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Abstract

In this paper, we study the structure of the local components of the (shallow, i.e. without U_p) Hecke algebras acting on the space of modular forms modulo p of level 1, and relate them to pseudo-deformation rings. In many cases, we prove that those local components are regular complete local algebras of dimension 2, generalizing a recent result of Nicolas and Serre for the case $p = 2$.

1. Introduction

1.1 General notation

In this paper we fix a prime number p. We shall denote by K a finite extension of $\mathbb{Q}_p$, by $\mathcal{O}$ the ring of integers of K, by $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, by π a uniformizer of $\mathcal{O}$ and by $\mathbb{F}$ the finite residue field $\mathcal{O}/\pi$.

We call $G_{\mathbb{Q},p}$ the Galois group of a maximal algebraic extension of $\mathbb{Q}$ unramified outside p and ∞. For ℓ a prime $\neq p$, we denote by $\mathrm{Frob}_\ell \in G_{\mathbb{Q},p}$ a Frobenius element at ℓ. We denote by c a complex conjugation in $G_{\mathbb{Q},p}$. We write $G_{\mathbb{Q}_p}$ for $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. There is a natural map $G_{\mathbb{Q}_p} \to G_{\mathbb{Q},p}$, well defined up to conjugacy. For ρ a representation of $G_{\mathbb{Q},p}$, we shall denote by $\rho_{|G_{\mathbb{Q}_p}}$ the composition of that map with ρ: this is a representation of $G_{\mathbb{Q}_p}$, well-defined up to isomorphism. We denote by $\omega_p : G_{\mathbb{Q},p} \to \mathbb{F}^*$ the cyclotomic character modulo p.

1.2 Definition of the Hecke algebras modulo p

We shall denote by $S_k(\mathcal{O})$ the module of cuspidal modular forms of weight k and level 1 with coefficients in $\mathcal{O}$, that we see by the q-expansion map as a sub-module of $\mathcal{O}[[q]]$. For $f \in S_k(\mathcal{O})$, we shall denote by $\sum_{n\geqslant 1} a_n(f)q^n$ its image in $\mathcal{O}[[q]]$. We denote by $S_{\leqslant k}(\mathcal{O})$ the sub-module $\sum_{i=0}^{k} S_i(\mathcal{O})$ (the sum is direct, cf. [Miy06, Lemma 2.1.1]) of $\mathcal{O}[[q]]$. We denote by $S_k(\mathbb{F})$ the space of cuspidal modular forms of weight k and level 1 over $\mathbb{F}$ in the sense of Swinnerton-Dyer and Serre, that is the image of $S_k(\mathcal{O})$ by the reduction map $\mathcal{O}[[q]] \to \mathbb{F}[[q]]$, $f \mapsto \tilde{f}$ which reduces each coefficient modulo $\mathfrak{p}$. It is clear that the natural map $S_k(\mathcal{O})/\mathfrak{p}S_k(\mathcal{O}) \to S_k(\mathbb{F})$ is an isomorphism. Similarly, we denote by $S_{\leqslant k}(\mathbb{F})$ the image of $S_{\leqslant k}(\mathcal{O})$ by the reduction map $f \mapsto \tilde{f}$. The reader should be aware that the natural map $S_{\leqslant k}(\mathcal{O})/\mathfrak{p}S_{\leqslant k}(\mathcal{O}) \to S_{\leqslant k}(\mathbb{F})$ is surjective but not an isomorphism in general, or in other words, that $S_{\leqslant k}(\mathbb{F}) = \sum_{i=0}^{k} S_i(\mathbb{F})$ but that the sum is not direct in general.

All of the modules considered above have a natural action of the Hecke operators T_n for $p \nmid n$. We call $\mathbb{T}_k$ the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S_{\leqslant k}(\mathcal{O}))$ generated by the T_n, $p \nmid n$, and similarly A_k the $\mathbb{F}$-subalgebra of $\mathrm{End}_{\mathbb{F}}(S_{\leqslant k}(\mathbb{F}))$ generated by the T_n, $p \nmid n$. Note that it would amount to

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the same to define $\mathbb{T}_k$ or A_k as generated by the Hecke operators T_ℓ and S_ℓ for $\ell \neq p$, ℓ prime, instead, where S_ℓ is the operator acting on forms of weight k as the multiplication by ℓ^{k-2}: this equivalence is clear from the relations

$$T_{mn} = T_m T_n \quad \text{when } (m,n) = 1, \tag{1}$$
$$T_{\ell^{n+1}} = T_\ell T_{\ell^n} - \ell S_\ell T_{\ell^{n-1}} \quad \text{for } n \geqslant 1. \tag{2}$$

Recall that on q-expansions, for a form $f \in S_k(\mathcal{O})$, one has for every $n \geqslant 1$,

$$a_n(T_\ell f) = a_{\ell n}(f) \quad \text{if } \ell \nmid n, \tag{3}$$
$$a_n(T_\ell f) = a_{\ell n}(f) + \ell^{k-1} a_{n/\ell}(f) \quad \text{if } \ell \mid n, \tag{4}$$
$$a_n(S_\ell f) = \ell^{k-2} a_n(f). \tag{5}$$

Since the actions of the operators T_ℓ on the various modules considered above are compatible in an obvious sense, one has a natural morphism of $\mathbb{F}$-algebras $\mathbb{T}_k/\mathfrak{p}\mathbb{T}_k \to A_k$ which is obviously surjective, but in general not an isomorphism as will be clear from the sequel. One also has surjective morphisms $\mathbb{T}_{k+1} \to \mathbb{T}_k$ and $A_{k+1} \to A_k$ given by restriction, and we can consider the projective limit:

$$\mathbb{T} = \varprojlim \mathbb{T}_k, \quad A = \varprojlim A_k.$$

By passage to the limit we obtain a surjective map $\mathbb{T}/\mathfrak{p}\mathbb{T} \to A$.

A well-known important fact is that $\mathbb{T}$ and A are complete semi-local rings. More precisely, if $\mathbb{F}$ is large enough, the maximal ideals, hence the local components, of both $\mathbb{T}$ and A are in bijection with $\mathbb{F}$-valued systems of eigenvalues for all the operators T_ℓ and ℓS_ℓ (for ℓ prime not dividing p) which have a non-trivial eigenspace in $S_{\leqslant k}(\mathbb{F})$ for k large enough, and those systems are finitely many. By Deligne's theorem on the existence of Galois representations attached to eigenforms and by the Deligne–Serre lemma, those systems of eigenvalues are in bijection with the set of isomorphism classes of modular Galois representations $\bar{\rho} : G_{\mathbb{Q},p} \to \mathrm{GL}_2(\mathbb{F})$ (the bijection being: eigenvalue of $T_\ell \leftrightarrow \operatorname{tr} \bar{\rho}(\mathrm{Frob}_\ell)$, eigenvalue of $\ell S_\ell \leftrightarrow \det \bar{\rho}(\mathrm{Frob}_\ell)$). Here and below, *modular* means that $\bar{\rho}$ is the semi-simplified reduction of a stable lattice for the Galois representation $\rho : G_{\mathbb{Q},p} \to \mathrm{GL}_2(K)$ attached by Deligne's construction to an eigenform in $S_k(\mathcal{O})$ for some integer k. We stress that *by definition, our modular representations $\bar{\rho}$ are semi-simple*. Moreover, if such a representation $\bar{\rho}$ is irreducible, then p is odd[1] and since $\bar{\rho}$ is odd, it is absolutely irreducible.

We call $\mathbb{T}_{\bar{\rho}}$ and $A_{\bar{\rho}}$ the local components of $\mathbb{T}$ and A corresponding to a modular representation $\bar{\rho}$. These rings are complete local rings. By definition, the image of $T_\ell \in \mathbb{T}$ in the residue field $\mathbb{F}$ of $\mathbb{T}_{\bar{\rho}}$ or $A_{\bar{\rho}}$ is $\operatorname{tr} \bar{\rho}(\mathrm{Frob}_\ell)$. The surjective map $\mathbb{T}/\mathfrak{p}\mathbb{T} \to A$ sends $\mathbb{T}_{\bar{\rho}}/\mathfrak{p}\mathbb{T}_{\bar{\rho}}$ onto $A_{\bar{\rho}}$.

1.3 Aim of the paper

The aim of this paper is to study the local components $A_{\bar{\rho}}$ of the Hecke algebra A modulo p, and their relation with deformation rings defined below. The study of the local components $A_{\bar{\rho}}$

[1] Indeed, as was observed by Serre, since $\Delta \equiv 1 \pmod 2$, $S_{k-12}(\mathbb{F}_2)$ is a subspace of $S_k(\mathbb{F}_2)$; this subspace has codimension at most 1 by the standard dimension formula. Hence, any eigenform in $S_k(\mathbb{F}_2)$ which is not already in $S_{k-12}(\mathbb{F}_2)$ is an eigenvector in the one-dimensional space $S_k(\mathbb{F}_2)/S_{k-12}(\mathbb{F}_2)$, hence has eigenvalues in $\mathbb{F}_2$. Therefore, any modular representation $G_{\mathbb{Q},2} \to \mathrm{GL}_2(\mathbb{F})$ is rational over $\mathbb{F}_2$, hence defined over $\mathbb{F}_2$ since finite fields have trivial Brauer group, and it is a simple exercise to show that there is only one semi-simple representation $G_{\mathbb{Q},2} \to \mathrm{GL}_2(\mathbb{F}_2)$ up to isomorphism, the direct sum of two copies of the trivial character.

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was initiated by Jochnowitz [Joc82] who proved that $A_{\bar\rho}$ is infinite-dimensional as a vector space over $\mathbb{F}$, and continued by Khare [Kha98], who proved, under the hypothesis that $\bar\rho$ is absolutely irreducible, that $A_{\bar\rho}$ is noetherian and has Krull's dimension at least 1. Recently, the structure of A in the case $p = 2$ has been determined by Nicolas and Serre. Let us explain their result, which is the direct motivation for this work. When $p = 2$, we can take $\mathcal{O} = \mathbb{Z}_2$, $\mathbb{F} = \mathbb{F}_2$ and there is only one modular representation (see the previous footnote). In other words, the $\mathbb{F}_2$-algebra A is local. Nicolas and Serre show that A is isomorphic to a power series ring in two variables: $A \simeq \mathbb{F}_2[[x, y]]$. Their proof of this result is long and difficult, but elementary. The result was extended to $p = 3$ by the first-named author of this article with Anna Medvedovsky: in this case $A = A_{\bar\rho} = \mathbb{F}_3[[x, y]]$ for the unique modular representation $\bar\rho = 1 + \omega_3$ (a sketch of the proof is given in the appendix of this paper). In this paper we shall give a generalization of these results for an arbitrary prime $p > 3$, using results of Böckle, Katz, Hida, Gouvêa–Mazur, Kisin, Wiles and Taylor–Wiles.

1.4 Deformations rings of pseudo-representations

To state our results, we need to recall Chenevier's notion of *pseudo-representations*,[2] restricted to dimension 2 for simplicity: if S is a commutative ring, and G is a group, a *pseudo-representation* (cf. [Che14, Lemma 1.9]) of G on S is an ordered pair of functions (t, d) from G to S, such that:

(a) d is a morphism of groups $G \to S^*$;
(b) $t(1) = 2$;
(c) for all g and h in G, $t(gh) = t(hg)$;
(d) for all g and h in G, $d(g)t(g^{-1}h) + t(gh) = t(g)t(h)$.

When G and S have a topology, we say that (t, d) is continuous if both t and d are.

If $\rho : G \to \mathrm{GL}_2(S)$ is a representation (i.e. a morphism of groups), then $(t_\rho := \operatorname{tr}\rho, d_\rho := \det\rho)$ is a pseudo-representation of G to S, called *the pseudo-representation attached to* ρ. Conversely, a pseudo-representation (t, d) of G to K, where K is an algebraically closed field, is attached to a unique semi-simple representation $\rho : G \to \mathrm{GL}_2(K)$ (see [Che14, Theorem A]). When 2 is invertible in S, (t, d) is determined by t, which is a pseudo-character of dimension 2 in the sense of Rouquier [Rou96], and the theory of pseudo-representations reduces to the more classical theory of pseudo-characters.

Let S be a henselian local ring, with algebraically closed residue field $K = S/m$, and $(t, d) : G \to S$ a pseudo-representation whose residual pseudo-representation $(t \pmod m), d \pmod m)$ of G to K is attached to a representation $\bar\rho : G \to \mathrm{GL}_2(K)$ which is absolutely irreducible. Then a theorem of Chenevier [Che14, Theorem B] asserts that (t, d) is attached to a unique (up to isomorphism) representation $\rho : G \to \mathrm{GL}_2(S)$.

Let $\mathcal{C}$ be the category of local profinite[3] $\mathcal{O}$-algebras S with maximal ideal m_S such that $S/m_S = \mathbb{F}$, the morphisms being the continuous local morphisms of $\mathcal{O}$-algebras, and let $\tilde{\mathcal{C}}$ be the full subcategory of $\mathcal{C}$ whose objects are the local profinite $\mathbb{F}$-algebras S. Let us fix an odd continuous representation $\bar\rho : G_{\mathbb{Q},p} \to \mathrm{GL}_2(\mathbb{F})$. We define a functor $D_{\bar\rho}$ from $\mathcal{C}$ to $\mathcal{SETS}$ by sending

[2] Cf. [Che14], where this notion has the not very convenient name *determinant*.

[3] By a profinite algebra we mean a topological algebra which is the directed projective limit of finite algebras, endowed with the discrete topology, with surjective morphisms. Note that we do not require that the topology on the local profinite algebra S is that defined by its maximal ideal m_S. However, this is the case when S is noetherian, for in this case the identity map $S \to S$, where the source is provided with the profinite topology and the target with the m_S-adic topology, is continuous, hence an homeomorphism since S is compact.

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S to the set of continuous pseudo-representations (t,d) of $G_{\mathbb{Q},p}$ to S such that $t \pmod{m_S} = \operatorname{tr}\bar\rho$, $d \pmod{m_S} = \det\bar\rho$, and $t(c)=0$.[4]

We denote by $\tilde D_{\bar\rho}$ the restriction of $D_{\bar\rho}$ to the subcategory $\tilde{\mathcal{C}}$, and by $\tilde D^0_{\bar\rho}$ the sub-functor of $\tilde D_{\bar\rho}$ of deformations (t,d) with constant determinant, that is such that $d=\bar d$. By [Che14, Proposition 3.3], the functor $D_{\bar\rho}$ is representable by a local profinite algebra $R_{\bar\rho}$, and by [Che14, Proposition 3.7], $R_{\bar\rho}$ is noetherian. It follows that the functor $\tilde D_{\bar\rho}$ is also representable, by $\tilde R_{\bar\rho} = R_{\bar\rho}/\mathfrak{p}R_{\bar\rho}$, and that $\tilde D^0_{\bar\rho}$ is representable as well, by a local algebra $\tilde R^0_{\bar\rho}$ which is a quotient of $\tilde R_{\bar\rho}$.

Definition 1. Let $\bar\rho$ be a modular representation $G_{\mathbb{Q},p}\to \mathrm{GL}_2(\mathbb{F})$. We shall say that $\bar\rho$ is *unobstructed* if the tangent space $\operatorname{Tan}\tilde D^0_{\bar\rho}$ to the functor $\tilde D^0_{\bar\rho}$ has dimension 2.

When $\bar\rho$ is irreducible, by the result of Chenevier [Che14, Theorem B] mentioned above, the functor $\tilde D^0_{\bar\rho}$ is just the usual functor of deformations of $\bar\rho$ *as a representation*, and with constant determinant, on the category $\tilde{\mathcal{C}}$. Hence, $\operatorname{Tan}\tilde D^0_{\bar\rho} = H^1(G_{\mathbb{Q},p},\mathrm{ad}^0\bar\rho)$ and we see that $\bar\rho$ is unobstructed in our sense if and only if $H^1(G_{\mathbb{Q},p},\mathrm{ad}^0\bar\rho)$ has dimension 2. By Tate's computation of the Euler's characteristic of global Galois representation, $\dim H^1(G_{\mathbb{Q},p},\mathrm{ad}^0\bar\rho)\geqslant 2$ with equality if and only if $H^2(G_{\mathbb{Q},p},\mathrm{ad}^0\bar\rho)=0$, which is equivalent to $H^2(G_{\mathbb{Q},p},\mathrm{ad}\bar\rho)=0$. The latter is exactly Mazur's definition of *unobstructed* (cf. [Maz89, § 1.6]), which is thus seen, in the irreducible case, to coincide with ours. For examples of irreducible $\bar\rho$ which are unobstructed, or obstructed, see [Bos91] and other works of the same author. By contrast, reducible representations are often, or, assuming the Vandiver conjecture, always unobstructed: when $p=2$, then $\bar\rho = 1\oplus 1$ is unobstructed by [Che14, Lemma 5.3] (see also [Bel12a, Proposition 1]), and when $p>2$ see Theorem 22. Note that in any case, the dimension of $\operatorname{Tan}\tilde D^0_{\bar\rho}$ is at least 2 (in the case $p>2$ and $\bar\rho$ reducible, this follows from Proposition 20 and Lemma 21).

It is easy to glue all of the pseudo-representations attached to the representations associated to modular eigenforms of level 1 and all weights, in order to prove the following proposition.

Proposition 2. *Fix a modular representation $\bar\rho : G_{\mathbb{Q},p}\to \mathrm{GL}_2(\mathbb{F})$. There exists a unique continuous pseudo-representation $(\tau,\delta): G_{\mathbb{Q},p}\to \mathbb{T}_{\bar\rho}$ such that $\tau(\mathrm{Frob}_\ell)=T_\ell$ for all $\ell\neq p$. It also satisfies $\delta(\mathrm{Frob}_\ell)=\ell S_\ell$ for all primes $\ell\neq p$, $\tau(c)=0$, and we have $\tau \pmod{m_{\mathbb{T}_{\bar\rho}}} = \operatorname{tr}\bar\rho$, $\delta \pmod{m_{\mathbb{T}_{\bar\rho}}} = \det\bar\rho$.*

Let $(\tilde\tau,\tilde\delta)$ be the pseudo-representation obtained by composing (τ,δ) with the natural morphisms $\mathbb{T}_{\bar\rho}\to A_{\bar\rho}$. Thus, $\tilde\tau \pmod{m_{A_{\bar\rho}}} = \operatorname{tr}\bar\rho$, $\tilde\delta \pmod{m_{A_{\bar\rho}}} = \det\bar\rho$. Moreover, the determinant $\tilde\delta: G_{\mathbb{Q},p}\to A^*_{\bar\rho}$ is constant (more precisely equal to ω_p^{k-1} where k is the weight of a modular form associated to $\bar\rho$).

We omit the proof of that proposition, which is simple and exactly similar as the case $p=2$ which can be found in [Bel12a, Step 1 of the proof of Theorem 1].

The pseudo-representation (τ,δ) (respectively $(\tilde\tau,\tilde\delta)$) of the proposition is an element of $D_{\bar\rho}(\mathbb{T}_{\bar\rho})$ (respectively of $\tilde D^0_{\bar\rho}(A_{\bar\rho})$) hence defines a morphism $R_{\bar\rho}\to\mathbb{T}_{\bar\rho}$ in the category $\mathcal{C}$ (respectively a morphism $\tilde R^0_{\bar\rho}\to A_{\bar\rho}$ in the category $\tilde{\mathcal{C}}$). These morphisms are surjective, because their images contain T_ℓ and ℓS_ℓ, as the image of the trace and determinant of Frob_ℓ for the universal pseudo-representation, respectively, for all $\ell\neq p$.

[4] When $p>2$, this condition is automatic and can be forgotten. Indeed, since $d(c) \pmod{m_S} = \det\bar\rho(c)=-1$, and $d(c)^2=1$ it follows from Hensel's lemma that $d(c)=-1$. Then part (d) applied to $g=c$, $h=1$, implies that $0=2t(c)$, hence $t(c)=0$.

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1.5 Statement of the main results

We shall prove the following three results concerning the Hecke algebra $A_{\bar\rho}$ and its relation with the deformation ring $\tilde{R}^0_{\bar\rho}$.

Theorem I. *Assume that $\bar\rho$ is unobstructed. Then the morphism $\tilde{R}^0_{\bar\rho} \to A_{\bar\rho}$ is an isomorphism, and $A_{\bar\rho}$ is isomorphic to a power series ring in two variables $\mathbb{F}[[x,y]]$.*

Theorem II. *Assume that $\bar\rho$ is absolutely irreducible after restriction to the Galois group of $\mathbb{Q}(\zeta_p)$. If $\bar\rho_{|G_{\mathbb{Q}_p}}$ is reducible, assume in addition that $\bar\rho_{|G_{\mathbb{Q}_p}}$ is not isomorphic to $\chi \otimes \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ nor to $\chi \otimes \left(\begin{smallmatrix} 1 & * \\ 0 & \omega_p \end{smallmatrix}\right)$, where χ is any character $G_{\mathbb{Q}_p} \to \mathbb{F}^*$ and $*$ may be trivial or not. Then $\tilde{R}^0_{\bar\rho} \to A_{\bar\rho}$ is an isomorphism and both rings have dimension 2.*

Theorem III. *In any case, $\tilde{R}^0_{\bar\rho}$ and $A_{\bar\rho}$ have dimension at least 2.*

When $p = 2$, Theorem I is proved (hence Theorem III as well, and Theorem II is empty) in [NS12a, NS12b] (for the part concerning the structure of A) and [Bel12a] (for the relation with $\tilde{R}^0_{\bar\rho}$). For $p = 3$, we sketch a proof in the appendix of this paper.

Let us give an idea of the proof of Theorems III and II in the case $p > 3$ (as Theorem I follows easily from Theorem III and the definition of *unobstructed*, see § 8). For Theorem III, we start with results of Gouvêa–Mazur obtained with the 'infinite fern' method (hence, using the deep results of Coleman on the existence of p-adic families of finite slope modular forms), which proves that $\dim \mathbb{T}_{\bar\rho} \geqslant 4$. We need to relate the characteristic 0 Hecke algebra $\mathbb{T}_{\bar\rho}$ with the characteristic p Hecke algebra $A_{\bar\rho}$. There are various obstacles to a direct comparison. First $\mathbb{T}_{\bar\rho}$ is obtained by the action of the Hecke operators on $S(\mathcal{O})$ while $A_{\bar\rho}$ is obtained by the action of the same operators on $S(\mathbb{F})$, and $S(\mathbb{F})$ is *not* equal to $S(\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F}$. To circumvent this problem, we work with the larger rings of divided congruences in the sense of Katz, $D(\mathcal{O})$ and $D(\mathbb{F})$, for which we do have, by construction, $D(\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} = D(\mathbb{F})$. We then need to control the changes introduced in our Hecke algebras by the change of modules of modular forms. By definition, the Hecke algebra that acts on $D(\mathcal{O})$ is the same as that, $\mathbb{T}$, constructed on $S(\mathcal{O})$. Moreover, a result of Katz allows us to compare $D(\mathbb{F})$ and $S(\mathbb{F})$, and from this the Hecke algebras on $D(\mathbb{F})$ with the Hecke algebra A constructed on $S(\mathbb{F})$. It therefore remains to compare the Hecke algebra $\mathbb{T}$ on $D(\mathcal{O})$ with the Hecke algebra on $D(\mathbb{F}) = D(\mathcal{O}) \otimes \mathbb{F}$. The main difficulty here is that the formation of Hecke algebras needs not commute with non-flat base changes. To solve the difficulty, we need to change again the Hecke algebras and replace them by their *full* counterpart, defined by the action of the T_ℓ, ℓS_ℓ for $\ell \neq p$ *and* U_p. As the full Hecke algebras are in duality with $D(\mathcal{O})$, those do commute with non-flat base change.

Then it remains to control how the addition of the U_p operator changes our Hecke algebras, both in characteristic p and 0, which we do by using (respectively generalizing) a result of Jochnowitz. At the end of these comparisons of many Hecke algebras, we conclude that $A_{\bar\rho} = \mathbb{T}_{\bar\rho}/(\mathfrak{p}, T)$ where T is in the maximal ideal of $\mathbb{T}$. It follows that $\dim A_{\bar\rho} \geqslant 2$, since as we have said $\dim \mathbb{T}_{\bar\rho} \geqslant 4$.

To prove Theorem II, we need to use, in addition to what have already been said, a result of Böckle, according to which $\mathbb{T}_{\bar\rho}$ is of dimension exactly 4 under the hypotheses of Theorem II. To conclude, we prove that $\mathbb{T}_{\bar\rho}$ is also flat over the Iwasawa algebra $\mathbb{Z}_p[[T]]$, which implies that $\dim \mathbb{T}_\rho/(p, T)$ is of dimension 2.

In §§ 2–9 we assume $p > 3$.

2. The divided congruences modules of Katz

This section and the next one intend to be a short exposition, without the proofs of the main results, of the theory of divided congruences of Katz, while introducing some objects and notation important for this article.

The *divided congruences module of cuspidal forms of weight at most k and level* 1, that we shall denote by $D_{\leqslant k}(\mathcal{O})$ is defined as the $\mathcal{O}$-sub-module of $S_{\leqslant k}(K) = \bigoplus_{i=0}^{k} S_i(K) \subset K[[q]]$ of forms whose q-expansion lies in $\mathcal{O}[[q]]$. Thus, $D_{\leqslant k}(\mathcal{O})$ is a free of finite rank $\mathcal{O}$-module, which contains $S_{\leqslant k}(\mathcal{O})$ as a co-torsion sub-module. We define the *divided congruence module of cuspidal forms of level* 1, denoted by $D(\mathcal{O})$, as $\bigcup_{k=0}^{\infty} D_{\leqslant k}(\mathcal{O})$ (the union being taken in $\mathcal{O}[[q]]$). The module $D(\mathcal{O})$ contains $S(\mathcal{O})$ as a co-torsion sub-module.

Remark 3. Let us recall that the name *divided congruences* is justified by the following elementary fact: let $f_i \in S_{k_i}(\mathcal{O})$ be a finite set of forms of distinct weights k_i. Assume that these forms satisfy a congruence

$$\sum_i f_i \equiv 0 \pmod{\mathfrak{p}^n} \tag{6}$$

for some integer n. Then $(\sum_i f_i)/\pi^n$ belongs to $D(\mathcal{O})$. And it is clear that every element of $D(\mathcal{O})$ is of this form.

Observe that the algebra $D(\mathcal{O})$ is not a *graded* sub-algebra of the graded (by the weight) algebra $S(K) = \sum_{i=0}^{\infty} S_i(K)$. In the example above, $(\sum_i f_i)/\pi^n$ belongs to $D(\mathcal{O})$ but the individual terms f_i/π^n in general do not.

A fundamental result on divided congruences is the following.

THEOREM 4 (Katz). *There exists a unique action of $\mathbb{Z}_p^*$ on $D(\mathcal{O})$ denoted by $(x, f) \mapsto x \cdot f$, such that for $x \in \mathbb{Z}_p^*$, and $f \in S_k(\mathcal{O}) \subset D(\mathcal{O})$, $x \cdot f = x^k f$.*

Note that the uniqueness is easy, as knowing the action of x on each $S_k(\mathcal{O})$ implies knowing it on $S(\mathcal{O})$, which is a co-torsion sub-module of the torsion-free module $D(\mathcal{O})$. Concretely, if the f_i are as in (6), and $f = (\sum_i f_i)/\pi^n \in D(\mathcal{O})$, then $x \cdot f$ has to be equal to $(\sum_i x^{k_i} f_i)/\pi^n$. But the existence of the action is much more difficult, because, as noted by Katz and Hida, it is not clear that the q-expansion of $(\sum_i x^{k_i} f_i)/\pi^n$ is in $\mathcal{O}[[q]]$. Instead the construction of this action by Katz uses the geometric interpretation of divided congruences and the Igusa tower. See [Kat75, Corollary 1.7] for the proof.

We define $D_{\leqslant k}(\mathbb{F})$ as the image of $D_{\leqslant k}(\mathcal{O})$ by the reduction map $f \mapsto \tilde{f}$, $\mathcal{O}[[q]] \to \mathbb{F}[[q]]$. This is similar to the definition of $S_{\leqslant k}(\mathbb{F})$ (which is a sub-module of $D_{\leqslant k}(\mathbb{F})$), but the following lemma provides an alternative definition of $D_{\leqslant k}(\mathbb{F})$ which does not hold for $S_{\leqslant k}(\mathbb{F})$.

LEMMA 5. *The natural map $D_{\leqslant k}(\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \to D_{\leqslant k}(\mathbb{F})$ is an isomorphism.*

Proof. This is because $\mathcal{O}[[q]]/D_{\leqslant k}(\mathcal{O})$ is without torsion, which follows from the definition of $D_{\leqslant k}(\mathcal{O})$. □

We define $D(\mathbb{F})$ as the union of the $D_{\leqslant k}(\mathbb{F})$ for $k = 1, 2, 3, \ldots$. By definition, $S(\mathbb{F}) \subset D(\mathbb{F})$. By the lemma, $\mathbb{Z}_p^*$ acts on $D(\mathbb{F})$. It is clear that this action preserves $S(\mathbb{F})$, since any $\tilde{f}$ in $S(\mathbb{F})$ is a sum $\sum_{i \in (\mathbb{Z}/p\mathbb{Z})^*} \tilde{f}_i$, where $\tilde{f}_i$ is the image of an $f_i \in \bigoplus_{k \equiv i \pmod{p-1}} S_k(\mathcal{O})$, and $x \in \mathbb{Z}_p^*$ acts on $\tilde{f}_i$ by $x \cdot \tilde{f}_i = \bar{x}^i \tilde{f}^i$, where $\bar{x}$ is the reduction mod x in $\mathbb{F}_p^*$. In particular, $1 + p\mathbb{Z}_p$ acts trivially on $S(\mathbb{F})$.

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Theorem 6 (Katz). *The space $S(\mathbb{F})$ is the space of invariants of $1+p\mathbb{Z}_p$ acting on $D(\mathbb{F})$.*

This is proved in [Kat75, § 4]; see also [Hid86, Theorem 1.1].

We observe that the two results of Katz recalled in this section are proved only there for $p>3$. We do not know whether they also hold for $p=2,3$.

3. Hecke operators on divided congruences

An important consequence of the existence of the action of $\mathbb{Z}_p^*$ on $D(\mathcal{O})$ (Theorem 4) is the possibility of defining Hecke operators on that module.

Corollary and Definition 7 (Hida, cf. [Hid86, p. 243]). *For ℓ a prime $\neq p$, and $f=\sum_{n\geqslant 0} a_n q^n \in D(\mathcal{O})$, define $S_\ell(f)=\ell^{-2}(\ell\cdot f)\in D(\mathcal{O})$, and define an element $T_\ell f = \sum_{n\geqslant 1} a_n(T_\ell f)q^n \in \mathcal{O}[[q]]$ with*

$$a_n(T_\ell f) = a_{n\ell}(f) \quad \text{if } \ell \nmid n, \tag{7}$$

$$a_n(T_\ell f) = a_{n\ell}(f) + \ell^{-1} a_{n/\ell}(\ell\cdot f) \quad \text{if } \ell \mid n. \tag{8}$$

Then $T_\ell f\in D(\mathcal{O})$. The operators T_ℓ and S_ℓ of $D(\mathcal{O})$ for every $\ell\neq p$ commute and act on the stable sub-module $S(\mathcal{O})$ as the usual T_ℓ and S_ℓ.

We also define operators T_n for $(n,p)=1$ by the formulas (1) and (2).

Proof. Let $f\in D(\mathcal{O})$. Since $\ell\cdot f\in D(\mathcal{O})$, and ℓ is invertible in $\mathcal{O}$, $S_\ell f\in D(\mathcal{O})$. Also T_ℓ coincides with the usual operator T_ℓ for any form in $S_k(\mathcal{O})$. Hence, if $f=(\sum_i f_i)/\pi^n$ with $f_i\in S_{k_i}(\mathcal{O})$, then $\pi^n T_\ell f=\sum_i T_\ell f_i$ by linearity hence $T_\ell f$ lies in $S_{\leqslant k}(K)$ for $k=\max(k_i)$. On the other hand, the coefficients of the q-expansion of $T_\ell f$ are in $\mathcal{O}$ by definition, so $T_\ell f\in D_{\leqslant k}(\mathcal{O})\subset D(\mathcal{O})$. The other assertions are clear. □

We define the operators T_ℓ and S_ℓ, and T_n for n coprime to p, on $D(\mathbb{F})$ by reducing the operators with the same name on $D(\mathcal{O})$ modulo $\mathfrak{p}$.

Lemma 8. *The $\mathcal{O}$-sub-algebra of $\operatorname{End}_{\mathcal{O}}(D_{\leqslant k}(\mathcal{O}))$ generated by the Hecke operators T_n for $p\nmid n$ is naturally isomorphic to $\mathbb{T}_k$ (defined in § 1.2).*

Proof. Denote temporarily by $\mathbb{T}'_k$ the sub-algebra of $\operatorname{End}_{\mathcal{O}}(D_{\leqslant k}(\mathcal{O}))$ generated by the Hecke operators T_n for $p\nmid n$. The restriction from $D_{\leqslant k}(\mathcal{O})$ to $S_{\leqslant k}(\mathcal{O})$ defines a morphism of $\mathcal{O}$-algebras $\mathbb{T}'_k\to\mathbb{T}_k$, which is surjective because its image contains all of the T_n for $p\nmid n$. Let $u\in\mathbb{T}'_k$ be an element of the kernel of that map. Then, by definition, u acts trivially on $S_{\leqslant k}(\mathcal{O})$. Therefore, u factors as a map $D_{\leqslant k}(\mathcal{O})/S_{\leqslant k}(\mathcal{O})\to D_{\leqslant k}(\mathcal{O})$. Since the source of this map is torsion while the target is torsion free, $u=0$, and the map $\mathbb{T}'_k\to\mathbb{T}_k$ is an isomorphism. □

In particular, the algebra $\mathbb{T}$ (see § 1.2) acts faithfully on $D(\mathcal{O})$, and we can see it as an $\mathcal{O}$-sub-algebra of $\operatorname{End}_{\mathcal{O}}(D(\mathcal{O}))$.

Lemma 9. *The homomorphism $\phi:\mathbb{Z}_p^*\to\operatorname{End}_{\mathcal{O}}(D(\mathcal{O}))$, defined by $\phi(x)f=x\cdot f$, $f\in D(\mathcal{O})$, takes values in the sub-algebra $\mathbb{T}$.*

Proof. Let us provide $D(\mathcal{O})$ with the sup norm $|\sum a_n q^n|=\sup_n|a_n|$, and $\operatorname{End}_{\mathcal{O}}(D(\mathcal{O}))$ with the weak topology, so that a sequence of operators $u_m\in\operatorname{End}_{\mathcal{O}}(D(\mathcal{O}))$ converges to u if and only if for every $f\in D(\mathcal{O})$, $|u_m f-uf|$ converges to 0. We claim that $\mathbb{T}$ is closed in $\operatorname{End}_{\mathcal{O}}(D(\mathcal{O}))$. Indeed, if $u_m\in\mathbb{T}$, and (u_m) converges to $u\in\operatorname{End}_{\mathcal{O}}(D(\mathcal{O}))$, then for every $k\geqslant 0$, the restriction of u to $D_{\leqslant k}(\mathcal{O})$ is the limit of the restriction of u_n to $D_{\leqslant k}(\mathcal{O})$, hence is in $\mathbb{T}_k$ since $\mathbb{T}_k$ is closed

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(even compact) in the finite-type $\mathcal{O}$-module $\mathrm{End}_{\mathcal{O}}(D_{\leqslant k}(\mathcal{O}))$. This shows that u is in $\varprojlim \mathbb{T}_k = \mathbb{T}$, hence the claim.

For the weak topology, the map $\phi : \mathbb{Z}_p^* \to \mathrm{End}_{\mathcal{O}}(D(\mathcal{O}))$ is continuous, since for any given $f \in D(\mathcal{O})$, we can write $f = (\sum_i f_i)/\pi^n \in D(\mathcal{O})$ with $f_i \in S_i(\mathcal{O})$, and $x \cdot f = (\sum_i x^i f_i)/\pi^n$ which makes clear that if x_m converges to x in $\mathbb{Z}_p^*$, then $|x_m f - xf|$ converges to 0.

For $\ell \neq p$ a prime number, one has $\phi(\ell) = \ell^2 S_\ell \in \mathbb{T}$. If $x \in \mathbb{Z}_p^*$, there exists by Dirichlet's theorem on primes in arithmetic progressions a sequence of primes ℓ_m (different from p) converging to x p-adically. Therefore, $\phi(\ell_n)$ converges to $\phi(x)$ in $\mathrm{End}_{\mathcal{O}}(D(\mathcal{O}))$. Hence, $\phi(x) \in \mathbb{T}$. □

Remark 10. The proof of the lemma shows that the two natural topologies one can consider on $\mathbb{T}$ are the same: the topology of the projective limit $\varprojlim \mathbb{T}_k$, each $\mathbb{T}_k$ having its natural topology of finite rank $\mathcal{O}$-module; and the topology obtained by restriction of the weak topology on $\mathrm{End}_{\mathcal{O}}(D(\mathcal{O}))$. Hence, an equivalent definition of $\mathbb{T}$ would be as *the closed subalgebra of* $\mathrm{End}_{\mathcal{O}}(D(\mathcal{O}))$ *(for its weak topology) generated by the* T_ℓ *and* S_ℓ.

Let us define the Iwasawa algebra $\Lambda = \mathcal{O}[[1+p\mathbb{Z}_p]]$. By choosing a topological generator (say $1+p$) of $1+p\mathbb{Z}_p$ one determines an isomorphism $\Lambda \simeq \mathcal{O}[[T]]$ under which the maximal ideal m_Λ of Λ becomes (π, T). The group homomorphism $\phi : 1+p\mathbb{Z}_p \to \mathbb{T}^*$ defines a morphism of $\mathcal{O}$-algebras $\psi : \Lambda \to \mathbb{T}$. Using that morphism, one regards $\mathbb{T}$ as a Λ-algebra.

4. Divided congruences of level $\Gamma_0(p)$

We shall also need a variant with level $\Gamma_0(p)$: $S_k(\Gamma_0(p), K)$ denotes the space of cusp forms of weight k and level $\Gamma_0(p)$ with coefficients in K. The *divided congruence module of cuspidal forms of weight at most* k *and level* $\Gamma_0(p)$, that we shall denote by $D_{\leqslant k}(\Gamma_0(p), \mathcal{O})$ is defined as the $\mathcal{O}$-sub-module of $S_{\leqslant k}(\Gamma_0(p), K) = \bigoplus_{i=0}^k S_i(\Gamma_0(p), K)$ of forms whose q-expansion lies in $\mathcal{O}[[q]]$. Similarly, the *divided congruence module of cuspidal forms of level* $\Gamma_0(p)$ is defined as $D(\Gamma_0(p), \mathcal{O}) = \bigcup_{k=0}^\infty D_{\leqslant k}(\Gamma_0(p), \mathcal{O})$. The theorems and corollary above also hold (with the same references) for $D(\Gamma_0(p), \mathcal{O})$ instead of $D(\mathcal{O})$.

Proposition 11. *The closures of $D(\mathcal{O})$ and $D(\Gamma_0(p), \mathcal{O})$ in $\mathcal{O}[[q]]$ (provided with the topology of uniform convergence) are equal.*

Proof. Their common closure is the space of p-adic modular functions of tame level 1, denoted $V_{\mathrm{par}}(\mathcal{O}, 1)$ in [Gou88, see Proposition I.3.9]. □

Corollary 12. *There is an isomorphism preserving q-expansions $D(\Gamma_0(p), \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \simeq D(\mathbb{F})$.*

We call $\mathbb{T}_k(\Gamma_0(p))$ the sub-algebra of $\mathrm{End}_{\mathcal{O}}(\Gamma_0(p), D_{\leqslant k}(\mathcal{O}))$ generated by the Hecke operators T_n for $p \nmid n$, and we define $\mathbb{T}(\Gamma_0(p))$ as $\varprojlim_k \mathbb{T}_k(\Gamma_0(p))$.

Corollary 13. *The algebras $\mathbb{T}(\Gamma_0(p))$ and $\mathbb{T}$ are naturally isomorphic.*

Proof. The natural morphism $r : \mathbb{T}(\Gamma_0(p)) \to \mathbb{T}$ is obtained by the projective limit of the surjective restriction maps $\mathbb{T}_k(\Gamma_0(p)) \to \mathbb{T}_k$. Hence, r is surjective. The algebra $\mathbb{T}(\Gamma_0(p))$ acts faithfully on $D(\Gamma_0(p), \mathcal{O})$. By continuity of the Hecke operators (for the topology of uniform convergence on q-expansions), it also acts on its closure in $\mathcal{O}[[q]]$, and its action is of course still faithful. The action of $\mathbb{T}(\Gamma_0(p))$ on $D(\mathcal{O})$ is also faithful, for if it was not, some operator $0 \neq u \in \mathbb{T}(\Gamma_0(p))$ would act by 0 on $D(\mathcal{O})$, hence by 0 on its closure by continuity, hence by 0 on the closure of $D(\Gamma_0(p), \mathcal{O})$ by the proposition above, contradicting what we just saw. Since that action factors through r, the map r must be injective. □

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5. Full Hecke algebras

Let us recall that the operator U_p is defined on q-expansions by $U_p(\sum a_n q^n) = \sum a_{pn} q^n$. This operator leaves stable the subspace of modular forms $S_{\leqslant k}(\Gamma_0(p), K)$ of $K[[q]]$, and since it also preserves the integrality of coefficients, it leaves stable $D_{\leqslant k}(\Gamma_0(p), \mathcal{O})$ and $D(\Gamma_0(p), \mathcal{O})$. By reduction, the subspace $D_{\leqslant k}(\Gamma_0(p), \mathbb{F})$ of $\mathbb{F}[[q]]$ is also stable by U_p, and so is the subspace $D(\Gamma_0(p), \mathbb{F}) = D(\mathbb{F})$ (cf. Corollary 12) of $\mathbb{F}[[q]]$. Note that the subspaces $S_{\leqslant k}(\mathbb{F}) \subset D_{\leqslant k}(\mathbb{F})$ of $\mathbb{F}[[q]]$ are also stable by U_p, since they are stable by T_p and in characteristic p, the operators T_p and U_p coincide.

We define the *full Hecke algebra* $\mathbb{T}_k^{\text{full}}$ as the subalgebra of $\text{End}_{\mathcal{O}}(D_{\leqslant k}(\Gamma_0(p), \mathcal{O}))$ generated by the Hecke operators T_ℓ and S_ℓ for ℓ prime different form p, *and by* U_p. We let $\mathbb{T}^{\text{full}} = \varprojlim \mathbb{T}_k^{\text{full}}$. We define the following full Hecke algebras in characteristic p: DA_k^{full} (respectively A_k^{full}) is the sub-algebra of $\text{End}_{\mathbb{F}}(D_{\leqslant k}(\mathbb{F}))$ (respectively of $\text{End}_{\mathbb{F}}(S_{\leqslant k}(\mathbb{F}))$) generated by the Hecke operators T_ℓ and S_ℓ for ℓ prime different form p, *and by* U_p. We let $DA^{\text{full}} = \varprojlim DA_k^{\text{full}}$ and $A^{\text{full}} = \varprojlim A_k^{\text{full}}$.

We have natural surjective morphisms of algebras (sending Hecke operators to Hecke operators with the same name) $\mathbb{T}_k^{\text{full}} \otimes_{\mathcal{O}} \mathbb{F} \to DA_k^{\text{full}}$ (induced by the reduction map $D_{\leqslant k}(\Gamma_0(p), \mathcal{O}) \to D_{\leqslant k}(\Gamma_0(p), \mathbb{F})$) and $DA_k^{\text{full}} \to A_k^{\text{full}}$ (induced by the inclusion $S_{\leqslant k}(\mathbb{F}) \subset D_{\leqslant k}(\mathbb{F}) \subset D_{\leqslant k}(\Gamma_0(p), \mathbb{F})$), hence by passage to the limit, surjective maps $\mathbb{T}^{\text{full}} \to DA^{\text{full}} \to A^{\text{full}}$.

Proposition 14. *The pairings* $\mathbb{T}_k^{\text{full}} \times D_{\leqslant k}(\mathcal{O}) \to \mathcal{O}$, $DA_k^{\text{full}} \times D_{\leqslant k}(\mathbb{F}) \to \mathbb{F}$ *and* $A_k^{\text{full}} \times S_{\leqslant k}(\mathbb{F})$ *given by* $(t, f) \mapsto a_1(tf)$ *are perfect.*

Proof. This is elementary and well known. □

Corollary 15. *The map* $\mathbb{T}_k^{\text{full}} \to DA_k^{\text{full}}$ *induces an isomorphism* $\mathbb{T}_k^{\text{full}} \otimes_{\mathcal{O}} \mathbb{F} \to DA_k^{\text{full}}$, *hence an isomorphism* $\mathbb{T}^{\text{full}} \otimes_{\mathcal{O}} \mathbb{F} \simeq DA^{\text{full}}$.

Proof. It is clear that the map $\mathbb{T}_k^{\text{full}} \otimes_{\mathcal{O}} \mathbb{F} \to DA_k^{\text{full}}$ is surjective. The preceding proposition assures that its source has dimension the rank of $D_{\leqslant k}(\mathcal{O})$ and that its image has dimension $\dim D_{\leqslant k}(\mathbb{F})$. These two numbers are equal since $D_k(\mathcal{O})$ is torsion-free. This proves the corollary. □

The composition $\Lambda \to \mathbb{T} \to \mathbb{T}^{\text{full}}$ defines a structure of Λ-algebra on $\mathbb{T}^{\text{full}}$. Those results combined with the main results of Katz yield the following result.

Proposition 16. *One has* $\mathbb{T}^{\text{full}} / m_\Lambda \mathbb{T}^{\text{full}} \simeq A^{\text{full}}$.

Proof. Set $\tilde{\Lambda} = \Lambda / \pi\Lambda$ and $m_{\tilde{\Lambda}}$ its maximal ideal (which is principal). By the above corollary, $\mathbb{T}^{\text{full}} / \pi \mathbb{T}^{\text{full}} \simeq DA^{\text{full}}$ as $\tilde{\Lambda}$-module. By Theorem 6, $S(\mathbb{F}) = D(\mathbb{F})[m_{\tilde{\Lambda}}]$. By Proposition 14, this implies $A^{\text{full}} = DA^{\text{full}} / m_{\tilde{\Lambda}} DA^{\text{full}} = \mathbb{T}^{\text{full}} / m_\Lambda \mathbb{T}^{\text{full}}$. □

6. Local components of normal and full Hecke algebras

The Hecke algebras $\mathbb{T}^{\text{full}}$ and A^{full} are semi-local, and for both of them, their local components are in bijection with the set of $\mathbb{F}$-valued systems of eigenvalues of all of the Hecke operators T_n that appear in $S(\mathbb{F})$, or what amounts to the same, the pairs $(\bar{\rho}, \lambda)$, where $\bar{\rho} : G_{\mathbb{Q},p} \to \text{GL}_2(\mathbb{F})$ is a modular representation attached to some eigenform $f \in S(\mathbb{F})$, and λ is the eigenvalue of U_p on f. We shall denote by $\mathbb{T}_{\bar{\rho},\lambda}^{\text{full}}$ and $A_{\bar{\rho},\lambda}^{\text{full}}$ the corresponding local algebras.

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Proposition 17. *One has a natural isomorphism of $A_{\bar\rho}$-algebras $A_{\bar\rho}[[U_p]] \simeq A^{\mathrm{full}}_{\bar\rho,0}$, and a natural isomorphism of $\mathbb{T}_{\bar\rho}$-algebras $\mathbb{T}_{\rho}[[U_p]] \simeq \mathbb{T}^{\mathrm{full}}_{\bar\rho,0}$.*

Proof. The first isomorphism is due to Jochnowitz, see [Joc82]. For the second we mimic her proof with some adaptations.

We have a natural surjective map $\mathbb{T}_k[U_p] \to \mathbb{T}^{\mathrm{full}}_k$. Since U_p is topologically nilpotent in $\mathbb{T}^{\mathrm{full}}_{\bar\rho,0}$, this map induces by passage to the limit and localization a surjective map $\mathbb{T}_{\bar\rho}[[U_p]] \to \mathbb{T}^{\mathrm{full}}_{\bar\rho,0}$. By Corollary 13, the algebra $\mathbb{T}$ (respectively $\mathbb{T}^{\mathrm{full}}$) acts faithfully on $D(\Gamma_0(p),\mathcal{O})$, and the quotient $\mathbb{T}_{\bar\rho}$ of $\mathbb{T}$ (respectively $\mathbb{T}^{\mathrm{full}}_{\bar\rho,0}$ of $\mathbb{T}^{\mathrm{full}}$) is the largest quotient that acts faithfully on $D(\Gamma_0(p),\mathcal{O})_{\bar\rho}$ (respectively $D(\Gamma_0(p),\mathcal{O})_{\bar\rho,0}$) where $D(\Gamma_0(p),\mathcal{O})_{\bar\rho}$ (respectively $D(\Gamma_0(p),\mathcal{O})_{\bar\rho,0}$) is the (direct) sum of the generalized eigenspaces for all of the T_ℓ and S_ℓ for $\ell \neq p$ (respectively and for U_p) with system of eigenvalues in $\mathcal{O}$ that lifts the $\mathbb{F}$-valued system attached to $\bar\rho$ (respectively to $(\bar\rho,0)$). Therefore, to prove that $\mathbb{T}_{\bar\rho}[[U_p]] \to \mathbb{T}^{\mathrm{full}}_{\bar\rho,0}$ is an isomorphism it is sufficient to prove that $\mathbb{T}_{\bar\rho}[[U_p]]$ acts faithfully on $D(\Gamma_0(p),\mathcal{O})_{\bar\rho,0}$.

Recall that on $D(\Gamma_0(p),\mathcal{O})_{\bar\rho,0}$ we have an operator V which acts on q-expansions as $\sum a_n q^n \mapsto \sum a_n q^{pn}$ (cf. e.g. [Gou88, § II.2]). One sees immediately on q-expansions that $U_p V = \mathrm{Id}$ and that VU_p is a projector. Thus, V is injective on $D(\Gamma_0(p),\mathcal{O})_{\bar\rho,0}$, U_p is surjective, and VU_p is the projector of kernel $\operatorname{Ker} U_p$ and image $\operatorname{Im} V$.

We claim that for every $t \in \mathbb{T}_{\bar\rho}$, $t \neq 0$, there exists $g \in D(\Gamma_0(p),\mathcal{O})_{\bar\rho,0}$ such that $U_p g = 0$, but $tg \neq 0$. Indeed there exists an f in $D(\Gamma_0(p),\mathcal{O})_{\bar\rho}$, such that $tf \neq 0$. Let i be the smallest integer such that tf has a coefficient $a_n \neq 0$ with $p^i \mid\mid n$. Then $U_p^i tf$ has a coefficient $a_n \neq 0$ with n relatively prime to p. Therefore $U_p^i tf$ is not in the image of V, hence is not in the image of the projector VU_p, hence is not in the kernel of the projector $1 - VU_p$. That is, $(1-VU_p)U_p^i tf \neq 0$. Define $g = (1-VU_p)U_p^i f$. Then clearly $U_p g = 0$ (so $g \in D(\Gamma_0(p),\mathcal{O})_{\bar\rho,0}$) and because t commutes with U_p and V, $tg \neq 0$, and the claim is proved.

Now let us prove that $\mathbb{T}_{\bar\rho}[[U_p]]$ acts faithfully on $D(\Gamma_0(p),\mathcal{O})_{\bar\rho,0}$. Let $\sum_{j=n}^\infty t_j U_p^j \in \mathbb{T}_{\bar\rho}[[U_p]]$ with $t_j \in \mathbb{T}_{\bar\rho}$ and $t_n \neq 0$. Then by the claim, there is $g \in D(\Gamma_0(p),\mathcal{O})_{\bar\rho,0}$ such that $t_n g \neq 0$ but $U_p g = 0$. Let $h = V^n g$, so that $U_p^n h = g$. Then $t_n(U_p^n h) = t_n g \neq 0$, but $U_p^{n+1} h = U_p g = 0$, and so $U_p^i h = 0$ for all $i > n$, and $(\sum_{j=n}^\infty t_j U_p^j)h \neq 0$. On the other hand, since $U_p^{n+1}h = 0$, h is in the generalized eigenspace of U_p with eigenvalue 0, hence $h \in D(\Gamma_0(p),\mathcal{O})_{\bar\rho,0}$, which proves the faithfulness. □

7. Proof of Theorem III

We need to prove that $\dim A_{\bar\rho} \geqslant 2$. It is equivalent to prove $\dim A_{\bar\rho}[[U_p]] \geqslant 3$, that is by Proposition 17, $\dim A^{\mathrm{full}}_{\bar\rho,0} \geqslant 3$. By Proposition 16, $A^{\mathrm{full}}_{\bar\rho,0}$ is isomorphic to $\mathbb{T}^{\mathrm{full}}_{\bar\rho,0}/m_\Lambda \mathbb{T}^{\mathrm{full}}_{\bar\rho,0}$. Since the ideal m_Λ is generated by two elements, one has by the *hauptidealsatz* that $\dim A^{\mathrm{full}}_{\bar\rho,0} \geqslant \dim \mathbb{T}^{\mathrm{full}}_{\bar\rho,0} - 2$ so it suffices to prove that $\dim \mathbb{T}^{\mathrm{full}}_{\bar\rho,0} \geqslant 5$. But by Proposition 17, that is $\dim \mathbb{T}_{\bar\rho}[[U_p]] = \dim \mathbb{T}_{\bar\rho} + 1$. It therefore suffices to prove that $\dim \mathbb{T}_{\bar\rho} \geqslant 4$. But that is precisely the result given by Gouvêa–Mazur's infinite fern argument, cf. [GM98, Eme11].

8. Proof of Theorem I

Assuming that $\bar\rho$ is unobstructed, we need to prove that the surjective map $\tilde{R}^0_{\bar\rho} \to A_{\bar\rho}$ is an isomorphism of local regular rings of dimension 2. But, by assumption, the cotangent space of

$\tilde{R}^0_{\bar\rho}$ has dimension 2, while the Krull dimension of $A_{\bar\rho}$ is at least 2 by Theorem III. The result follows.

9. Proof of Theorem II

THEOREM 18 (Böckle, Diamond–Flach–Guo, Gouvêa–Mazur, Kisin). *Under the hypotheses of Theorem II, the natural map $R_{\bar\rho} \to \mathbb{T}_{\bar\rho}$ is an isomorphism between local rings of dimension 4.*

Proof (Compare [Eme11]). Under more restrictive assumptions than Theorem II, namely under the assumption that $\bar\rho_{|G_{\mathbb{Q}_p}}$, if irreducible, is flat up to torsion by a character (cf. [Boc01, Assumption (2.1)]), the fact that $R_{\bar\rho} \to \mathbb{T}_{\bar\rho}$ is an isomorphism is proved in [Boc01, Theorems 3.1 and 3.9]. In [Boc01], this assertion is used only to ensure the validity of Theorem 2.8 there, due to Diamond [Dia96, Theorem 1.1]. However, Diamond, with Flach and Guo, later generalized [Boc01, Theorem 2.8], proving it under the hypotheses (ii) of our Theorem II: cf. [DFG04, Theorem 3.6]. Therefore, Böckle's results [Boc01, Theorems 3.1 and 3.9] are true, with the same proof, under the hypotheses of Theorem II.

Once we know that $R_{\bar\rho} \to \mathbb{T}_{\bar\rho}$ is an isomorphism, we conclude by recalling that $\mathbb{T}_{\bar\rho}$ has dimension at least 4 by the infinite fern argument of Gouvêa–Mazur (cf. [GM98]), and that $R_{\bar\rho}$ has dimension at most 4 by a theorem of Kisin (under weaker assumptions than ours), cf. [Kis04, Main Theorem]. □

Recall the Iwasawa algebra Λ and the morphism of algebras $\psi : \Lambda \to \mathbb{T}_{\bar\rho}$ introduced at the end of § 3.

LEMMA 19. *The algebra $\mathbb{T}_{\bar\rho}$ is flat over Λ.*

Proof. We first observe that $\mathbb{T}_{\bar\rho}$ is flat over $\mathcal{O}$, because it is torsion-free as a sub-module of $\mathbb{T}$, which is itself torsion-free as the projective limit of the $\mathbb{T}_k$, which are sub-modules of the torsion-free modules $\mathrm{End}_{\mathcal{O}}(S_{\leqslant k}(\mathcal{O}))$. Hence, $R_{\bar\rho}$ is flat over $\mathcal{O}$.

Second, let $\chi : G_{\mathbb{Q},p} \to \mathcal{O}^*$ be the Teichmüller lift of the character $\det\bar\rho$. Let us call $D^0_{\bar\rho}$ the functor which parametrizes the deformations of $\bar\rho$ with constant determinant χ. Let $R^0_{\bar\rho}$ be the ring representing $D^0_{\bar\rho}$. Let us call $D_{\det\bar\rho}$ the deformation functor of the character $\det\bar\rho$. This functor is representable by the Iwasawa algebra Λ (cf. [Maz89, § 1.4]).

Consider the morphism of functors $D_{\bar\rho} \to D_{\det\bar\rho} \times D^0_{\bar\rho}$, which to a deformation $\rho : G_{\mathbb{Q},p} \to \mathrm{GL}_2(S)$, attaches the pair $(\det\rho, \rho\otimes((\det\rho)^{-1}\chi)^{1/2})$. Here, note that $(\det\rho)^{-1}\chi \equiv 1 \pmod{m_S}$ by definition of χ, hence since $p > 2$ the character $(\det\rho)^{-1}\chi$ has a unique square root by Hensel's lemma. One checks easily that this morphism of functor is an isomorphism. Hence, we get a natural isomorphism

$$\Lambda \otimes_{\mathcal{O}} R^0_{\bar\rho} \to R_{\bar\rho}.$$

From that isomorphism follows the fact that $R^0_{\bar\rho}$ is flat over $\mathcal{O}$ (for if it had torsion, so would have $R^0_{\bar\rho} \otimes_{\mathcal{O}} \Lambda$ since Λ is flat over $\mathcal{O}$, contradicting the fact that $R_{\bar\rho}$ has no $\mathcal{O}$-torsion). Therefore, $R_{\bar\rho} = R^0_{\bar\rho} \otimes_{\mathcal{O}} \Lambda$ is flat over Λ by universality of flatness.

To conclude, we observe that the natural diagram

$$\begin{array}{ccc} R_{\bar\rho} & \longrightarrow & \mathbb{T}_{\bar\rho} \\ & \nwarrow \quad \nearrow_{\psi} & \\ & \Lambda & \end{array} \tag{9}$$

is commutative. Indeed, the determinant $\delta : G_{\mathbb{Q},p} \to \mathbb{T}_{\bar\rho}^*$ (see Proposition 2) is a deformation of $\det\bar\rho$, hence defines a morphism $\psi' : \Lambda \to \mathbb{T}_{\bar\rho}$ since Λ is the universal deformation ring of $\det\bar\rho$. Moreover, it is clear from the definition of the map $R_{\bar\rho} \to \mathbb{T}_{\bar\rho}$ that the diagram above commutes if the map $\Lambda \to \mathbb{T}_{\bar\rho}$ is replaced by ψ'. Hence, we are reduced to showing that ψ' is the same map as the map ψ defined in § 3, and to do so, it is enough to do so after composition by any map $\lambda_f : \mathbb{T}_\rho \to \bar K$ attached to a modular eigenform $f \in S_k(\Gamma)$ for some integer k. But $(\lambda_f \circ \psi)(x) = x^{k-1}$ for $x \in \mathbb{Z}_p^*$ by definition of ψ and since f is of weight k, while $(\lambda_f \circ \psi')(\ell) = \det \rho_f(\mathrm{Frob}_\ell) = \ell^{k-1}$ for any prime $\ell \neq p$. By continuity, it follows that $\psi = \psi'$

Since the upper horizontal map of (9) is an isomorphism, it follows that $\mathbb{T}_{\bar\rho}$ is flat over Λ. □

It follows from the lemma that $\mathbb{T}^{\mathrm{full}}_{\bar\rho,0} = \mathbb{T}_{\bar\rho}[[U_p]]$ is also flat over Λ. Thus, the dimension of $\mathbb{T}^{\mathrm{full}}_{\bar\rho,0}/m_\Lambda\mathbb{T}^{\mathrm{full}}_{\bar\rho,0}$ is equal by [Eis95, Theorem 10.10] to $\dim \mathbb{T}^{\mathrm{full}}_{\bar\rho,0} - 2 = 5 - 2 = 3$, hence $A^{\mathrm{full}}_{\bar\rho,0} = A_{\bar\rho}[[U_p]]$ has dimension 3 and $A_{\bar\rho}$ has dimension 2. This concludes the proof.

10. When are the reducible modular representations unobstructed?

In this section, we discuss the condition, for a modular representation $\bar\rho$, of being *unobstructed* (Definition 1). As noted just after the definition, when $\bar\rho$ is absolutely irreducible, this notion coincides with Mazur's notion of being *unobstructed*, and has thus been discussed extensively in the literature. This is why we restrict ourselves in this section to a $\bar\rho$ which is *reducible*. We shall see that in this case, many (conjecturally all) representations are unobstructed.

In the case $p = 2$, we have already noted that the only modular representation $\bar\rho = 1 \oplus 1$ was unobstructed. Let us assume that $p > 2$ for the rest of this section.

Since our modular representation $\bar\rho$ is reducible, and by definition semi-simple, it is the direct sum of two characters $G_{\mathbb{Q},p} \to \mathbb{F}^*$. By class field theory, any character $G_{\mathbb{Q},p} \to \mathbb{F}^*$ is of the form ω_p^a where $\omega_p : G_{\mathbb{Q},p} \to \mathbb{F}_p^*$ is the cyclotomic character modulo p and a is an integer in $\{0, 1, \dots, p-2\}$. Hence, $\bar\rho$ is, up to a twist, of the form $1 \oplus \omega_p^a$, with a odd since $\bar\rho$ is, and one has $\mathrm{tr}\,\bar\rho = 1 + \omega_p^a$, $\det\bar\rho = \omega_p^a$.

The functor $\tilde D^0_{\bar\rho}$ is the functor of deformations of $(\bar t, \bar d)$ as pseudo-representations (t, d) in the sense of Chenevier, with the condition d constant, that is $d = \det\bar\rho = \omega_p^a$. (Since $p > 2$, it is by [Che14] the same functor as the functor which attaches to S a pseudo-character $t : G_{Q,p} \to S$ of dimension 2, deforming $\bar t$ and satisfying the condition $d(g) := (t^2(g) - t(g^2))/2 = 1$ for all $g \in G_{\mathbb{Q},p}$.) Let $\mathrm{Tan}\,\tilde D^0_{\bar\rho}$ be the tangent space of that functor.

Proposition 20. *The dimension of* $\mathrm{Tan}\,\tilde D^0_{\bar\rho}$ *is* $1 + \dim H^1(G_{\mathbb{Q},p}, \omega_p^a) \dim H^1(G_{\mathbb{Q},p}, \omega_p^{-a})$

Proof. By the main theorem of [Bel12b], that tangent space of $\tilde D_{\bar\rho}$ lies in an exact sequence:

$$0 \to \mathrm{Tan}\,(\tilde D_{\omega_p^a} \oplus \tilde D_1) \xrightarrow{\iota} \mathrm{Tan}\,(\tilde D_{\bar\rho}) \to H^1(G_{\mathbb{Q},p}, \omega_p^a) \otimes H^1(G_{\mathbb{Q},p}, \omega_p^{-a}) \to H^2(G_{\mathbb{Q},p}, 1)^2.$$

Here, $\tilde D_{\omega_p^a}$ (respectively $\tilde D_1$) is the deformation functor of ω_p^a (respectively 1) as a character of $G_{\mathbb{Q},p}$, and ι is the map which to a pair of deformations (χ, χ') of $(\omega_p^a, 1)$ on $\mathbb{F}_p[\epsilon]/(\epsilon^2)$ attaches the deformation $\chi + \chi'$ of $\omega_p^a + 1 = \mathrm{tr}\,\bar\rho$. It is well known (e.g. as a consequence of Tate's global Euler–Poincaré's formula) that $H^2(G_{\mathbb{Q},p}, 1) = 0$. Hence, an exact sequence

$$0 \to \mathrm{Tan}\,(\tilde D_{\omega_p^a} \oplus \tilde D_1) \to \mathrm{Tan}\,(\tilde D_{\bar\rho}) \to H^1(G_{\mathbb{Q},p}, \omega^a) \otimes H^1(G_{\mathbb{Q},p}, \omega^{-a}) \to 0.$$

Recall that the functor $\tilde D^0_{\bar\rho}$ is the sub-functor of $\tilde D_{\bar\rho}$ of deformations which have constant determinant. We claim that the restriction to $\mathrm{Tan}\,(\tilde D^0_{\bar\rho})$ of the surjective map $\mathrm{Tan}\,(\tilde D_{\bar\rho}) \to$

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$H^1(G_{\mathbb{Q},p},\omega_p^a)\otimes H^1(G_{\mathbb{Q},p},\omega_p^{-a})$ is also surjective. Indeed, if an element x of $H^1(G_{\mathbb{Q},p},\omega^a)\otimes H^1(G_{\mathbb{Q},p},\omega_p^{-a})$ is the image of a deformation ρ of $\bar\rho$ to $\mathbb{F}[\epsilon]/(\epsilon^2)$, then $\det\rho$ is a deformation of $\det\bar\rho=\omega_p^a$, hence an element of $\mathrm{Tan}\,(\tilde D_{\omega_p^a})\subset\mathrm{Tan}\,(\tilde D_{\omega_p^a}\oplus\tilde D_1)$, and removing to $\bar\rho$ the image by ι of that element gives an element of $\mathrm{Tan}\,(\tilde D^0_{\bar\rho})$ whose image in $H^1(G_{\mathbb{Q},p},\omega_p^a)\otimes H^1(G_{\mathbb{Q},p},\omega_p^{-a})$ is still x.

We therefore have an exact sequence:

$$0\to\mathrm{Tan}\,((\tilde D_{\omega_p^a}\oplus\tilde D_1)^0)\xrightarrow{\iota}\mathrm{Tan}\,(\tilde D^0_{\bar\rho})\to H^1(G_{\mathbb{Q},p},\omega_p^a)\otimes H^1(G_{\mathbb{Q},p},\omega_p^{-a})\to 0$$

where $(\tilde D_{\omega_p^a}\oplus\tilde D_1)^0$ is the sub-functor of the functor of $\tilde D_{\omega_p^a}\oplus\tilde D_1$ parameterizing deformations (χ,χ') of $(\omega_p^a,1)$ such that $\chi\chi'$ is constant. The dimension of $\mathrm{Tan}\,((\tilde D_{\omega_p^a}\oplus\tilde D_1)^0)$ is easily computed to be 1, and the proposition follows. □

We shall use the following rather standard notation: for χ a character of $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$, we denote by $A(\chi)$ the part of the p-torsion subgroup of the class group $\mathrm{Cl}(\mathbb{Q}(\mu_p))$ on which $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ acts by χ.

LEMMA 21. *For every odd a, one has* $\dim H^1(G_{\mathbb{Q},p},\omega_p^a)=1+\dim A(\omega_p^{p-a})$ *and* $\dim H^1(G_{\mathbb{Q},p},\omega_p^{-a})=1+\dim A(\omega_p^{a+1})$.

Proof. The second equality is the same as the first since $\omega_p^{-a}=\omega_p^{p-1-a}$. We shall therefore only prove the first equality.

When $a=1$, a class in $H^1(G_{\mathbb{Q},p},\omega_p)$ is represented by a cocycle of the form $g\mapsto c_\alpha(g):=g(\alpha)/\alpha$, with $\alpha\in\bar{\mathbb{Q}}$, $\alpha^p\in\mathbb{Q}$ and $v_\ell(\alpha^p)=0$ for all prime $\ell\neq p$, and the cocycle c_α is a coboundary if and only if $\alpha\in\mathbb{Q}$ (cf. [Was97] for this simple application of the Kümmer exact sequence). Therefore, the dimension of $H^1(G_{\mathbb{Q},p},\omega_p)$ is 1 and this space is generated by the cocycle c_α for $\alpha=p$. On the other hand, $A(\omega_p^{p-1})=A(1)$ is the p-torsion of the class group of $\mathbb{Q}$, so has dimension 0, and the equality is proved.

When $a>1$ by Greenberg–Wiles version of Poitou–Tate duality [Was97, Theorem 2], one has $\dim H^1(G_{\mathbb{Q},p},\omega_p^a)=\dim H^1_0(G_{\mathbb{Q},p},\omega_p^{1-a})+1$ where $H^1_0(G_{\mathbb{Q},p},\omega_p^{1-a})=\mathrm{Ker}\,(H^1(G_{\mathbb{Q},p},\omega_p^{1-a})\to H^1(G_{\mathbb{Q}_p},\omega_p^{1-a}))$. Since $H^1(G_{\mathbb{Q},p},\omega_p^{1-a})=H^1(I_p,\omega_p^{1-a})$ (cf. [Was97, Theorem 2]) so $H^1_0(G_{\mathbb{Q},p},\omega_p^{1-a})$ parametrizes extensions of $G_{\mathbb{Q}}$-representations $0\to\omega_p^{1-a}\to V\to 1\to 0$ that are unramified everywhere (in the sense that applying the functor of I_ℓ-invariants to this short exact sequence yields a sequence which is still exact, including for $\ell=p$), and this space, by Hilbert class field theory, cf. [Rub00, ch. I], is the space $A(\omega_p^{1-a})=A(\omega_p^{p-a})$. □

THEOREM 22. *The residual representation $\bar\rho=1\oplus\omega_p^a$ ($1\leqslant a\leqslant p-2$, a odd) is unobstructed if either of the following conditions holds:*

(i) *$a>1$ and p does not divide $B_{a+1}B_{p-a}$ where B_n is the nth Bernoulli number;*
(ii) *$a=1$;*
(iii) *Vandiver's conjecture holds for p.*

Therefore, in any of those cases, $A_{\bar\rho}$ is a regular local complete algebra of dimension 2 over $\bar{\mathbb{F}}_p$.

Proof. Let us assume that condition (i) holds. By the reflection theorem [Was82, Theorem 10.9] one has p-rank $A(\omega_p^{p-a})\leqslant p$-rank $A(\omega_p^a)$ and p-rank $A(\omega_p^{a+1})\leqslant p$-rank $A(\omega_p^{p-1-a})$. The p-rank of $A(\omega_p^a)$ (respectively $A(\omega_p^{p-1-a})$ is 0 if $p\nmid B_{p-a}$ and $p\nmid B_{a+1}$ by Herbrand's theorem [Was82].

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Hence by the Lemma, $\dim H^1(G_{\mathbb{Q},p}, \omega_p^a) \leqslant 1$ and $\dim H^1(G_{\mathbb{Q},p}, \omega_p^{-a}) \leqslant 1$ and $\bar\rho$ is unobstructed by Proposition 20.

In case (ii), that is $a = 1$, we have already seen that $\dim H^1(G_{\mathbb{Q},p}, \omega_p) \leqslant 1$ in the course of the proof of the lemma, and by the lemma again, $\dim H^1(G_{\mathbb{Q},p}, \omega_p^{-1}) \leqslant 1 + \dim A(\omega_p^2) \leqslant 1 + \dim A(\omega_p^{p-2})$ by the reflexion theorem. By Herbrand's theorem, $A(\omega_p^{p-2})$ is 0 (since $B_2 = 1/6$ is not divisible by any p), and the result follows again from the proposition.

Finally, in case (iii), one just needs to recall that Vandiver's conjecture is the statement that $A(\omega_p^n)$ is 0 for every even n, so the result follows again from the lemma and the proposition. □

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Appendix A. The case $p = 3$

This appendix is devoted to the case $p = 3$ of Theorem I. We give a complete treatment of the Galois-theoretic part of the proof, which allows us to obtain a more precise result than Theorem I, with explicit determination of systems of generators of A: it is a concrete illustration of the Galois-theoretical methods used in this article. But we only sketch the second part of the proof, concerning the order of nilpotence of modular forms modulo 3, which uses completely different methods than those of this paper, more akin to Nicolas and Serre's methods in characteristic 2. The details of this second part are due to Anna Medvedovsky and will appear in her forthcoming thesis, as part of her general study of modular forms modulo 3.

A.1 Results

If $p = 3$, the only modular Galois representation is $\bar\rho = 1 \oplus \omega_3$, which is unobstructed. We therefore only need to prove Theorem I, that is that $A = A_{\bar\rho}$ is isomorphic to $\mathbb{F}_3[[x, y]]$. If ℓ is a prime $\neq 3$, $\operatorname{tr}\bar\rho(\operatorname{Frob}_\ell) = 1 + \ell \pmod 3$. It follows that the operators T_ℓ for $\ell \equiv 2 \pmod 3$, and $1 + T_\ell$ for $\ell \equiv 1 \pmod 3$ belongs to the maximal ideal m_A of A. We shall write T'_ℓ for $1 + T_\ell$ when $\ell \equiv 1 \pmod 3$ and for T_ℓ when $\ell \equiv 2 \pmod 3$ so that the Hecke operators T'_ℓ are always locally nilpotent on $S(\mathbb{F}_3)$.

Let us also recall that by [SD73], $S(\mathbb{F}_3)$ is the $\mathbb{F}_3$-vector space of basis $(\tilde\Delta^k)_{k=0,1,2,\ldots}$, where $\tilde\Delta$ is the reduction modulo 3 of the q-expansion of the modular form Δ. We restrict our attention to the subspace M of $S(\mathbb{F}_3)$ generated by $\tilde\Delta^k$ for $k \geqslant 1$, $k \not\equiv 0 \pmod 3$.

Lemma 23. (i) *The space M is stable by all of the T_ℓ, $\ell \neq 3$, hence by A.*

(ii) *The action of A on M is faithful.*

(iii) *For every cofinite ideal I in A, the pairing $A/I \times M[I] \to \mathbb{F}_3$, $(T, f) \mapsto a_1(Tf)$ is perfect.*

Proof. Note that if $\tilde\Delta^k = \sum a_n q^n \in \mathbb{F}_3[[q]]$, and $a_n \neq 0$, then $n \equiv k \pmod 3$: this follows from the case $k = 1$, which results from the known congruences about $\tau(n)$ (see [SD73]). The space M is therefore the subspace of $S(\mathbb{F}_3)$ of forms $\sum a_n q^n$ which satisfy $a_{3n} = 0$ for every integer n, and part (i) follows then from the formula giving the action of T_ℓ on q-expansions. Also one has $S(\mathbb{F}_3) = \bigoplus_{n=0}^{\infty} M^{3^n}$, and since for $f \in S(\mathbb{F}_3)$, and $\ell \neq 3$ a prime, one has $T_\ell(f^3) = (T_\ell f)^3$, this decomposition is stable by A and $S(\mathbb{F}_3)$ is, as an A-module, isomorphic to a countable direct sum of copies of M. Since A acts faithfully on $S(\mathbb{F}_3)$ by construction, part (ii) follows, and part (iii) is then routine. □

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Let $\mathcal{P}_1$ be the set of primes ℓ which are congruent to 1 modulo 3 but not split in the splitting field L of $X^3 - 3$. Let $\mathcal{P}_2$ be the set of primes ℓ which are congruent to 2 modulo 3 but not to 8 modulo 9. Note that $\mathcal{P}_1$ and $\mathcal{P}_2$ both have density 1/3 and are disjoint.

We shall sketch the proof of the following more precise version of Theorem 1.

THEOREM 24. *There is a (unique) isomorphism of algebras $\mathbb{F}_3[[x,y]] \to A$ which sends x to $T'_2 = T_2$ and y to $T'_7 = 1 + T_7$. For $\ell \neq 3$ a prime, one has $T'_\ell \equiv x \pmod{m_A^2}$ if and only if $\ell \in \mathcal{P}_2$, $T'_\ell \equiv y \pmod{m_A^2}$ if and only if $\ell \in \mathcal{P}_1$, and $T'_\ell \equiv 0 \pmod{m_A^2}$ if and only if $\ell \notin \mathcal{P}_1 \cup \mathcal{P}_2$.*

As in [NS12b], one deduces immediately from the theorem, using Lemma 23(iii), that the following result holds.

COROLLARY 25. *There exists a unique basis $m(a,b)_{a\in\mathbb{N},b\in\mathbb{N}}$ of M,* adapted *to T'_2 and T'_7 in the following sense:*

(i) $m(0,0) = \tilde{\Delta}$;
(ii) $T'_2 m(a,b) = m(a-1,b)$ *if* $a \geqslant 1$, *and* $T'_2 m(0,b) = 0$;
(iii) $T'_7 m(a,b) = m(a,b-1)$ *if* $b \geqslant 1$, *and* $T'_7 m(a,0) = 0$;
(iv) *the first coefficient a_1 of $m(a,b)$ is zero except if $(a,b) = (0,0)$.*

Example 26. With simple computations, one checks that

$$\begin{aligned}
m(0,1) &= \tilde{\Delta}^7 + 2\tilde{\Delta}^{10},\\
m(0,2) &= \tilde{\Delta}^{13} + 2\tilde{\Delta}^{16} + \tilde{\Delta}^{19} + \tilde{\Delta}^{28},\\
m(1,0) &= \tilde{\Delta}^2,\\
m(2,0) &= \tilde{\Delta}^4 + 2\tilde{\Delta}^7 + \tilde{\Delta}^{10},\\
m(3,0) &= \tilde{\Delta}^8 + 2\tilde{\Delta}^{11},\\
m(4,0) &= 2\tilde{\Delta}^{13} + 2\tilde{\Delta}^{16} + \tilde{\Delta}^{19} + \tilde{\Delta}^{28},\\
m(1,1) &= 2\tilde{\Delta}^5 + 2\tilde{\Delta}^8 + \tilde{\Delta}^{11}.
\end{aligned}$$

The proof of the theorem rests on two propositions, one concerning deformation theory, and one of elementary nature, similar to some arguments of Nicolas and Serre about the order of nilpotence of modular forms.

PROPOSITION 27. *The tangent space $\tilde{D}^0_{\bar\rho}(\mathbb{F}_3[\epsilon])$ to the functor $\tilde{D}^0_{\bar\rho}$ has dimension 2. This space has a basis of pseudo-characters $\tau_1, \tau_2 : G_{\mathbb{Q},3} \to \mathbb{F}_3[\epsilon]$ (deforming $\bar{t} = 1 + \omega_3 : G_{\mathbb{Q},3} \to \mathbb{F}_3$) such that for $i = 1, 2$, and any prime $\ell \neq 3$, $\tau_i(\mathrm{Frob}_\ell)$ is non-constant (that is, lies in $\mathbb{F}_3[\epsilon] - \mathbb{F}_3$) if and only if $\ell \in \mathcal{P}_i$*

To state the second proposition, we need two definitions.

DEFINITION 28. For every form $f \in S(\mathbb{F}_3)$, the *index of nilpotence* of f, denoted by $g(f)$, is the smallest integer n such that $T'_{\ell_1} \dots T'_{\ell_n} f = 0$ for any choice of n primes $\ell_1, \dots, \ell_n$ (not necessarily distinct) different form 3.

DEFINITION 29. Let $k \geqslant 1$ be an integer. Write k in base 3, that is $k = \sum_{i=0}^r a_i 3^i$, with the $a_i \in \{0,1,2\}$, $a_r \neq 0$. We define the *content* of k by $c(k) = \sum_{i=0}^r a_i 2^i$.

Let us note for later use the following estimate of $c(k)$.

LEMMA 30. *One has $c(k) \leqslant 2^{2+[\log k/\log 3]}$, where $[\log k/\log 3]$ is the integral part of $\log k/\log 3$.*

Proof. Let $r = [\log k/\log 3]$, so that $3^r \leqslant k < 3^{r+1}$, and $k = \sum_{i=0}^r a_i 3^i$, with the $a_i \in \{0,1,2\}$, $a_r \neq 0$. Then $c(k) = \sum_{i=0}^r a_i 2^i \leqslant 2\sum_{i=0}^r 2^i = 2^{r+2} - 2 \leqslant 2^{r+2}$. □

Proposition 31. *The index of nilpotence of the form $\tilde{\Delta}^k$ is at most its content, that is $g(\tilde{\Delta}^k) \leqslant c(k)$.*

A.2 Proof of the theorem assuming Propositions 27 and 31

Let us denote by $\tau_{\rm univ} : G_{\mathbb{Q},3} \to \tilde{R}^0_{\bar\rho}$ the universal pseudo-character and by m the maximal ideal of the deformation ring $\tilde{R}^0_{\bar\rho}$. For $\ell \neq 3$ a prime, let us denote by $t_\ell \in \tilde{R}^0_{\bar\rho}$ the element $\tau_{\rm univ}(\mathrm{Frob}_\ell)$ and by $t'_\ell \in m$ the element $t_\ell - \mathrm{tr}\,\bar\rho(\mathrm{Frob}_\ell)$. By definition of the map $\tilde{R}^0_{\bar\rho} \to A$, one sees that this map sends t_ℓ on T_ℓ and t'_ℓ on T'_ℓ.

Since $t'_\ell \in m$, one can see $t'_\ell \in m/m^2$ as an element of the cotangent space of $\mathrm{Spec}\,\tilde{R}^0_{\bar\rho}$, that is as a linear form on the tangent space $\tilde{D}^0_{\bar\rho}(\mathbb{F}_3[\epsilon])$. Proposition 27 can be translated as $t'_\ell(\tau_1) \neq 0$ if and only if $\ell \in \mathcal{P}_1$, $t'_\ell(\tau_2) \neq 0$ if and only if $\ell \in \mathcal{P}_2$, where τ_1, τ_2 is the basis of $\tilde{D}^0_{\bar\rho}(\mathbb{F}_3[\epsilon])$ introduced in the proposition. Hence, it is clear that t'_{ℓ_1}, t'_{ℓ_2} form a basis of m/m^2 if and only if $\ell_1 \in \mathcal{P}_1$, $\ell_2 \in \mathcal{P}_2$ (up to exchanging ℓ_1 and ℓ_2). In particular, t'_2 and t'_7 is a basis of m/m^2.

Since the map $R^0_{\bar\rho} \to A$ is surjective, it follows that T'_2 and T'_7 generate m_A/m_A^2, hence by Nakayama, generate the topological algebra A. Hence, to show that the morphism $\mathbb{F}_2[[x,y]] \to A$ that sends x and T'_2 and y on T'_7 is an isomorphism, it suffices to prove that A has Krull dimension at least 2.

Thus, $\dim A/m_A^n = \dim M[m_A^n] = \dim\{f \in M, g(f) \leqslant n\}$. By Proposition 31, the latter space contains all of the forms Δ^k for $c(k) \leqslant n$, hence by Lemma 30 all of the forms $\tilde{\Delta}^k$ for $2^{2+[\log k/\log 3]} \leqslant n$, in particular for $k \leqslant \frac{1}{9} n^{\log 3/\log 2}$, $k \not\equiv 0 \pmod 3$. As the forms $\tilde{\Delta}^k$ are linearly independent, one deduces that $\dim A/m_A^n \geqslant \frac{2}{27} n^{\log 3/\log 2}$. If A was of Krull's dimension 1 (respectively 0), then $\dim A/m_A^n$ would be linear in n (respectively constant) for n large enough, which would contradict the above estimate since $\log 3/\log 2 > 1$. Therefore, A has Krull dimension at least 2, hence is isomorphic to $\mathbb{F}_3[[x,y]]$ by the isomorphism described above, hence has dimension 2.

A.3 Proof of Proposition 27

It suffices to construct two pseudo-characters τ_1 and τ_2 satisfying *for $i = 1,2$, $\tau_i(\mathrm{Frob}_\ell)$ is non-constant if and only if $\ell \in \mathcal{P}_i$* because two such pseudo-characters are clearly linearly independent, hence a basis of $D^0_{\bar\rho}(\mathbb{F}_3[\epsilon])$ since this space has dimension 2 by point (ii) of Theorem 22.

A.3.1 *Construction of τ_2.* We define an additive non-trivial character $\alpha : G_{\mathbb{Q},3} \to \mathrm{Gal}(\mathbb{Q}(\mu_9)/\mathbb{Q}) = (\mathbb{Z}/9\mathbb{Z})^* \to \mathbb{F}_3$, by $\alpha(1) = \alpha(8) = 0$, $\alpha(2) = \alpha(7) = 1$, and $\alpha(4) = \alpha(5) = 2$. (Note that by class field theory, any non-trivial character $G \to \mathbb{F}_3$ factors through the quotient $\mathrm{Gal}(\mathbb{Q}(\mu_9)/\mathbb{Q})$ of $G_{\mathbb{Q},3}$ which is cyclic of order 6, hence is either α or $-\alpha$). Define $\tau_2 : G_{\mathbb{Q},3} \to \mathbb{F}_3$ by $(1+\epsilon\alpha) + \omega_3(1-\epsilon\alpha)$. As the sum of two characters $G_{\mathbb{Q},3} \to \mathbb{F}_3[\epsilon]^*$, τ_w is a pseudo-character of $G_{\mathbb{Q},3}$ of dimension 2, clearly deforming $1+\omega_3$. Its determinant is $(1+\epsilon\alpha)\times\omega_3(1-\epsilon\alpha) = \omega_3$, hence is constant. Thus, τ_2 is an element of $\tilde{D}^0_{\bar\rho}(\mathbb{F}_3[\epsilon])$. One has for $\ell \neq 3$ a prime, $\tau_2(\mathrm{Frob}_\ell) = 1 + \omega_3(\ell) + \epsilon\alpha(\ell)(1-\omega_3(\ell))$. So $\tau_2(\ell)$ is non-constant if and only if $\alpha(\ell) \neq 0$ and $\omega_3(\ell) \neq 1$ in $\mathbb{F}_3$, that is if and only if $\ell \pmod 9$ is not 1 and 8, and $\ell \pmod 3$ is 2, which means $\ell \in \mathcal{P}_2$.

A.3.2 *Construction of τ_1.* The splitting field of $X^3 - 3$ in $\mathbb{C}$ is $L = \mathbb{Q}(u,j)$, where $u = 3^{1/3}$ and j is the cubic root of unity $(-1+i\sqrt{3})/2$. The Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the symmetric group S_3 by sending $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ to its action on $\{u, ju, j^2u\}$ which we identify

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with $\{1,2,3\}$ by sending u on 1, uj on 2, and uj^2 on 3. With this identification, the character $\omega_3 : \mathrm{Gal}(L/\mathbb{Q}) \to (\mathbb{Z}/3\mathbb{Z})^* = \{1,-1\}$ is identified with the sign character of S_3.

We also identify the multiplicative subgroup $\mu_3(L) = \{1, j, j^2\}$ of L^* with the additive subgroup of $\mathbb{F}_3$ by sending 1 on 0, j on 1, j^2 on 2. We define a map $\tau_1 : S_3 \to \mathbb{F}_3[\epsilon]$ as in the third column of the table below:

σ	$\omega_3(\sigma)$	$\tau_1(\sigma)$	$\frac{1}{2}(\tau_1(\sigma)^2 - \tau_1(\sigma^2))$
Id	1	2	1
(12)	-1	0	-1
(13)	-1	0	-1
(23)	-1	0	-1
(123)	1	$2+\epsilon$	1
(132)	1	$2+\epsilon$	1

One checks by straightforward computations that τ_1 is a pseudo-character on S_3 of dimension 2. (A more conceptual proof can be obtained by the arguments of the proof of the main theorem in [Bel12b].) The determinant of this pseudo-character has been computed in the last column of the above table, and is seen to be equal to ω_3. Therefore $\tau_1 : G \to \mathrm{Gal}(L/\mathbb{Q}) = S_3 \to \mathbb{F}_3[\epsilon]$ is in $D^0_{\bar\rho}(\mathbb{F}_3[\epsilon])$, and one sees from the table that $\tau_1(\mathrm{Frob}_\ell)$ is non-constant if and only if $\mathrm{Frob}\,\ell$ in S_3 is a 3-cycle, that is if and only if Frob_ℓ is an element of sign 1 but not the identity, that is if and only if $\ell \equiv 1 \pmod 3$ but ℓ does not split in L.

A.4 Sketch of the proof of Proposition 31

We have seen during the proof of Theorem 24, before using Proposition 31, that the ideal m_A was generated by T'_2 and T'_7. Hence, the following lemma holds.

LEMMA 32. *For $f \in M$, the index of nilpotence $g(f)$ of f is the smallest n such that $T'_{\ell_0} \dots T'_{\ell_n} f = 0$ for any choice of primes $\ell_0, \dots, \ell_n$ in the set $\{2, 7\}$.*

Thus, we are reduced to study the operators T'_2 and T'_7.

LEMMA 33 (Nicolas–Serre). *One has the following recurrence relations:*

$$\begin{aligned}
T'_2\tilde\Delta^k &= \tilde\Delta T'_2\tilde\Delta^{k-2} - \tilde\Delta^3 T'_2\tilde\Delta^{k-3} \quad \text{for } k \geqslant 3,\\
T_7\tilde\Delta^k &= -\tilde\Delta T_7(\Delta^{k-1}) - \tilde\Delta^2 T_7(\tilde\Delta^{k-2}) - \tilde\Delta^3 T_7(\tilde\Delta^{k-3}) + (\tilde\Delta^4 - \tilde\Delta)T_7(\tilde\Delta^{k-4})\\
&\quad - (\tilde\Delta^5 + \tilde\Delta^2)T_7(\tilde\Delta^{k-5}) - (\tilde\Delta^6 + \tilde\Delta^3)T_7(\tilde\Delta^{k-6})\\
&\quad + (\tilde\Delta + \tilde\Delta^4 - \tilde\Delta^7)T_7(\tilde\Delta^{k-7}) - \tilde\Delta^8 T_7(\tilde\Delta^{k-8}) \quad \text{for } k \geqslant 8.
\end{aligned}$$

More generally, Nicolas and Serre prove that $T_p\tilde\Delta^k = \sum_{i=0}^p c_{p,i}(\tilde\Delta)T_p(\Delta^{k-i})$ for all $k \geqslant p+1$, with the $c_{p,i}$ polynomials of one variable of degree at most i. The proof is similar to that of [NS12a, Theorem 3.1]. Nicolas has computed the $c_{p,i}$ for small values of p (up to $p = 37$). The details have not yet been published.

DEFINITION 34. If $0 \neq f = \sum a_k\tilde\Delta^k \in M$, we define the *content* $c(f)$ *of* f, as $c(f) = \max_{k, a_k \neq 0} c(k)$. If $f = 0$ we define $c(0) = -\infty$.

PROPOSITION 35 (Medvedovsky). *For every $f \in M$, one has $c(T'_2 f) \leqslant c(f) - 1$, and $c(T'_7 f) \leqslant c(f) - 2$.*

The proof will appear in Medvedovsky's thesis.

Let $n \geqslant c(\Delta^k)$ and $g = T'_{\ell_0} \dots T'_{\ell_n}\tilde\Delta^k = 0$. By applying the lemma $n+1$ times, one obtains $c(g) < 0$, hence $c(g) = -\infty$ and $g = 0$. This proves Proposition 31.

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