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A NOTE ON FIBONACCI TYPE GROUPS

BY
C. M. CAMPBELL AND E. F. ROBERTSON

1. **Introduction.** Let F_n be the free group on $\{a_i : i \in \mathbb{Z}_n\}$ where the set of congruence classes mod n is used as an index set for the generators. The permutation $(1, 2, 3, \ldots, n)$ of $\mathbb{Z}_n$ induces an automorphism θ of F_n by permuting the subscripts of the generators. Suppose w is a word in F_n and let $N(w)$ denote the normal closure of $\{w\theta^{i-1} : 1 \le i \le n\}$. Define the group $G_n(w)$ by $G_n(w) = F_n/N(w)$ and call $w\theta^{i-1} = 1$ the relation (i) of $G_n(w)$.

In this note we consider the group $G_n(w)$ where w is the word

$$w = a_h a_{2h} \cdots a_{rh}(a_{rh+k}^{-1})$$

and r, h, k are integers such that $k \ge 0$, $h \ge 1$, $r \ge 2$. For this particular choice of w we denote $G_n(w)$ by $\mathbf{R}(r, n, k, h)$. The groups $\mathbf{R}(2, n, n-1, 2)$ are discussed in [6] while the groups $\mathbf{R}(2, n, k, h)$ have been investigated by Johnson and Mawdesley. The groups $\mathbf{R}(r, n, k, 1)$ are the *generalized Fibonacci groups* $\mathbf{F}(r, n, k)$ discussed in [2], [3], [4] and [7] while the groups $\mathbf{R}(r, n, 1, 1)$ are the ordinary *Fibonacci groups* $\mathbf{F}(r, n)$ discussed in [5] and [8]. We exhibit some isomorphisms, showing that more of the groups $\mathbf{R}(r, n, k, h)$ are generalized Fibonacci groups than are indicated above. We also discuss the group $\mathbf{R}(3, 6, 5, 2)$, a finite non-metacyclic group which is not a generalized Fibonacci group.

2. **Some isomorphisms.** It follows immediately from the definition that if $k_1 \equiv k_2 \bmod n$ and $h_1 \equiv h_2 \bmod n$ then $\mathbf{R}(r, n, k_1, h_1) \cong \mathbf{R}(r, n, k_2, h_2)$ so that when we write $\mathbf{R}(r, n, k, h)$ we shall assume that k and h have been reduced mod n.

LEMMA 1.

$$\begin{aligned}\mathbf{R}(r, n, k, h) &\cong \mathbf{R}(r, n, k+(r-1)h, -h)\\ &\cong \mathbf{R}(r, n, -k, -h)\\ &\cong \mathbf{R}(r, n, -k-(r-1)h, h).\end{aligned}$$

Proof. The isomorphisms are immediate on considering the maps ϕ_1, ϕ_2, ϕ_3 from the free group F_n on $\{x_i : i \in \mathbb{Z}_n\}$ to $\mathbf{R}(r, n, k, h)$ induced by $x_i\phi_1 = a_i^{-1}$, $x_i\phi_2 = a_{-i}$ and $x_i\phi_3 = a_{-i}^{-1}$.

LEMMA 2. *If α is an integer coprime to n then*

$$\mathbf{R}(r, n, k, h) \cong \mathbf{R}(r, n, k/\alpha, h/\alpha).$$

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Proof. This isomorphism follows from considering the map ϕ from the free group on $\{x_i : i \in \mathbb{Z}_n\}$ to $\mathbf{R}(r, n, k, h)$ induced by $x_i\phi = x_{i/\alpha}$.

Notice that it follows from this result that if h is coprime to n, $\mathbf{R}(r, n, k, h) \cong \mathbf{F}(r, n, k/h)$.

THEOREM 3. *Suppose that* $(r-1)h \equiv 0 \bmod n$ *and* k *is coprime to* n, *then*

$$\mathbf{R}(r, n, k, h) \cong \mathbf{F}(r^{(n,h)}, d, \gamma)$$

where $d = n/(n, h)$ *and* γ *is such that* $(n, h) = \beta n + \gamma h$.

Proof. By Lemma 2 we can assume without loss of generality that $k=1$. The first relation of $\mathbf{R}(r, n, 1, h)$ reduces to

$$(a_h a_{2h} \cdots a_{dh})^{(r-1)/d} a_h = a_{h+1}$$

where the generators $a_h, a_{2h}, \ldots, a_{dh}$ are distinct. This allows us to express a_{h+1} in terms of $a_h, a_{2h}, \ldots, a_{dh}$ and relation (ih) allows us to express $a_{(i+1)h+1}$ also in terms of $a_h, a_{2h}, \ldots, a_{dh}$ for $1 \le i \le d-1$. Substituting these expressions in relation (2) gives

$$(a_h a_{2h} \cdots a_{dh})^{(r^2-1)/d} a_h = a_{h+2}.$$

Continuing in this way we obtain

$$(a_h a_{2h} \cdots a_{dh})^{(r^j-1)/d} a_h = a_{h+j}, \qquad 1 \le j \le (n, h),$$

since a_{h+j}, $1 \le j \le (n, h)$ are distinct and $a_{h+(n,h)} \in \{a_h, a_{2h}, \ldots, a_{nh}\}$. At this stage the n relations for $\mathbf{R}(r, n, 1, h)$ have been reduced to the d relations

$$((a_h a_{2h} \cdots a_{dh})^{(r^{(n,h)}-1)/d} a_h a_{h+(n,h)}^{-1})\theta^{(i-1)h} = 1, \qquad 1 \le i \le d.$$

Putting $x_i = a_{ih}$, $1 \le i \le d$ we obtain the relations

$$((x_1 x_2 \cdots x_d)^{(r^{(n,h)}-1)/d} x_1 x_{1+\gamma}^{-1})\bar{\theta}^{i-1} = 1, \qquad 1 \le i \le d,$$

where $\bar{\theta}$ permutes the subscripts of x_i, $1 \le i \le d$, according to the permutation $(1, 2, \ldots, d)$. The result now follows.

COROLLARY. *With the conditions on* r, n, k, h *as in the statement of Theorem* 3, $\mathbf{R}(r, n, k, h)$ *is metacyclic of order* $r^n - 1$.

Proof. This follows from Theorem 1 of [3] and Theorem 3 on showing that $r^{(n,h)} \equiv 1 \bmod d$ and γ is coprime to n. These are straightforward applications of elementary number theory.

Notice, using the results of [4], that if $\mathbf{R}(r, n, k_1, h_1)$ and $\mathbf{R}(r, n, k_2, h_2)$ satisfy the conditions of the above theorem then they are isomorphic if, and only if, $(n, h_1) = (n, h_2)$.

Next we show that if $(n, k, h) \ne 1$, then $\mathbf{R}(r, n, k, h)$ is infinite.

THEOREM 4. *If* $(n, k, h) = d \neq 1$, *then*

$$\mathbf{R}(r, n, k, h) \cong \mathop{*}_{d} \mathbf{R}(r, n/d, k/d, h/d),$$

the free product of d copies of $\mathbf{R}(r, n/d, k/d, h/d)$.

Proof. Let $\alpha = n/d$, $\beta = k/d$, $\gamma = h/d$ and fix t with $0 \leq t \leq d-1$. With $x_j = a_{jd+t}$ the relations $(id+t)$, $1 \leq i \leq \alpha$, reduce to

$$(x_\gamma x_{2\gamma} \cdots x_{r\gamma} x_{r\gamma+\beta}^{-1})\bar{\theta}^{i-1} = 1, \qquad 1 \leq i \leq \alpha,$$

where the subscripts of the x_i are reduced mod α and permuted by $\bar{\theta}$ according to the permutation $(1, 2, \ldots, \alpha)$. The result now follows.

3. **The group** $\mathbf{R}(3, 6, 5, 2)$. The only Fibonacci group known to be finite and not metacyclic is $\mathbf{F}(3, 6)$, a group of order 1512, see [2], where the three known finite non-metacyclic generalized Fibonacci groups are discussed. The only finite non-metacyclic group which we have discovered in the class $\mathbf{R}(r, n, k, h)$ other than these generalized Fibonacci groups is $\mathbf{R}(3, 6, 5, 2)$.

Using Tietze transformations the following 2-generator, 2-relation presentation is obtained.

$$\mathbf{R}(3, 6, 5, 2) = \langle a, b \mid a^{-1}ba^2b^{-1}ab^2 = (ba^{-1}b^{-1}a^{-1})^2ba^{-1}bab^{-1}a = 1\rangle.$$

We have investigated this group using the coset enumeration programme [1] which shows that $|\mathbf{R}(3, 6, 5, 2)| = 1512 = 2^3 \cdot 3^3 \cdot 7$. It is soluble but not metabelian and has the following Sylow structure. A Sylow 2-subgroup is cyclic and generated by a. It is not normal. Both the Sylow 3-subgroup and the Sylow 7-subgroup are normal, the Sylow 3-subgroup being the non-abelian group of order 27 with exponent 3. Despite the coincidence in the orders $\mathbf{R}(3, 6, 5, 2)$ is not isomorphic to $\mathbf{F}(3, 6)$ since, for example, $\mathbf{F}(3, 6)$ has Q_8 as a Sylow 2-subgroup.

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MATHEMATICAL INSTITUTE,
UNIVERSITY OF ST. ANDREWS,
ST. ANDREWS, KY16 9SS,
SCOTLAND.