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ISOMETRIC MULTIPLICATION OF HARDY-ORLICZ SPACES

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For a modulus function ϕ, we define the Hardy-Orlicz space $H(\phi)$. Two main questions are discussed in this paper. First, when is a linear map $m_g : H(\phi) \to H(\phi)$, $m_g(f) = g.f$ an isometry? Second, when is $H(\phi) = H^1$?

0. Introduction.

Let Δ be the open unit disc in $\mathbb{C}$, the set of complex numbers, and $H(\Delta)$ be the space of analytic functions in Δ. If T is the unit circle, then $L^p(T)$ denotes the space of p-Lebesgue integrable functions, $0 < p \leq \infty$. The classical Hardy spaces will be denoted by

$$H^p = \left\{ f \in H(\Delta) : \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty \right\}, 0 < p < \infty ,$$

and H^∞ is the space of bounded analytic functions in Δ.

It is very well-known that every $f \in H^p$ has a radial limit function, also denoted by f, in $L^p(T)$. Further, H^p can be considered as a closed subspace of $L^p(T)$, when equipped with the metric:

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$$d(f,g) = \begin{cases} \left(\int_T |f(x) - g(x)|^p \, dx\right)^{\frac{1}{p}} & p \geq 1 \\ \int_T |f(x) - g(x)|^p \, dx & 0 < p < 1 . \end{cases}$$

Another important and interesting class in $H(\Delta)$ is the Nevalinna class:

$$N = \left\{ f \in H(\Delta) : \sup_{0 \le r < 1} \int_0^{2\pi} \ln(1 + |f(re^{i\theta})|) d\theta < \infty \right\} .$$

By N^+ we mean as usual the space

$$N^+ = \left\{ f \in N : \sup_{0 \le r < 1} \int_0^{2\pi} \ln(1 + |f(re^{i\theta})|) d\theta = \int_0^{2\pi} \ln(1 + |f(e^{i\theta})|) d\theta \right\} .$$

We may think of the metric for H^p $0<p<1$ and N as given by

$$d(f,g) = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} \phi(|f(re^{i\theta}) - g(re^{i\theta})|) d\theta$$

where

$$\phi(x) = x^p \quad \text{for} \quad H^p$$

and
$$\phi(x) = \ln(1 + |x|) \quad \text{for} \quad N .$$

It is clear that in both of these cases ϕ is continuous, increasing, subadditive and zero only at zero. Moreover $\phi(|u|)$ is subharmonic for every $u \in H(\Delta)$. Assuming that we have a function ϕ as described above we can define the space

$$H(\phi) = \left\{ f \in H^+(\Delta) : \int_0^{2\pi} \phi|f(e^{i\theta})| d\theta = \sup_{0 \le r < 1} \int_0^{2\pi} \phi(|f(re^{i\theta})|) d\theta \right\}$$

where H^+ denoted the subspace of $H(\Delta)$ consisting of functions which have radial limits almost everywhere. See [2] for more details about these spaces. See [9] and [10] for general related results.

In this paper we consider the following question: if $g \in H(\phi)$ and the map $m_g(f) = f.g$ is an isometry what can we say about g ? Our result in this regard generalizes the known one for H^p , see [4]. In

Section 3 we consider the very natural question: what condition on ϕ would guarantee that $H(\phi) = H^1$? Finally we consider the projective tensor product of $H(\phi)$ with itself.

1. Preliminaries and Notation.

A function $\phi : [0,\infty) \to R$ is called a modulus function if:

(i) ϕ is continuous and increasing,

(ii) $\phi(x) = 0$ if and only if $x = 0$,

(iii) $\phi(x + y) \le \phi(x) + \phi(y)$.

Examples of such functions are: $\phi(x) = x^p$ $0 < p \le 1$, $\phi(x) = 1n(1 + x)$. In fact if ϕ is a modulus function then $\psi(x) = \dfrac{\phi(x)}{1 + \phi(x)}$ is a modulus function, and for modulus functions ϕ_1, ϕ_2 , the function $w = \phi_1 \circ \phi_2$ is a modulus function.

Throughout the paper, we will assume that our modulus function satisfies the additional conditions that $\phi(|u|)$ is a subharmonic function whenever $u \in H(\Delta)$, and ϕ is strictly increasing.

Let $H^+(\Delta) = \{f \in H(\Delta) : \lim_{r \to 1} f(re^{i\theta})$ exists $a.e.\}$. Thus $H^+(\Delta)$ can be viewed as a space of functions on T .

Now, we define the Hardy-Orlicz space:

$$H(\phi) = \left\{ f \in H^+(\Delta) : \sup_{0 \le r < 1} \int_0^{2\pi} \phi|f(re^{i\theta})|d\theta = \int_0^{2\pi} \phi|f(e^{i\theta})|d\theta < \infty \right\},$$

where ϕ is a modulus function. On $H(\phi)$ we define a metric

$$d(f,g) = ||f - g||_\phi = \int_0^{2\pi} \phi|f(e^{i\theta}) - g(e^{i\theta})|d\theta .$$

With this metric, $H(\phi)$ is a topological vector space. Further, since we are assuming that $\phi(|u|)$ is subharmonic for $u \in H(\Delta)$, the space $H(\phi)$ is an F-space, [2]. If ϕ is bounded, then $H(\phi) = H(\Delta)$. If $\phi(x) = 1n(1 + x^p), 0 < p \le 1$, then we write N_p for $H(\phi)$. Clearly $N_1 = N^+ \subseteq N_p$, noting that $1n(1 + x^p) = \phi_1 \circ \phi_2(x)$, where

$\phi_1(x) = \ln(1+x)$, $\phi_2(x) = x^p$.

In [2], it was shown that $H^1 \subseteq H(\phi)$ for all modulus functions ϕ.

Throughout the paper, we write $||f||_\phi$ for $\int_0^{2\pi} \phi|f(e^{i\theta})|d\theta$, $f \in H(\phi)$.

2. Multiplication on $H(\phi)$.

Throughout this section, we will view $H(\phi)$ as a space of functions defined on T . A function g defined on T is called a (Schur) multiplier of $H(\phi)$ if $g.f \in H(\phi)$ for all $f \in H(\phi)$, where $(g.f)(x) = g(x).f(x)$.

The set of all multipliers of $H(\phi)$ will be denoted by $M(H(\phi))$. It is well known (and easy to prove) that $M(H^p) = H^\infty$. In this section we characterize $M(H(\phi))$ for a large class of modulus functions. First we need the following

LEMMA 2.1. *Let ϕ be a modulus function which satisfies:*

(i) for any $f \in H^1$ there exists $g \in H(\phi)$ such that $\phi(|g|) = |f|$;

(ii) $\phi(x).\phi(y) \le \phi(x.y)$ for all $x \ge 1$ and $y \ge 0$.

If $g \in M(H(\phi))$, and $f \in H^1$, then $\int_0^{2\pi} (\phi|g(e^{i\theta})|).f(e^{i\theta})d\theta < \infty$.

Proof. Since $1 \in H(\phi)$, it follows that $g \in H(\phi)$. Let $E = \{\theta : |g(e^{i\theta})| \ge 1\}$. Then

$$\int_0^{2\pi} \phi|g(e^{i\theta})||f(e^{i\theta})|d\theta = \int_E \phi|g(e^{i\theta})||f(e^{i\theta})|d\theta + \int_{E^c} \phi|g(e^{i\theta})||f(e^{i\theta})|d\theta .$$

By the first assumption on ϕ , there exists $h \in H(\phi)$ such that $|f| = \phi|h|$. Consequently, using assumption (ii) on ϕ , we have

$$\int_0^{2\pi} \phi|g(e^{i\theta})||f(e^{i\theta})|d\theta \le \int_E \phi|g(e^{i\theta}).h(e^{i\theta})|d\theta + \phi(1).||h||_\phi ,$$

$$\le ||h.g||_\phi + \phi(1)||h||_\phi < \infty ,$$

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since $g \in M(H(\phi))$. □

THEOREM 2.2. *Let* ϕ *be a modulus function satisfying the conditions in Lemma 2.1. If* $\lim_{x\to\infty} \phi(x) = \infty$, *then* $M(H(\phi)) = H^\infty$.

Proof. Let $g \in M(H(\phi))$. Lemma 2.1 implies that $\int_0^{2\pi} \phi\left[|g(e^{i\theta})| + 1\right] . |f(e^{i\theta})| d\theta < \infty$ for all $f \in H^1$. It follows easily that $\ln(1 + \phi|g(e^{i\theta})|) \in L^1(T)$. Hence, [5, p.53], there exists $u \in H^1$ such that $1 + \phi|g| = |u|$. Consequently $\int_0^{2\pi} |u(e^{i\theta})| . |f(e^{i\theta})| d\theta < \infty$ for all $f \in H^1$. This implies that $u \in M(H^1) = H^\infty$. Thus $1 + \phi|g| \in H^\infty$. Since $\lim_{x\to\infty} \phi(x) = \infty$, it follows that $g \in H^\infty$. Hence $M(H(\phi)) \subseteq H^\infty$. That $H^\infty \subseteq M(H(\phi))$ is clear. Hence $H^\infty = M(H(\phi))$. □

COROLLARY 2.3. $M(H^p) = H^\infty$, $0 < p < 1$.

Proof. The modulus function defining H^p, $0 < p < 1$, is $\phi(x) = x^p$. For $f \in H^p$, one can write $f = u.v$, where u is an inner function, and v is an outer function, [3], such that $|f|^p = |v|^p$. Thus $v^p \in H^1$. Hence ϕ satisfies condition (i) of Lemma 2.1. Condition (ii) of Lemma 2.1 is clearly verified for ϕ. So the result in Lemma 2.1 is true for H^p, and therefore by Theorem 2.2, $M(H^p) = H^\infty$.

THEOREM 2.4. *Let* ϕ *be a modulus function such that* $\phi(x.y) \le \phi(x) + \phi(y)$ *for all* $x,y \in [0,\infty)$. *Then* $M(H(\phi)) = H(\phi)$.

Proof. Since $1 \in H(\phi)$, clearly, $M(H(\phi)) \subseteq H(\phi)$. Let $g \in H(\phi)$. Then for any $f \in H(\phi)$ we have

$$\int_0^{2\pi} \phi|f(e^{i\theta}).g(e^{i\theta})| d\theta \le \int_0^{2\pi} \phi|f(e^{i\theta})| d\theta + \int_0^{2\pi} \phi|g(e^{i\theta})| d\theta$$

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$$\leq ||f||_\phi + ||g||_\phi < \infty .$$

Hence $g \quad M(H(\phi))$. □

COROLLARY 2.5. $M(N_p) = N_p \quad 0 < p \leq 1$.

Proof. This modulus function defining N_p is $\phi(x) = \ln(1 + x^p)$. Since

$$\phi(x.y) = \ln(1 + (x.y)^p) \leq \ln(1 + x^p) + \ln(1 + y^p) ,$$

theroem 2.4 applies to give $M(N_p) = N_p$. □

For $g \in M(H(\phi))$, one defines a linear map m_g on $H(\phi)$ by $m_g(f) = g.f$ for $f \in H(\phi)$. The map m_g will be called an isometry if $||m_g(f)||_\phi = ||f||_\phi$ for all $f \in H(\phi)$.

THEOREM 2.6. *Let* ϕ *be a modulus function such that* $\lim_{x\to\infty} \phi(x) = \infty$. *Let* $g \in M(H(\phi))$. *Then* m_g *is an isometry on* $H(\phi)$ *if and only if* $|g| = 1$ *for almost all* θ.

Proof. If $|g| = 1$, then it is easily seen that m_g is an isometry.

Let m_g be an isometry on $H(\phi)$. Then $g^n \in H(\phi)$ for all n , and $||g^n||_\phi = ||g||_\phi$.

Let $E = \{\theta : |g(e^{i\theta})| > 1\}$. First we show that E has Lebesgue measure zero. Suppose E has a positive Lebesgue measure, so that

$$\int_E \phi|g^n(e^{i\theta})|d\theta \leq ||g^n||_\phi = ||g||_\phi .$$

Since $|g| > 1$ on E , by Fatou's lemma we get

$$\infty = \int_E \lim_n \phi|g^n(e^{i\theta})|d\theta \leq \underline{\lim}_n \int_E \phi|g^n(e^{i\theta})|d\theta \leq ||g||_\phi .$$

This contradiction implies that E has Lebesgue measure zero.

Let $B = \{\theta : |g(e^{i\theta})| < 1\}$. Then $\lim_{n\to\infty} \phi|g^n(e^{i\theta})| = 0$ on B .

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Since $||g^n||_\phi = ||g||_\phi$, and $|g| = 1$ on B^c , it follows that

$$\int_B \phi|g(e^{i\theta})\ d\theta = \int_B \phi|g^n(e^{i\theta})|d\theta \ .$$

The Lebesgue domoninated convergence theorem implies that

$$\int_B \phi|g(e^{i\theta})|d\theta = 0 \ .$$

Consequently, $\phi|g(e^{i\theta})| = 0$ on B , and hence $g = 0$ on B . But g is the radial limit of an analytic function in Δ . This implies that B has Lebesgue measure zero. From this we conclude that $|g| = 1$ a.e on T . □

3. Equality of H^1 and $H(\phi)$.

In [2], it was shown that $H^1 \subseteq H(\phi)$ for all ϕ . Further, if $\int_0^\infty \frac{\phi(x)}{x^2} dx < \infty$, then $H^1 \subsetneq H(\phi)$. In this section we study the question of when $H^1 = H(\phi)$.

THEOREM 3.1. *Let ϕ be a given modulus function. Then the following are equivalent:*

(i) $\underline{\lim}_{x\to 0} \frac{\phi(x)}{x} = \delta$, $\underline{\lim}_{x\to\infty} \frac{\phi(x)}{x} = \varepsilon$, δ , $\varepsilon \in R^+$;

(ii) $H(\phi) = H^1$ *and* $||f||_1 \le \lambda||f||_\phi \le \eta||f||_1$. *for all* $f \in H(\phi)$, *and some constants* λ , $\eta \in R^+$.

Proof. (i) → (ii) . Choose $0 < a < b$ such that $\frac{\phi(x)}{x} \ge r$ on $[0,a)$ and $\frac{\phi(x)}{x} \ge s$ on (b,∞) , for some r, $s \in R^+$.

Let $f \in H(\phi)$, and

$$E(a) = \{\theta : 0 < |f| < a\}$$
$$E(b) = \{\theta : |f| > b\}$$
$$E(a,b) = \{\theta : a \le |f| \le b\} \ .$$

Then

$$||f||_1 = \int_{E(a)} |f(e^{i\theta})|d\theta + \int_{E(a,b)} |f(e^{i\theta})|d\theta + \int_{E(b)} |f(e^{i\theta})|d\theta$$

$$\leq \frac{1}{r}\,||f||_\phi + \int_{E(a,b)} |f(e^{i\theta})|d\theta + \frac{1}{s}\,||f||_\phi .$$

The function $\frac{\phi(x)}{x}$ is continuous on $[a,b]$, consequently there exists $c \in (a,b)$ such that $\frac{\phi(x)}{x} \geq \frac{\phi(c)}{c}$ for all $x \in [a,b]$. Hence $\phi(x) \geq t.x$ on $[a,b]$. It follows

$$||f||_1 \leq \frac{1}{r}\,||f||_\phi + \frac{1}{t}\,||f||_\phi + \frac{1}{s}\,||f||_\phi$$

$$\leq \lambda||f||_\phi < \infty .$$

where $\lambda = \max\,(\frac{1}{r}, \frac{1}{t}, \frac{1}{s})$.

Similarly one can show that $||f||_\phi \leq \lambda||f||_1$. Thus (ii) is proved.

(ii) → (i). Consider the map $f(e^{i\theta}) = x\,e^{i\theta}$, $x > 0$. Then $||f||_1 = x$, $||f||_\phi = \phi(x)$. From (ii) we get

$$x \leq \lambda\phi(x) \leq \eta x .$$

Hence $\frac{1}{\lambda} \leq \frac{\phi(x)}{x} \leq \frac{\eta}{\lambda}$.

Consequently $\lim\limits_{x\to 0} \frac{\phi(x)}{x}$ is finite and $\lim\limits_{x\to\infty} \frac{\phi(x)}{x}$ is finite. This proves (i) . □

THEOREM 3.2. *If* $\lim\limits_{x\to\infty} \frac{\phi(x)}{x} = 0$, *then* $H^1 \subsetneq H(\phi)$.

Proof. Since $\lim\limits_{x\to\infty} \frac{\phi(x)}{x} = 0$, one can choose a sequence $x_n > \sqrt{n}$ such that $\phi(x_n) < \frac{x_n}{n}$. With no loss of generality, we can choose $x_{n+1} > x_n$ for all n . Since $x_n < \sqrt{n}$, then $H = \sum_1^\infty \frac{1}{n\,x_n} < \infty$.

Choose points $y_m \in [0,2\pi)$ such that $y_0 = 2\pi$, and $y_n > y_{n+1}$

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for all n , and $y_n - y_{n-1} = \frac{2\pi}{n\, x_n . H}$. The interval $[0, 2\pi]$ is then partitioned into disjoint intervals $I_n = (y_{n-1}, y_n]$, $\sum_{n=1}^{\infty} |y_n - y_{n-1}| = 2\pi$.

Define the function f on $(0,2\pi]$ such that $f(x) = x_n$ on I_n .

$$||f||_\phi = \sum_{n=1}^{\infty} \int_{I_n} \phi|f(x)|dx \ ,$$

$$= \sum_{n=1}^{\infty} \phi(x_n)|y_n - y_{n-1}| \ ,$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \ .$$

Hence $f \in L(\phi) = \left\{f : T \to \mathbb{C} : \int_T \phi|f| < \infty\right\}$. But

$$||f||_1 = \sum_{n=0}^{\infty} \int_{I_n} |f(x)|dx$$

$$= \sum_{n=1}^{\infty} x_n . |y_n - y_{n-1}|$$

$$= \Sigma\, x_n \cdot \frac{1}{n . x_n . H} = \infty \ .$$

Hence $f \notin L^1$.

It is not difficult to see that one can construct f to be continuous, without changing the facts that $f \in H(\phi)$ but $f \notin H^1$.

Now, consider the following sequence of functions:

$$f_n(x) = f(x) \text{ if } x \in \bigcup_{j=1}^{n} I_j = E_n \text{ , define } f_n \text{ on}$$

$[0, y_n)$ by $f_n(x) = x_n$. Then $f_n \in C(T)$ for all n , and $f_n(x) \leq f(x)$ for all $x \in [0, 2\pi]$. Thus

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$||f_n||_\phi \leq ||f||_\phi$, for all n . Further

$$||f_n||_1 > \int_{E_n} |f|dx \geq \sum_{k=1}^{n} \frac{1}{k} \qquad (*)$$

Let $\varepsilon > 0$. By Wermer's maximality theorem, [8] , for each n , one can find $g_n \in C(T)$ such that $||f_n - g_n||_\infty \leq \varepsilon$. Further, there exists $G_n \in H(\Delta)$ such that

$$\lim_{r\to 1} G_n\,(re^{i\theta}) = g_n(e^{i\theta}) \text{ for almost all } \theta .$$

Now:

$$||G_n||_\phi = ||g_n||_\phi < ||f_n||_\phi + \varepsilon ,$$

for all n . Since ϕ is assumed to satisfy the condition that $\phi|u|$ is subharmonic for $u \in H(\Delta)$, it follows that, [2],

$$|G_n(z)| \leq \phi^{-1}\left(\frac{4||G_n||_\phi}{1-r}\right) , \text{ for all } z = re^{i\theta} \in \Delta .$$

Consequently, the sequence $\{G_n\}$ is uniformly bounded on compact subsets of Δ , and so it is a normal family.

Hence there exists a subsequence (G_{n_j}) which converges uniformly on compact sets to some analytic function $G \in H(\Delta)$. Since

$$\int_0^{2\pi} \phi|G(re^{i\theta})|d\theta = \lim_{n\to\infty} \int_0^{2\pi} \phi|G_n\,(re^{i\theta})|d\theta ,$$

$$\leq ||f||_\phi + \varepsilon ,$$

it follows that $G \in H(\phi)$.

$$||G_n||_1 > ||f_n||_1 - \varepsilon ,$$

$$> \left(\sum_1^n \frac{1}{k}\right) - \varepsilon .$$

Since $||G_n||_1 = \lim_{r\to 1} \int_0^{2\pi} |G_n(re^{i\theta})|d\theta$, we get

$$\int_0^{2\pi} |G(re^{i\theta})|d\theta \geq \left(\sum_{k=1}^{\infty} \frac{1}{k}\right) - \varepsilon ,$$

for some $r \in (0,1)$. Hence $G \notin H^1$. Hence $H^1 \subsetneq H(\phi)$. □

4. $H(\phi) \hat{\otimes} H(\phi)$.

Let $H(\phi) \otimes H(\phi)$ be the space of all analytic functions f on $\Delta^2 = \Delta \times \Delta$, such that $f(z,w) = \sum_{i=1}^{n} u_i(z)\, v_i(w)$, $u_i, v_i \in H(\phi)$, for some modulus function ϕ. We will assume that ϕ satisfies the condition that $\phi|u|$ is subharmonic if $u \in H(\Delta)$. Let us define the metric d on $H(\phi) \otimes H(\phi)$ by:

$$d(f,g) = \inf \left\{ \sum_{i=1}^{n} ||u_i||_\phi \cdot ||v_i||_\phi \right\},$$

where the infimum is taken over all representations of $f - g$ in $H(\phi) \otimes H(\phi)$. Once can easily check that d is a metric on $H(\phi) \otimes H(\phi)$, and we write

$$||f - g||_\phi \quad \text{for} \quad d(f,g) .$$

The space $H(\phi) \otimes H(\phi)$ with the metric d is not complete. We write $H(\phi) \hat{\otimes} H(\phi)$ for the completion. Following [1], one can show that every element in $H(\phi) \hat{\otimes} H(\phi)$ has a representation $f = \sum_{i=1}^{\infty} u_i \otimes v_i$,

$$\sum_{i=1}^{\infty} ||u_i||_\phi \cdot ||v_i||_\phi < \infty , \text{ and } ||f||_\phi = d(f) = \inf\left\{ \sum_{i=1}^{\infty} ||u_i||_\phi \cdot ||v_i||_\phi \right\} .$$

The space $H(\phi) \hat{\otimes} H(\phi)$ will be called the projective tensor product of $H(\phi)$ with itself.

Tensor product is usually defined for locally convex topological vector spaces. The space $H(\phi)$ is not locally convex in general. The main result of this section is:

THEOREM 4.1. *$H(\phi) \hat{\otimes} H(\phi)$ is a topological vector space.*

Proof. First, we remark that d is a quasi-norm on $H(\phi) \hat{\otimes} H(\phi)$. That is:

(i) $d(f,0) = 0$ if and only if $f = 0$

(ii) $d(0,-f) = d(0,f)$

(iii) $d(f+g, 0) \leq d(f, 0) + d(g, 0)$.

These follow easily from the properties of the metric d and the representations of functions in $H(\phi) \hat{\otimes} H(\phi)$.

From Proposition 1 of [7, p. 38], it remains to show that:

(i) if $\alpha_n \to 0$, then $d(\alpha_n . f,0) \to 0$ for all $f \in H(\phi) \hat{\otimes} H(\phi)$;

(ii) if $d(f_n) \to 0$, then $d(\alpha f_n,0) \to 0$ for all $\alpha \in R$.

To prove (i): let $f = \sum_{i=1}^{n} u_i \otimes v_i \in H(\phi) \hat{\otimes} H(\Phi)$. Then

$$0 \leq d(\alpha_n f,0) \leq \sum_{i=1}^{k} ||\alpha_n u_i||_\phi . ||v_i||_\phi ,$$

and

$$\lim_n d(\alpha_n f,0) \leq \sum_{i=1}^{k} \lim_n ||\alpha_n u_i||_\phi . ||v_i||_\phi = 0 .$$

Now, for $f = \sum_{i=1}^{\infty} u_i \otimes v_i$, $\sum_{i=1}^{\infty} ||u_i||_\phi . ||v_i||_\phi < \infty$, define the sequence of functions

$$g_n(i) = ||\alpha_n u_i||_\phi . ||v_i||_\phi .$$

The Lebesgue dominated convergence theorem on the set on natural numbers with the counting measure implies:

$$\begin{aligned} \lim_{n\to\infty} d(\alpha_n f,0) &\leq \lim_{n\to\infty} \sum_{i=1}^{\infty} ||\alpha_n u_i||_\phi . ||v_i||_\phi \\ &\leq \sum_{i=1}^{\infty} \lim_{n\to\infty} ||\alpha_n u_i||_\phi . ||v_i||_\phi \\ &= 0 . \end{aligned}$$

To prove (ii): let $f_n \in H(\phi) \hat{\otimes} H(\phi)$, $d(f_n,0)\to 0$. Let k be a positive integer such that $k > \alpha$. Then

$$d(\alpha . f_n,0) \leq d(k f_n,0) \leq k.d(f_n,0) \to 0$$

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This completes the proof of the theorem. □

Characterization of the (Schur) multipliers of $H(\phi) \hat{\otimes} H(\phi)$ would be an interesting problem.

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