alpha, \delta x_2) \in \text{int}(B_X).$$

As  $(x_1, x_2), (0, x_2)$  and  $(0, \delta)$  belong to  $B_X$ , it suffices to show that

$$(\delta x_1 - \delta \alpha, \delta x_2) \in \text{conv}\{(x_1, x_2), (0, x_2), (0, \delta)\}.$$

For this, we need  $\delta x_1 - \delta \alpha \in ]0, x_1[$ . This is equivalent to  $x_1 < \alpha\delta/(\delta - 1)$ , and this inequality is true since  $c = (\alpha\delta/(\delta - 1), 0)$  is the only point in  $r^+ \cap r^-$ .

Since  $\delta x_2 > x_2$ , the only other thing we need to show is that  $(\delta x_1 - \delta \alpha, \delta x_2)$  lies below the line defined by  $(0, \delta)$  and  $(x_1, x_2)$ . This line is the graph of the function  $y(t) = x_2t/x_1 + \delta - \delta t/x_1$ , and so we need

$$\delta x_2 < x_2(\delta x_1 - \delta \alpha)/x_1 + \delta - \delta(\delta x_1 - \delta \alpha)/x_1.$$

After some elementary computations, we see that this inequality is equivalent to

$$0 < x_1 - x_2\alpha + \alpha\delta - \delta x_1.$$

To finish the proof of the first part we only need to observe that the second part of Lemma 2.4 implies that  $(\alpha, 1)$  is above the line defined by  $(0, \delta)$  and  $(x_1, x_2)$ , so that  $1 > x_2\alpha/x_1 + \delta - \delta\alpha/x_1$ . This is also equivalent to  $0 < x_1 - x_2\alpha + \alpha\delta - \delta x_1$ , and so we are done.

For the second inclusion, take  $y = (y_1, y_2) \in (\alpha, 0) + \delta^{-1}S_X$ , with  $y_1 < \alpha$ , and  $y' = (y'_1, y'_2) = (2\alpha - y_1, -y_2)$  symmetric to  $y$  with respect to  $(\alpha, 0)$ , and suppose that  $y \in S_X$ . As both  $(\alpha, 1)$  and  $(\alpha, 0) + \delta^{-1}(\alpha, 1)$  belong to  $((\alpha, 0) + \delta^{-1}S_X) \cap r^+$ , there are no more points in this intersection, and this means that  $(\alpha, 0) + \delta^{-1}S_X$  lies below  $r^+$  outside the interval  $[\alpha, \alpha(1 + \delta)]$ . As  $S_X$  lies above this line in  $[0, \alpha]$ , we get  $y_1 < 0$ . Now  $y' \in (\alpha, 0) + \delta^{-1}S_X$  and  $y'_1 = 2\alpha - y_1 > 2\alpha$  together imply  $|y_2| = |y'_2| < \delta^{-1} < 1$ , and from this we get  $y_1 < -\alpha$ . We also have  $\|(y_1 - \alpha, y_2)\| = \delta^{-1}$ , and so

$$(y_1 - \alpha, y_2), (\alpha, y_2) \in \text{int}(B_X), (y_1, y_2) \in S_X \quad \text{and} \quad (y_1, y_2) \in [(y_1 - \alpha, y_2), (\alpha, y_2)],$$

a contradiction.  $\square$

**LEMMA 2.6.** *With the previous notation,  $\beta > \alpha$  implies  $\|b - b'\| < \|a - a'\|$ .*

**PROOF.** We may suppose that  $\|z\| = 1$ . Recall that, in the basis we are dealing with,  $\frac{1}{2}(a + a') = (\alpha, 0)$  and  $\frac{1}{2}(b + b') = (\beta, 0)$ .

Let  $b = (b_1, b_2), b' = (b'_1, b'_2)$  be the expressions in coordinates of  $b, b'$  in the basis  $\{z, (a - a')/2\}$ . It is clear that  $(b'_1, b'_2) = (2\beta - b_1, -b_2)$ , and we may suppose  $b_1 \geq \beta > \alpha$ . Then, with  $\beta' = r^+(\beta) < r^+(\alpha) = 1$ , Lemma 2.4 implies

$$b \in B_X \cap ([\beta, \infty[ \times \mathbb{R}) \subset [(\beta, 0) + \beta' B_X] \cap ([\beta, \infty[ \times \mathbb{R}),$$

so  $\|b - (\beta, 0)\| < \|a - (\alpha, 0)\|$ , and we are done. □

**LEMMA 2.7.** *Let  $z \in \text{int}(B_X) \setminus \{0\}$ . There exists exactly one pair  $x, x' \in S_X$  such that  $z = \frac{1}{2}(x + x')$ .*

**PROOF.** For the existence, we will define some auxiliary functions. For  $t \in [0, 2\pi]$ , first let  $x(t)$  be defined as the only point in  $S_X \cap \{\lambda(\cos(t), \sin(t)) : \lambda \in ]0, \infty[ \}$ . Then take  $z \in \text{int}(B_X) \setminus \{0\}$  and define  $f(t)$  as  $\|z - y(t)\|$ , where  $y(t)$  is the only point in  $S_X \cap \{z + \lambda x(t) : \lambda \in ]0, \infty[ \}$ . It is clear that all these functions are continuous and, moreover,  $f(2\pi) = f(0)$ . So there exists  $t \in [0, \pi[$  such that  $f(t + \pi) = f(t)$ . For this  $t$ , we have  $z = \frac{1}{2}(y(t) + y(t + \pi))$ .

For the uniqueness, suppose that we have four different points  $x, x', y, y' \in S_X$  such that  $x + x' = y + y' = 2z$ . Take as a basis  $\{z, 1/2(x - x')\}$ , so that  $x + x' = y + y' = (2, 0)$ ,  $x = (1, 1), x' = (1, -1)$  and  $y = (y_1, y_2), y' = (y'_1, y'_2)$ . As usual,  $\delta = 1/\|(0, 1)\|$ .

Now suppose  $y_1 > 1$ . By the first inclusion in Proposition 2.5,  $y \in S_X$  implies  $y \in (1, 0) + \delta^{-1} \text{int}(B_X)$ . But the second inclusion in the same proposition implies that, then,  $y' \in \text{int}(B_X)$ , so we are done. □

To finish our proof of the remaining implication of Proposition 2.1, we only need to notice that, for  $\beta > \alpha > 0$ , Lemma 2.6 leads to a contradiction with our initial assumptions and, if  $\alpha = \beta$ , the contradiction arises from Lemma 2.7. □

### 3. Main result

We can now state and prove the last step before the main result. We are no longer assuming  $(X, \|\cdot\|)$  to be strictly convex.

**PROPOSITION 3.1.** *Let  $x, y, z$  be nonzero vectors in  $X$  and  $(\gamma_n), (\delta_n) \subset \mathbb{R}$  be a pair of positive sequences converging monotonically to 0. If  $\gamma_n x, \delta_n y \in B(-z, z)$  for every  $n \in \mathbb{N}$ , then  $y = \pm x$ .*

**PROOF.** We may suppose  $\|z\| = 1$ .

If the result does not hold, then we may take  $\{x, y\}$  as a basis of  $X$ . In coordinates,  $x = (1, 0), y = (0, 1), z = (z_1, z_2)$ , and we may suppose  $z_1, z_2 > 0$ . Indeed, if  $z_1 < 0$  then we may take  $-x$  instead of  $x$  and the case  $z_1 = 0$  is absurd.

As  $\gamma_n x, \delta_n y \in B(-z, z)$ , in coordinates,

$$\|(z_1 + \gamma_n, z_2)\| = \|(z_1 - \gamma_n, z_2)\|, \quad \|(z_1, z_2 + \delta_n)\| = \|(z_1, z_2 - \delta_n)\|, \quad \text{for all } n.$$

Set  $\alpha_n = \|(z_1 + \gamma_n, z_2)\|^{-1}$  and  $\beta_n = \|(z_1, z_2 + \delta_n)\|^{-1}$ . Observe that  $\alpha_n \rightarrow 1, \beta_n \rightarrow \|z\| = 1$ , and also that the convexity of  $\|\cdot\|$  implies that  $\|(z_1 \pm \gamma_n, z_2)\|$  and  $\|(z_1, z_2 \pm \delta_n)\|$  are at least 1, so  $\gamma_n \leq 1, \delta_n \leq 1$  for every  $n$ .

The choice of  $(\alpha_n)$  and  $(\beta_n)$  gives

$$\alpha_n(z_1 \pm \gamma_n, z_2), \beta_n(z_1, z_2 \pm \delta_n) \in S_X, \quad \text{for all } n,$$

so we have a pair of sequences  $(\alpha_n)z$  and  $(\beta_n)z$  that converge to  $z$  and such that each  $\alpha_n z$  is the midpoint of the segment  $(\{\alpha_n z_1\} \times \mathbb{R}) \cap B_X$  and each  $\beta_n z$  is the midpoint of the segment  $(\mathbb{R} \times \{\beta_n z_2\}) \cap B_X$ .

We will analyse the shape of the unit ball  $B_X$  of such a norm and will eventually rule out every possibility.

Suppose there is some  $v = (v_1, v_2) \in B_X$  with  $v_1 > z_1$ . Then  $B_X$  contains the triangle  $\text{conv}\{(v_1, v_2), (0, 0), (z_1, z_2)\}$ . Note that  $v_1 z_2 \neq z_1 v_2$ , because  $v_1 z_2 = z_1 v_2$  is absurd, so this is actually a triangle. If  $v_1 z_2 > v_2 z_1$  (respectively,  $v_1 z_2 < v_2 z_1$ ), then the interior of this triangle contains  $(z_1 + \gamma_n, z_2)$  (respectively,  $(z_1 - \gamma_n, z_2)$ ) for infinitely many  $n$ . As  $(z_1 \pm \gamma_n, z_2), (z_1, z_2 \pm \delta_n) \notin \text{int } B_X$  for every  $n$ , and applying the analogous reasoning to  $v_2$ , we get  $v_1 \leq z_1$  and  $v_2 \leq z_2$  for every  $(v_1, v_2) \in B_X$ . So

$$\{(v_1, v_2) \in B_X : v_1, v_2 \geq 0\} \subseteq \text{conv}\{(z_1, z_2), (0, z_2), (0, 0), (z_1, 0)\}.$$

Now consider  $r_2$  as the line that is *vertically symmetric* to  $r_1 = \{(t, z_2) : t \in \mathbb{R}\}$  with respect to  $r_0 = \{(t, tz_2/z_1) : t \in \mathbb{R}\} = \langle z \rangle$ , that is, the line  $r_2 = \{(t, 2tz_2/z_1 - z_2)\}$ . For every  $t$ , the (unique) point in  $r_2 \cap (\{t\} \times \mathbb{R})$  is symmetric to (the point in)  $r_1 \cap (\{t\} \times \mathbb{R})$  with respect to  $r_0 \cap (\{t\} \times \mathbb{R})$ . As  $B_X$  lies below  $r_1$ , it is readily seen that  $\beta_n(z_1, z_2 - \delta_n)$  lies above  $r_2$  for every  $n$ . So, with an argument similar to that in the previous paragraph, we can see that every  $(v_1, v_2) \in B_X$  lies above  $r_2$ . As  $r_2$  contains both  $(z_1, z_2)$  and  $(0, -z_2)$ , the point where  $r_2$  intersects the horizontal axis is  $(z_1/2, 0)$ , so we can describe the situation as follows:

$$\{(v_1, v_2) \in B_X : v_1, v_2 \geq 0\} \subseteq \text{conv}\{(z_1, z_2), (0, z_2), (0, 0), (z_1/2, 0)\}.$$

Now consider  $r_3$  as the line that is *horizontally symmetric* to  $r_2$  with respect to  $r_0$ , that is, the midpoint of  $r_3 \cap (\mathbb{R} \times \{t\})$  and  $r_2 \cap (\mathbb{R} \times \{t\})$  is  $r_0 \cap (\mathbb{R} \times \{t\})$  for every  $t$ . The same argument implies that  $B_X$  lies below  $r_3$ , and  $(0, z_2/3)$  is the point where  $r_3$  intersects the vertical axis. So

$$\{(v_1, v_2) \in B_X : v_1, v_2 \geq 0\} \subseteq \text{conv}\{(z_1, z_2), (0, z_2/3), (0, 0), (z_1/2, 0)\}.$$

By iterating this process, for every  $n$ ,

$$\{(v_1, v_2) \in B_X : v_1, v_2 \geq 0\} \subseteq \text{conv}\{(z_1, z_2), (0, z_2/(2n - 1)), (0, 0), (z_1/2n, 0)\}$$

and this is absurd. The proof is therefore complete. □

Finally, we have the following new characterisation (in the negative) of the Euclidean case among all Minkowski planes.

**THEOREM 3.2.** *The norm  $\|\cdot\|$  is not Euclidean if and only if, for some  $x \neq 0$  and for each  $\lambda \in (0, +\infty) \setminus \{1\}$ , there is a  $y$  such that  $\|y\| = \lambda\|x\|$ ,  $\langle x, y \rangle = \mathbb{R}^2$  and  $B(-x, x) \cap B(-y, y) \neq \emptyset$ .*

**PROOF.** ( $\Leftarrow$ ): This is the simple part. If  $\|\cdot\|$  is Euclidean, then  $B(-x, x)$  and  $B(-y, y)$  are both straight lines, and they are different provided that  $\langle x, y \rangle = \mathbb{R}^2$ .

( $\Rightarrow$ ): First, let us assume  $\|\cdot\|$  not to be strictly convex.

Let  $a, b \in S_X$  such that  $[a, b] \subset S_X$ . Take  $x = (3a + b)/4$  and  $z = (a + 3b)/4$  and observe that  $(x - z)/2 = (a - b)/4$  belongs to both  $B(-x, x)$  and  $B(-z, z)$ . Furthermore, for every  $\alpha \in [-1, 1]$ ,

$$\|z - \alpha(x - z)/2\| = \|(a + 3b)/4 - \alpha(a - b)/4\| = \|(1 - \alpha)a/4 + (3 + \alpha)b/4\| = 1$$

because the last is a convex combination of  $a$  and  $b$ . This means that the full segment  $[(z - x)/2, (x - z)/2]$  lies in  $B(-z, z)$ . By symmetry, it is also included in  $B(-x, x)$ .

Now let  $\lambda > 0$ . If  $\lambda \leq 1$ , then  $\lambda(z - x)/2 \in B(-x, x) \cap B(-\lambda z, \lambda z)$ . If  $\lambda \geq 1$ , then  $(z - x)/2 \in B(-x, x) \cap B(-\lambda z, \lambda z)$ , so in any case the result follows with  $y = \lambda z$ .

To deal with the case where  $\|\cdot\|$  is strictly convex, suppose that, for every  $x \neq 0$  and a certain  $\lambda \in (0, +\infty) \setminus \{1\}$ , there exists no  $y$  with  $\|y\| = \lambda\|x\|$  satisfying  $\langle x, y \rangle = \mathbb{R}^2$  and  $B(-x, x) \cap B(-y, y) \neq \emptyset$ .

Consider  $x_0$  such that  $B(-x_0, x_0)$  is not a straight line. (The existence of such a bisector is guaranteed in any non-Euclidean Minkowski plane; see [2, Theorem 5.5].) Then  $B(-x_0, x_0) = B(-\lambda x_0, \lambda x_0)$ .

Let us assume this is not the case: let  $p \in B(-x_0, x_0)$ ,  $p \notin B(-\lambda x_0, \lambda x_0)$ . Now, if we take the unique (see [7, Corollary 2.5])  $y = B(-p, p) \cap \lambda\|x_0\|S_X$ , we come to a contradiction, as  $p \in B(-x_0, x_0) \cap B(-y, y)$ .

On the other hand,  $B(-\lambda x_0, \lambda x_0) = \lambda B(-x_0, x_0)$ , as we said in Remark 1.4; therefore,  $B(-x_0, x_0) = \lambda B(-x_0, x_0)$ . Now, take linearly independent  $e, v \in B(-x_0, x_0)$ . As  $B(-x_0, x_0) = \lambda B(-x_0, x_0)$ , we have  $\lambda u \in B(-x_0, x_0)$  for every  $u \in B(-x_0, x_0)$  so  $\lambda^n u \in B(-x_0, x_0)$  and, in particular,

$$\lambda^n e, \lambda^n v \in B(-x_0, x_0) \quad \text{for every } n \in \mathbb{Z}. \tag{3.1}$$

As  $\mathcal{B} = \{e, v\}$  is a basis in  $X$ , we may take coordinates giving  $e = (1, 0)$ ,  $v = (0, 1)$  and  $x_0 = (\alpha, \beta)$ . For the sake of clarity, we will suppose  $x_1, x_2 > 0$ . (If we had  $x_1 < 0$  we could just take  $-e$  instead of  $e$  and the case  $x_1 = 0$  is absurd.) We may also suppose  $\lambda \in ]0, 1[$ . Indeed, if we have  $\lambda > 1$  we can take  $\mu = \lambda^{-1}$  and we are in exactly the same situation as before:  $B(-x_0, x_0) = \mu B(-x_0, x_0)$  with  $\mu < 1$ .

Rewriting (3.1) in coordinates we get

$$\begin{aligned} \|(\alpha - \lambda^n, \beta)\| &= \|(\alpha + \lambda^n, \beta)\| = \|(-\alpha + \lambda^n, -\beta)\| = \|(-\alpha - \lambda^n, -\beta)\|, \\ \|(\alpha, \beta - \lambda^n)\| &= \|(\alpha, \beta + \lambda^n)\| = \|(-\alpha, -\beta + \lambda^n)\| = \|(-\alpha, -\beta - \lambda^n)\|. \end{aligned}$$

But, by Proposition 3.1, this cannot happen and the proof is complete. □



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