# ON THE PRODUCT OF TWO LINEAR FORMS, ONE HOMOGENEOUS AND ONE INHOMOGENEOUS

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## PART I: FORMULATION OF PROBLEM

### 1. Introduction

This paper is devoted to a complete investigation into a problem initiated by Davenport [4], and further studied by Kanagasabapathy [6], [7], from whom I borrow the title. The question is a hybrid of the two classical results of Hurwitz and Minkowski on indefinite binary quadratic forms.

Suppose $(\alpha x+\beta y)$ and $(\gamma x+\delta y)$ are two linear forms that do not represent zero for integers $x, y$ not both zero. If $\Delta = |\alpha\delta-\beta\gamma|$ is the determinant of the two forms, and $\eta$ is any non-zero real number, we may ask the question: for what values of $k'$ are there integral solutions $x, y$ not both zero, for the following inequality?

$$|(\alpha x+\beta y)(\gamma x+\delta y+\eta)| \leqq k'\Delta. \tag{1.1}$$

The best published result to date is due to Kanagasabapathy [7], who proved that the *best possible constant* $k$ for this problem, satisfies the inequalities

$$\frac{1}{4.2847} \leqq k < \frac{1}{4.25777}.$$

The major results of this paper may be summarised by the following theorem.

THEOREM 1.1. *Suppose that we have two linear forms*

$$X = \alpha x+\beta y$$
$$Y = \gamma x+\delta y,$$

*which have determinant $\Delta$, and which do not non-trivially represent zero in integers. If $\eta$ is a non-zero constant, then:*

A. $$\inf_{(x,y)\neq(0,0)} |X(Y+\eta)| \leqq k\Delta,$$

*where the infimum is taken over all integer pairs* $(x, y)$, *not the origin, and where* $k$ *is given by*

$$k = \frac{(3/49)(366458018\,\varphi-7320551)}{(8238730\,\theta+392361)\varphi-(164581\,\theta+7838)}$$
$$= 0\cdot 234254343\cdots,$$

*with*

$$\varphi = \frac{147+\sqrt{21651}}{6}, \qquad \theta = \frac{104250+2\sqrt{10}}{9005}.$$

B. *Equality holds in A for forms equivalent by an integral unimodular transformation to the form*

$$X = \alpha x + y$$
$$Y = x + \delta y,$$

*with*

$$\alpha = \frac{2\sqrt{10}-5195}{2997},$$

$$\delta = \frac{91018391\,\varphi-1818229}{8238730\,\varphi-164581},$$

*and*

$$\eta = -\tfrac{1}{2}\left(\varphi + \frac{18014063\,\varphi-359856}{49(8238730\,\varphi-164581)}\right).$$

C. *For every* $k'$, *such that* $0 \leqq k' < k$, *there exist uncountably many pairs of linear forms* $X$, $Y$, *to each of which there corresponds at least one real non-zero constant* $\eta$, *with*

$$\inf_{(x,y)\neq(0,0)} |X(Y+\eta)| = k'\Delta.$$

In part I of this paper, we will determine a systematic arithmetic formulation for the problem in terms of divided cells of lattices, and semi-regular continued fractions. This will then allow, in Part II, an exact calculation of the numerical value of $k$. In Part III we will then determine further information about the distribution of minimum values taken by such products of linear forms.

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## 2. The divided cell method

We will give a brief account of the divided cell method, the details of which are described in [1], [2], [3], [8] and [9]. Let

$$f(x, y) = (\alpha x+\beta y)(\gamma x+\delta y) = \frac{\Delta(\theta x+y)(x+\varphi y)}{\theta\varphi-1} \tag{2.1}$$

be an indefinite binary quadratic form, which does not represent zero for integers $x$, $y$, not both zero (which implies that the ratios $\alpha/\beta$ and $\delta/\gamma$ are irrational). If $P$ is the two dimensional point $(x_0, y_0)$, then we may define the following functions.

$$M(f;\, P) = \inf_{x,y} |f(x+x_0, y+y_0)| = \inf_{x,y} |(\alpha x+\beta y+\xi_0)(\gamma x+\delta y+\eta_0)|, \tag{2.2}$$

where $\xi_0 = \alpha x_0+\beta y_0$, $\eta_0 = \gamma x_0+\delta y_0$ and $x$, $y$ are integral.

$$M(f) = \sup_P M(f;\, P), \tag{2.3}$$

where the supremum need only be taken over a complete set of points, incongruent mod 1. $M(f)$ is known as the *inhomogeneous* minimum of the form $f$.

If we change the variables of $f$ by the integral unimodular transformation

$$x \to px+qy$$
$$y \to rx+sy,$$

where $p, q, r, s$ are integers, and $ps-qr = \pm 1$, then the new form obtained is said to be *equivalent* to $f$; it is clear that equivalent forms have the same inhomogeneous minimum.

The divided cell method enables $M(f)$ to be explicitly calculated in terms of the determinant of the form $f$. The basic result on inhomogeneous minima was discovered by Minkowski at the turn of the century.

THEOREM 2.1. (*Minkowski*)

$$M(f) \leq \frac{\Delta}{4},$$

*where inequality holds for all forms which do not represent zero.*

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The set of points $\mathscr{L}$, in the $\xi$–$\eta$ plane, given by

$$\mathscr{L}: \begin{array}{l} \xi = \alpha x+\beta y+\xi_0 \\ \eta = \gamma x+\delta y+\eta_0, \end{array} \tag{2.4}$$

for integral $x, y$, form a two dimensional *inhomogeneous lattice* or grid. A *cell* of the grid is any parallelogram of area $\Delta$, whose vertices are lattice points. A *divided cell* has vertices lying strictly in different quadrants.

Let us suppose that there are no lattice points on the axes, then we may invoke a result by Delauney [5], which guarantees the existence of a divided cell for such grids. From this cell it is possible, by a simple algorithm, to define a doubly infinite chain of divided cells, say $\{S_n\}$, for $-\infty < n < \infty$ (see [2]).

Denote the vertices of the divided cell $S_n$ by $A_n$, $B_n$, $C_n$ and $D_n$, labelled in a clockwise direction. Then at the next step of the algorithm, we define a new divided cell, $S_{n+1}$, and a unique pair of integers $h_n$, $k_n$, by the following rules:

$$\begin{cases} A_{n+1} = A_n-(h_n+1)V_n \\ B_{n+1} = A_n-h_nV_n \\ C_{n+1} = C_n+(k_n+1)V_n \\ D_{n+1} = C_n+k_nV_n, \end{cases} \tag{2.4}$$

where $V_n = A_n-D_n = B_n-C_n$.

For reference, we take $A_0$ to be in the first quadrant. The vertices of the new divided cell are simply the end points of the two lattice steps (on the infinite lines $A_nD_n$ and $C_nB_n$) which straddle the $\xi$-axis. It is therefore clear from (2.4) that $A_n$ is either in the first or third quadrants and that $h_n$ and $k_n$ both have the sign of the slope of the line segment $|A_nD_n|$.

It is shown in [3], for example, that the integer pairs $h_n$, $k_n$ satisfy the conditions:

(i) $h_nk_n > 0$

(ii) neither $h_n$ nor $k_n$ can be constantly equal to $-1$ for all large positive (or negative) $n$.

(iii) it is impossible that $h_{n+2r} = k_{n+2r+1} = 1$, for some $n$, and all $r \geqq 0$, (or all $r \leqq 0$).

In fact, any contravention of (ii) or (iii) implies that there is a grid point on an axis, contradicting our original assumption. We change to a more convenient notation by putting

$$\begin{cases} a_{n+1} = h_n+k_n \\ \varepsilon_n = h_n-k_n. \end{cases} \tag{2.5}$$

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We may therefore transform the conditions (i), (ii) and (iii) above into:

(2.6)
$$\begin{cases} \text{(i) } |a_n| \geqq 2, \text{ and } a_n \text{ is not constantly equal to } 2 \text{ (or } -2) \text{ for large } n \text{ of either sign.} \\ \text{(ii) } |\varepsilon_n| \leqq |a_{n+1}|-2, \text{ and } \varepsilon_n \text{ has the same parity as } a_{n+1}. \\ \text{(iii) neither } a_{n+1}+\varepsilon_n \text{ nor } a_{n+1}-\varepsilon_n \text{ is constantly equal to } -2, \text{ for large } n \text{ of either sign.} \\ \text{(iv) for some } n, \text{ the relation} \\ \qquad a_{n+2r+1}+\varepsilon_{n+2r} = a_{n+2r+2}-\varepsilon_{n+2r+1} = 2 \\ \text{does not hold for all } r \geqq 0 \text{ (or } r \leqq 0). \end{cases}$$

We now briefly introduce the notion of a semi-regular continued fraction. In contrast to ordinary continued fractions, we usually allow at each step of the development, a choice of taking either the integer below or the integer above of the appropriate complete quotient. In fact, each irrational number has uncountably many expansions in semi-regular continued fractions.

If $\{a_n\}$ is a sequence of integers, $n \geqq 1$, for which the condition (i) of (2.6) holds, then we may define the sequence of convergents $p_n/q_n$ by:

$$p_0 = 1, \qquad p_1 = a_1, \qquad q_0 = 0, \qquad q_1 = 1,$$

and for $n \geqq 1$

$$\begin{cases} p_{n+1} = a_{n+1}p_n - p_{n-1} \\ q_{n+1} = a_{n+1}q_n - q_{n-1}. \end{cases} \tag{2.7}$$

Then if

$$\begin{aligned} \alpha &= [a_1, a_2, a_3, \cdots] \\ &= a_1 - \frac{1}{a_2-}\,\frac{1}{a_3-}\cdots. \\ \alpha &= \lim_{n\to\infty} [a_1, a_2, \cdots, a_n] \\ &= \lim_{n\to\infty} \frac{p_n}{q_n}. \end{aligned}$$

Now for the doubly infinite sequence pair $\{a_{n+1}, \varepsilon_n\}$, $-\infty < n < \infty$, arising from the grid $\mathscr{L}$, we may define the following variables:

$$\begin{cases} \varphi_n = [a_{n+1}, a_{n+2}, \cdots], \quad \theta_n = [a_n, a_{n-1}, \cdots], \\ \mu_n = \varepsilon_n + \sum_{r=1}^{\infty} \dfrac{(-1)^r \varepsilon_{n+r}}{\varphi_{n+1}\varphi_{n+2}\cdots\varphi_{n+r}}, \\ \lambda_n = \varepsilon_{n-1} + \sum_{r=1}^{\infty} \dfrac{(-1)^r \varepsilon_{n-r-1}}{\theta_{n-1}\theta_{n-2}\cdots\theta_{n-r}}. \end{cases} \tag{2.8}$$

We have the following result:

THEOREM 2.2. *For all n,*

$$|a_{n+1}|-2+\sum_{r=1}^{\infty}\frac{|a_{n+r+1}|-2}{|\varphi_{n+1}\varphi_{n+2}\cdots\varphi_{n+r}|}\leqq|\varphi_n|-1,$$

$$|a_n|-2+\sum_{r=1}^{\infty}\frac{|a_{n-r}|-2}{|\theta_{n-1}\theta_{n-2}\cdots\theta_{n-r}|}\leqq|\theta_n|-1,$$

*and equality holds if and only if all the relevant $a_n$ have constant sign.*

COROLLARY: *For all n,*

$$\left\{\begin{array}{l}|\mu_n|<|\varphi_n|-1\\ |\lambda_n|<|\theta_n|-1.\end{array}\right. \tag{2.9}$$

The proofs of these results may be found in [1].

Returning to the definition (2.2), it is clear that

$$M(f;P)=\inf_{(\xi,\eta)\in\mathscr{L}}|\xi\eta|.$$

One of the fundamental results in this theory is that it is, in fact, unnecessary to extend this infimum over all grid points of $\mathscr{L}$, but only over those which are vertices of divided cells. By means of the sequence pair $\{a_{n+1},\varepsilon_n\}$, we may explicitly evaluate the products of the coordinates at the vertices. We eventually obtain the following theorem [2].

THEOREM 2.3. *If $\{a_{n+1},\varepsilon_n\}$ is the sequence pair associated with f and P, then*

$$M(f;P)=\inf_n M_n(f;P)=\inf_n M_n,$$

*where*

$$M_n=\min_{1\leqq i\leqq 4}(M_n^{(i)}),$$

*and*

$$\left\{\begin{array}{l}
M_n^{(1)}=\dfrac{\Delta}{4|\theta_n\varphi_n-1|}\,|(\theta_n+1-\lambda_n)(\varphi_n+1-\mu_n)|\\[2ex]
M_n^{(2)}=\dfrac{\Delta}{4|\theta_n\varphi_n-1|}\,|(\theta_n-1-\lambda_n)(\varphi_n-1+\mu_n)|\\[2ex]
M_n^{(3)}=\dfrac{\Delta}{4|\theta_n\varphi_n-1|}\,|(\theta_n-1+\lambda_n)(\varphi_n-1-\mu_n)|\\[2ex]
M_n^{(4)}=\dfrac{\Delta}{4|\theta_n\varphi_n-1|}\,|(\theta_n+1+\lambda_n)(\varphi_n+1+\mu_n)|.
\end{array}\right. \tag{2.10}$$

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Thus given any form $f$ and point $P$, we may obtain (theoretically) the associated sequence of integer pairs $\{a_{n+1}, \varepsilon_n\}$ satisfying (2.6), and hence use Theorem 2.3 to evaluate $M(f; P)$. Conversely, any sequence of integer pairs satisfying the conditions (2.6), can be shown to correspond to the chain of divided cells of a grid (see [1], [8]).

The following result will be useful in seeking bounds for $M_n$.

THEOREM 2.4. *Under the hypothesis of Theorem* 2.3,

$$M_n \leqq \frac{\Delta}{4|\theta_n\varphi_n-1|}\min\{|(\theta_n-1)(\varphi_n-1)|,\ |(\theta_n+1)(\varphi_n+1)|,\ |\theta_n(\varphi_n\pm\mu_n)|,\ |(\theta_n\pm\lambda_n)\varphi_n|\}.$$

PROOF:

$$\begin{aligned} M_n &\leqq \min\{M_n^{(1)}, M_n^{(2)}\} \\ &\leqq (M_n^{(1)}M_n^{(2)})^{\frac{1}{2}} \\ &= \frac{\Delta}{4|\theta_n\varphi_n-1|}|(\theta_n+1-\lambda_n)(\theta_n-1-\lambda_n)(\varphi_n+1-\mu_n)(\varphi_n-1+\mu_n)|^{\frac{1}{2}}. \end{aligned}$$

From (2.9), and using the inequality between geometric and arithmetic means twice we obtain

$$M_n \leqq \frac{\Delta|(\theta_n-\lambda_n)\varphi_n|}{4|\theta_n\varphi_n-1|}.$$

The other results follow analogously by considering different pairings of $M_n^{(i)}$.

## 3. Modification of the method

For simplicity we will assume, without loss of generality, that the form has unit determinant. We may rearrange the problem described in § 1 as follows. Suppose $\theta$ and $\varphi$ are irrational, and $\alpha$ is real and non-zero; put

$$M(f;\alpha) = \inf_{(x,y)\neq(0,0)}\left|\frac{(\theta x+y)(x+\varphi y+\alpha)}{\theta\varphi-1}\right| \tag{3.1}$$

where

$$f(x,y) = \frac{(\theta x+y)(x+\varphi y)}{\theta\varphi-1}. \tag{3.2}$$

Let us say at the outset that we may suppose $x+\varphi y+\alpha \neq 0$, for integers $x, y$, or else trivially $M(f;\alpha) = 0$. We now quote a result from [4].

THEOREM 3.1 (*Davenport*). *If $X$ and $Y$ are homogeneous linear forms of unit determinant, and which do not represent zero for integral values of the*

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*variables and c is any real non-zero constant then there exists an integral, unimodular transformation into new variables, which transforms* $(X+c)Y$, *into*

$$\pm\frac{(x+\theta y-\alpha)(x-\varphi y)}{\theta+\varphi}$$

*where* $\theta > 1$, $0 < \varphi < 1$, *and* $1 \leqq \alpha < \theta$.

Rewriting this in the notation of (3.2), we need only consider such forms for which

$$\varphi > 1, \qquad \theta < -1, \qquad -\varphi < \alpha \leqq -1. \tag{3.3}$$

We may suppose that $\alpha < 1$, else $M(f; \alpha) = 0$.

Now for any $\beta > 0$, $\gamma > 0$ such that $\beta\gamma = 1/(|\theta\varphi|+1)$, consider the grid $\mathscr{L}$ defined by:

$$\mathscr{L}: \begin{array}{l} \xi = \beta(\theta x+y) \\ \eta = \gamma(x+\varphi y+\alpha). \end{array} \tag{3.4}$$

We will call such an inhomogeneous lattice a $p$-grid; it has one, and only one, point on the axes. We will say that a cell of a $p$-grid is *pseudo-divided* or *p-divided*, if three of its vertices are in different quadrants, and the fourth is on an axis.

The following four points form a cell, $S_0$, of the $p$-grid $\mathscr{L}$.

$$\begin{cases} C_0 = \{\beta\theta, \gamma(1+\alpha)\}, & B_0 = \{0, \gamma\alpha\}, \\ D_0 = \{\beta(1+\theta), \gamma(1+\alpha+\varphi)\}, & A_0 = \{\beta, \gamma(\varphi+\alpha)\}. \end{cases} \tag{3.5}$$

The conditions (3.3) imply that $S_0$ is a $p$-divided cell of $\mathscr{L}$. It has been shown, although this is not necessary for the following argument, that any $p$-grid has at least one $p$-divided cell.

Now we may apply the algorithm (2.4) to the $p$-divided cell $S_0$, and it follows that we obtain a sequence of genuine divided cells $\{S_n\}$, $n > 0$, together with a sequence pair $\{a_{n+1}, \varepsilon_n\}$, $n \geqq 0$, satisfying (2.6). In addition, $a_1 < 0$, since the conditions (3.3) imply that the lattice line segment $|A_0D_0|$ has negative slope.

The algorithm also works in the reverse direction, in that there exists a cell $S_{-1}$, such that $S_0$ is obtained from it by the formulae (2.4). However the sequence of cells $\{S_{-n}\}$, $n \geqq 0$, are all $p$-divided, since the point $B_0$ is a vertex of each one.

Consideration of the geometry of the $p$-grid, and the rules (2.4), indicates that, for negative $n$, $A_n$ is in the first quadrant when $n$ is even, and the third quadrant when $n$ is odd. It easily follows in fact that, for all $n \leqq 0$,

$$h_{2n-1} = k_{2n-2} = 1. \tag{3.6}$$

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Hence, for $n \leqq 0$,

$$a_n > 0. \tag{3.7}$$

Now by methods analogous to those of Barnes [3], it follows that $h_{2n} = k_{2n-1} = 1$ cannot constantly hold for all small negative $n$; this implies that $a_n \geqq 2$, with strict inequality for infinitely many negative $n$.

Consequently, there corresponds to the $p$-grid, a doubly infinite sequence of integer pairs $\{a_{n+1}, \varepsilon_n\}$, $-\infty < n < \infty$, such that the conditions (2.6) hold for positive $n$, together with

$$\left\{\begin{array}{ll} \text{(i)} & a_n \geqq 2, \text{ for } n \leqq 0, \text{ with strict inequality holding infinitely} \\ & \text{often,} \\ \text{(ii)} & \varepsilon_n = (-1)^n(a_{n+1}-2), \quad \text{for all} \quad n < 0. \end{array}\right. \tag{3.8}$$

It may also be shown, in a similar way to [1], [8], that to every sequence pair of integers satisfying all these conditions, there corresponds a $p$-grid, which is unique except for a constant multiple of each coordinate.

For convenience, we will display a particular chain pair in the following tableau notation:

$$\begin{array}{cccc|cccc} \cdots & a_{-2}, & a_{-1}, & a_0 & a_1, & a_2, & a_3, & \cdots \\ \cdots & \varepsilon_{-3}, & \varepsilon_{-2}, & \varepsilon_{-1} & \varepsilon_0, & \varepsilon_1, & \varepsilon_2, & \cdots \end{array}$$

We will call the central line the *centre* of the chain. Note also that, for $n < 0$, the value of $\varepsilon_n$ is automatically fixed from (3.8) by the value of $a_{n+1}$.

Now applying Theorem 2.2, we obtain from (3.8), for $n \leqq 0$,

$$\begin{aligned} \lambda_n &= (-1)^{n-1}(a_n-2)+\sum_{r=1}^{\infty} \frac{(-1)^r(-1)^{n-r-1}(a_{n-r}-2)}{\theta_{n-1}\theta_{n-2}\cdots\theta_{n-r}} \\ &= (-1)^{n-1}\left\{a_n-2+\sum_{r=1}^{\infty}\frac{a_{n-r}-2}{\theta_{n-1}\theta_{n-2}\cdots\theta_{n-r}}\right\} \\ &= (-1)^{n-1}(\theta_n-1), \end{aligned} \tag{3.9}$$

since the appropriate $a_r$ have constant sign. In a similar sort of way we would expect

$$|\mu_{-n}| \sim \varphi_{-n}-1, \qquad \text{as } n \to \infty.$$

The following stronger result is in fact true.

THEOREM 3.2 *For* $n < -1$,

$$\varphi_n - 1 = |\mu_n| + \frac{\varphi_{-1}-1+\mu_{-1}}{\varphi_{n+1}\varphi_{n+2}\cdots\varphi_{-1}}.$$

PROOF: By (2.8), (3.8)

$$|\mu_n| = a_{n+1}-2+\frac{a_{n+2}-2}{\varphi_{n+1}}+\cdots+\frac{a_{-1}-2}{\varphi_{n+1}\varphi_{n+2}\cdots\varphi_{-2}}+ \\ -\frac{\mu_{-1}}{\varphi_{n+1}\varphi_{n+2}\cdots\varphi_{-1}}. \tag{3.10}$$

Now since $\varphi_r = a_{r+1}-(1/\varphi_{r+1})$, then for all $r$,

$$\varphi_r-1 = a_{r+1}-2+\frac{\varphi_{r+1}-1}{\varphi_{r+1}}.$$

Hence

$$\varphi_n-1 = a_{n+1}-2+\frac{a_{n+2}-2}{\varphi_{n+1}}+\cdots+\frac{a_{-1}-2}{\varphi_{n+1}\varphi_{n+2}\cdots\varphi_{-2}}+ \\ +\frac{\varphi_{-1}-1}{\varphi_{n+1}\varphi_{n+2}\cdots\varphi_{-1}},$$

and so the result follows by (3.10).

Now it is clear that

$$M(f;\alpha) = \inf_{\substack{(\xi,\eta)\in\mathscr{L}\\ \xi\neq 0}} |\xi\eta|.$$

Since the arithmetic formulation of the vertices of the sequence of cells is identical with § 2, we may use (3.8) and (3.9) to simplify the formulae (2.10), giving the following result.

THEOREM 3.3. *Suppose* $\{a_{n+1}, \varepsilon_n\}$ *is the chain pair of integers of the p-grid associated with* $f$ *and* $\alpha$; *then*

$$M(f;\alpha) = \inf_n M_n,$$

*where*

$$M_n = \min_{1\leqq i\leqq 4} (M_n^{(i)}),$$

*and for* $n > 0$, $M_n^{(i)}$ *are given by* (2.10),

$$\begin{cases} M_0^{(1)} = \dfrac{\theta_0(|\varphi_0|-1+\mu_0)}{2(|\theta_0\varphi_0|+1)}, \\ M_0^{(2)} = \dfrac{(\theta_0-1)(|\varphi_0|+1-\mu_0)}{2(|\theta_0\varphi_0|+1)}, \\ M_0^{(3)} = \dfrac{(|\varphi_0|-1-\mu_0)}{2(|\theta_0\varphi_0|+1)}, \\ M_0^{(4)} = \infty, \end{cases} \tag{3.11}$$

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*and for $n < 0$,*

$$(3.12)\qquad \begin{cases} M_n^{(1)} = \dfrac{\theta_n(\varphi_n+1+(-1)^{n+1}\mu_n)}{2(\theta_n\varphi_n-1)}, \\ M_n^{(2)} = \dfrac{(\theta_n-1)(\varphi_n-1+(-1)^n\mu_n)}{2(\theta_n\varphi_n-1)}, \\ M_n^{(3)} = \dfrac{\varphi_n+1+(-1)^n\mu_n}{2(\theta_n\varphi_n-1)}, \\ M_n^{(4)} = \infty. \end{cases}$$

NOTE: We have already commented that $a_1 < 0$, and hence

$$|\theta_0\varphi_0-1| = |\theta_0\varphi_0|+1.$$

This theorem gives a completely general method for evaluating $M(f;\alpha)$.

## PART II: CALCULATION OF THE CONSTANT

### 4. Application of the method

We will apply the method described in Part I of this paper to evaluate the best possible constant $k$, referred to in the introduction. Clearly

$$(4.1)\qquad \sup_{f,\alpha} M(f;\alpha) = k,$$

where the supremum is taken over all binary quadratic forms that do not represent zero, and all real non-zero $\alpha$.

To each non-trivial case, we can associate a chain pair of integers $\{a_{n+1}, \varepsilon_n\}$, which satisfy (2.6) for positive $n$, and (3.8) for negative $n$. By Theorem 3.3,

$$M(f;\alpha) = M(\{a_{n+1}, \varepsilon_n\}),$$

and so

$$\sup M(\{a_{n+1}, \varepsilon_n\}) = k,$$

where the supremum extends over all sequences of integer pairs satisfying the required conditions. If there exists a chain pair for which $M(\{a_{n+1}, \varepsilon_n\}) = k$, then it is called a *critical chain*, and the corresponding $f$ and $\alpha$, a *critical form*. If we put $\theta = -\theta_0$, $\varphi = -\varphi_0$ and $\alpha = (\varphi_0-1-\mu_0)/2$, where $\theta_0$, $\varphi_0$ and $\mu_0$ are the values taken for the critical chain, then the corresponding critical form will be

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$$\pm\frac{(\theta x+y)(x+\varphi y+\alpha)}{\theta\varphi-1}. \tag{4.2}$$

If we change the variables in (4.2) by an integral unimodular transformation, then clearly the equivalent form obtained has the same infimum.

In this part of the paper we will prove that $k$ has the value

$$k = \frac{(3/49)(366458018\,\varphi-7320551)}{(8238730\,\theta+392361)\varphi-(164581\,\theta+7838)}$$
$$= 0.234254343\cdots,$$

where $$\varphi = \frac{147+\sqrt{21651}}{6}, \qquad \theta = \frac{104250+2\sqrt{10}}{9005}.$$

## 5. Introductory lemmas

We will seek chain pairs for which $M(f;\alpha) \geqq k$, and so we will assume at the outset that

$$M_n^{(i)} \geqq k$$

for all $n$, and all relevant $i$. By moving in a stepwise process from the centre of the chain, the values of each member of the chain pair will be isolated by the above conditions, eventually leading us to a unique chain pair for which $M(f;\alpha) = k$. For convenience, $M(f;\alpha)$ will be abbreviated to $M$, provided that there is no ambiguity.

We will make constant use, often without specific reference, of the fact that the linear fractional form

$$y(x) = \frac{ax+b}{cx+d} \tag{5.1}$$

is a monotonic function in any interval of $x$ which does not contain the point $x = -d/c$. $y(x)$ is increasing if $ad-bc > 0$, and decreasing if $ad-bc < 0$.

We will use the following notation.

$$\tau_n = \frac{\mu_n}{\varphi_n}, \qquad \sigma_n = \frac{\lambda_n}{\theta_n}. \tag{5.2}$$

LEMMA 5.1. For $n \geqq 1$,

(i) *if* $\theta_n\varphi_n > 0$, *then* $|\theta_n| > \dfrac{1}{1-4k}$, $|\varphi_n| > \dfrac{1}{1-4k} > 15.87$, $|\tau_n| < 0.0668$ and $|\sigma_n| < 0.0668$.

(ii) *if* $\theta_n\varphi_n < 0$, *then* $\varphi_n > 2$, $|\tau_n| < 1-4k < 0.063$, $|\sigma_n| < 1-4k$; *furthermore if* $|\theta_n| > 30$, *then* $|\varphi_n| > 10$.

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PROOF. (i) Theorem 2.4 holds for $n \geqq 1$; thus if

$$|\theta_n| \leqq \frac{1}{1-4k},$$

$$M_n \leqq \frac{(|\theta_n|-1)(|\varphi_n|-1)}{4(|\theta_n\varphi_n|-1)} < \frac{|\theta_n|-1}{4|\theta_n|} \leqq k.$$

Hence

$$|\theta_n| > \frac{1}{1-4k}.$$

By symmetry the same result holds for $|\varphi_n|$, and so we may assume that

$$|\theta_n\varphi_n| > (15.87)^2 > 250.$$

Now if $|\tau_n| \geqq 0.0668$, Theorem 2.4 implies

$$M_n \leqq \frac{|\theta_n|(|\varphi_n|-|\mu_n|)}{4(|\theta_n\varphi_n|-1)} = \frac{|\theta_n\varphi_n|(1-|\tau_n|)}{4(|\theta_n\varphi_n|-1)}$$

$$< \frac{(250)(0.9332)}{4(249)} < k.$$

By symmetry the same result holds for $|\sigma_n|$.

(ii) When $\theta_n\varphi_n < 0$, let us suppose for definiteness that $a_{n+1} > 0$; then

$$\varphi_n \geqq 2 - \frac{1}{\varphi_{n+1}} > 2$$

whenever $\varphi_{n+1} < 0$. If $\varphi_{n+1} > 0$, then by (i) $\theta_{n+1} > 15$, and the first result follows. When $|\theta_n| > 30$, $|\varphi_n| < 10$, then again by Theorem 2.4, and the remark (5.1),

$$M_n \leqq \frac{(|\theta_n|+1)(|\varphi_n|-1)}{4(|\theta_n\varphi_n|+1)} < \frac{(31)(9)}{4(301)} < k.$$

When $|\tau_n| \geqq 1-4k$, as in (i),

$$M_n < \frac{|\theta_n\varphi_n|(1-|\tau_n|)}{4(|\theta_n\varphi_n|+1)} < \frac{1-|\tau_n|}{4} < k.$$

Similarly for $|\sigma_n|$.

## 6. Chains with $\varepsilon_0 \geqq 0$

In this paragraph we will show that if a chain is to be critical, then we must have $\varepsilon_0 < 0$.

Suppose then that $\varepsilon_0 \geqq 0$, then by (2.8) and (2.9), $\mu_0 > -1$; thus whenever $\theta_0 > 2.14$, (3.11) implies,

$$M_0^{(3)} < \frac{|\varphi_0|}{2(\theta_0|\varphi_0|+1)} < \frac{1}{2|\theta_0|} < k.$$

Consequently $\theta_0 < 2.14$, and thus $a_0 = 2$ or 3.

*When* $a_0 = 3$,

$$\theta_{-1} = \frac{1}{3-\theta_0} < 1.17,$$

and from (3.8), $\varepsilon_{-1} = -1$, implying that $-2 < \mu_{-1} < 0$. Now $\varphi_{-1} > 3$, and so by (3.12),

$$M_{-1}^{(2)} < \frac{(\theta_{-1}-1)(\varphi_{-1}+1)}{2(\theta_{-1}\varphi_{-1}-1)} < \frac{(0.17)(4)}{2(2.51)} < k.$$

*When* $a_0 = 2$, then by (3.8) $\varepsilon_{-1} = 0$, and by the argument of Lemma 5.1 (ii), we have $|\varphi_0| > 2$. Now if $\mu_0 > 0$,

$$\mu_{-1} = -\frac{\mu_0}{\varphi_0} > 0;$$

if, however, $\mu_0 < 0$, then the hypothesis that $\varepsilon_0 \geqq 0$ implies $\varepsilon_0 = 0$, and Lemma 5.1 implies

$$|\mu_{-1}| = \left|\frac{\tau_1}{\varphi_0}\right| < \frac{0.07}{2} < 0.04.$$

Thus in both cases $\mu_{-1} > -0.04$, and since $2 < \varphi_{-1} < 3$, when $\theta_{-1} < 2.2$,

$$M_{-1}^{(2)} < \frac{(\theta_{-1}-1)(\varphi_{-1}-0.96)}{2(\theta_{-1}\varphi_{-1}-1)} < \frac{(1.2)(2.04)}{2(5.6)} < k,$$

and when $\theta_{-1} > 4$,

$$M_{-1}^{(3)} < \frac{\varphi_{-1}+1.04}{2(\theta_{-1}\varphi_{-1}-1)} < \frac{3.04}{14} < k.$$

Whenever $2.2 < \theta_{-1} < 4$, then $1.54 < \theta_0 < 1.75$, and since $|\tau_1| < 0.07$, when $|\varphi_0| < 5$, then

$$M_0^{(3)} < \frac{|\varphi_0|-0.93}{2(\theta_0|\varphi_0|+1)} < \frac{4.07}{2(8.7)} < k,$$

and when $|\varphi_0| > 5$ then

$$M_0^{(2)} < \frac{(\theta_0-1)(|\varphi_0|+1.07)}{2(\theta_0|\varphi_0|+1)} < \frac{(0.75)(6.07)}{2(9.75)} < k.$$

This completes the exclusion of all cases when $\varepsilon_0 \geqq 0$. We have therefore shown that $\varepsilon_0 < 0$, and so $\mu_0 < 0$. Now by (3.8), $\varepsilon_n = (-1)^n|\varepsilon_n|$, for $n < 0$, and it is easily checked that, in fact, $\mu_n = (-1)^n|\mu_n|$, since $\mu_0 < 0$. This enables us to rewrite the formulae (3.11) and (3.12), with the sign associated with $|\mu_n|$ determined;

$$(6.1)\quad \begin{cases} M_0^{(1)} = \dfrac{\theta_0(|\varphi_0|-1-|\mu_0|)}{2(\theta_0|\varphi_0|+1)}, \quad M_0^{(2)} = \dfrac{(\theta_0-1)(|\varphi_0|+1+|\mu_0|)}{2(\theta_0|\varphi_0|+1)}, \\ M_0^{(3)} = \dfrac{|\varphi_0|-1+|\mu_0|}{2(\theta_0|\varphi_0|+1)} \end{cases}$$

and for $n < 0$,

$$(6.2)\quad \begin{cases} M_n^{(1)} = \dfrac{\theta_n(\varphi_n+1-|\mu_n|)}{2(\theta_n\varphi_n-1)}, \quad M_n^{(2)} = \dfrac{(\theta_n-1)(\varphi_n-1+|\mu_n|)}{2(\theta_n\varphi_n-1)}, \\ M_n^{(3)} = \dfrac{\varphi_n+1+|\mu_n|}{2(\theta_n\varphi_n-1)}. \end{cases}$$

## 7. Evaluation of $M$ for a certain chain

Designate by (c) the following chain pair.

$$(7.1)\quad \begin{array}{l} {}_\infty\begin{pmatrix} 5, & 5, & 2, & 5, & 5, & 2 \\ 3, & -3, & 0, & -3, & 3, & 0 \end{pmatrix} \begin{matrix} 3, & 4, & 4, & 2 \\ 1, & -2, & 2, & 0 \end{matrix}\Bigg| \\ \begin{matrix} -11, & 21, & 461, & -17, & 50, \\ -1, & -1, & -29, & -1, & -2, \end{matrix} \begin{pmatrix} 49, & -42, & 49, & -42 \\ -3, & 0, & 3, & 0 \end{pmatrix}_\infty, \end{array}$$

where the recurring segments in each direction are enclosed by the brackets. This section will be devoted to a proof that, for the corresponding $f$ and $\alpha$, $M(f;\alpha) = k$. The rest of Part II will show that (c) is in fact the critical chain for this problem.

Now $\theta_{-4} = [\overline{2, 5, 5}]$ satisfies the equation $8x^2-16x+3 = 0$, implying that

$$(7.2)\qquad \theta_{-4} = \frac{4+\sqrt{10}}{4} = 1.790569\cdots,$$

and

$$(7.3)\qquad \theta_0 = [2, 4, 4, 3, \theta_{-4}] = \frac{71\,\theta_{-4}-26}{41\,\theta_{-4}-15} = \frac{5195-2\sqrt{10}}{2997}.$$

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Similarly $\varphi_5 = \overline{[49, -42]}$ satisfies the equation $42\,x^2 - 2058\,x - 49 = 0$, implying

$$\varphi_5 = \frac{147+\sqrt{21651}}{6}. \tag{7.4}$$

The following general lemma will also be useful in a later paragraph.

LEMMA 7.1. *If we have a half-chain of the following form:*

$$\left| \begin{array}{ccccc} a, & -a_2, & a, & -a_4, & a, \cdots \\ \varepsilon, & 0, & -\varepsilon, & 0, & \varepsilon, \cdots \end{array} \right.$$

*where* $a_{2n+1} = a$, $\varepsilon_{2n} = (-1)^n \varepsilon$, $\varepsilon_{2n+1} = 0$ *for all* $n \geqq 0$, *and* $\varepsilon, a, a_{2n}$ *are all positive, and of the correct parity and size, then*

$$\tau_0 = a/\varepsilon.$$

PROOF. It is clear that

$$\mu_0 = \varepsilon \left(1 + \sum_{n=1}^{\infty} \frac{1}{|\varphi_1 \varphi_2 \cdots \varphi_{2n}|}\right),$$

since

$$|\mu_{2n}| = \varepsilon + \left|\frac{\tau_{2n+2}}{\varphi_{2n+1}}\right|.$$

Thus

$$|\tau_{2n}| - \varepsilon/a = \frac{|\tau_{2n+2}| - \varepsilon/a}{a|\varphi_{2n+1}|+1},$$

and since $a|\varphi_{2n+1}|+1 > 5$, for all $n \geqq 0$, then

$$|\tau_0 - \varepsilon/a| < \frac{1}{5^r},$$

for all $r$, which implies the result.

COROLLARY. *For the chain* $(c)$, $|\tau_5| = 3/49$.

Now from (3.9), $\lambda_0 = 1 - \theta_0$; using also (7.2), (7.3) and (7.4), we may compute the following table of truncated values for $(c)$.

TABLE 1

| $n$ | $\lvert\theta_n\rvert$ | $\lvert\sigma_n\rvert$ | $\lvert\varphi_n\rvert$ | $\lvert\tau_n\rvert$ |
|---|---|---|---|---|
| 0 | 1.7312 | 0.4223 | 11.0476 | 0.0864 |
| 1 | 11.5776 | 0.0498 | 20.9978 | 0.0446 |
| 2 | 21.0863 | 0.0497 | 461.0587 | 0.0630 |
| 3 | 460.9525 | 0.0628 | 17.0200 | 0.0564 |
| 4 | 17.0021 | 0.0551 | 49.9796 | 0.0387 |
| 5 | 50.0588 | 0.0410 | 49.0237 | 0.0612 |
| 6 | 48.9800 | 0.0604 | 42.0203 | 0.0014 |

We will show that $M_n > k$, $n \neq 1$, for this chain $(c)$. Formulae (2.10), (6.1) and (6.2) enable the calculation of $M_n^{(i)}$, and hence the verification of this statement. By direct calculation one can show that $M_n > k$ for $n = -1, 0, 2, 3, 4, 5$. To demonstrate the method we will show that $M_2 > k$. The other cases follow in a similar way.

Clearly $M_2^{(1)} > \frac{1}{4}$. Now

$$M_2^{(3)} > M_2^{(2)} > M_2^{(4)},$$

whenever

$$\varphi_2 - |\mu_2| > \frac{\theta_2}{|\lambda_2|-1} \quad \text{and} \quad \frac{\theta_2 - 1}{|\lambda_2|} > \frac{\varphi_2 - 1}{|\mu_2|}.$$

Reference to Table 1 shows that both these inequalities hold for the chain (c). Hence it follows that

$$\begin{aligned} M_2 = M_2^{(4)} &= \frac{(\theta_2+1-|\lambda_2|)(\varphi_2+1-|\mu_2|)}{4(\theta_2\varphi_2-1)} \\ &= \frac{(\theta_2-|\sigma_1|)(\varphi_2-28-|\tau_3|)}{4(\theta_2\varphi_2-1)} \\ &> \frac{(21.0364)(433.002)}{38884.1} > k. \end{aligned}$$

There remain the following cases.

(i) *Proof that $M_{2m} > k$, $m \geqq 3$.* For the purpose of an observation in Part III of this paper, we will show that (i) is also true for $a_{2m+1} = -44$. From the sign pattern of the chain, it follows that the four alternatives at each step $M_{2m}$, $m \geqq 3$, are, in some order, say,

$$(7.5) \qquad \left\{ \begin{aligned} M_{2m}^{(1)} &= \frac{(\theta_{2m}+1+|\lambda_{2m}|)(|\varphi_{2m}|-1-|\mu_{2m}|)}{4(\theta_{2m}|\varphi_{2m}|+1)}, \\ M_{2m}^{(2)} &= \frac{(\theta_{2m}-1+|\lambda_{2m}|)(|\varphi_{2m}|+1+|\mu_{2m}|)}{4(\theta_{2m}|\varphi_{2m}|+1)}, \\ M_{2m}^{(3)} &= \frac{(\theta_{2m}-1-|\lambda_{2m}|)(|\varphi_{2m}|+1-|\mu_{2m}|)}{4(\theta_{2m}|\varphi_{2m}|+1)}, \\ M_{2m}^{(4)} &= \frac{(\theta_{2m}+1-|\lambda_{2m}|)(|\varphi_{2m}|-1+|\mu_{2m}|)}{4(\theta_{2m}|\varphi_{2m}|+1)}. \end{aligned} \right.$$

From (7.1), Lemma (7.1) and Table 1,

$$2.9589 < |\lambda_6| \leqq |\lambda_{2m}| < 3 + \frac{3.007}{(42)(49)} < 3.0015,$$

$$|\mu_{2m}| = |\tau_{2m+1}| = \tfrac{3}{49} = 0.0612\cdots,$$

$$48.98 < \theta_{2m} < 49.024 \text{ and } 42.02 < |\varphi_{2m}| < 44.03.$$

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Clearly $M_{2m}^{(2)} > \frac{1}{4}$, and

$$M_{2m}^{(1)} > \frac{(\theta_{2m}+3.9)(|\varphi_{2m}|-1.1)}{4(\theta_{2m}|\varphi_{2m}|+1)} > \frac{(53.9)(40.9)}{4[(50)(42)+1]} > k,$$

$$M_{2m}^{(3)} > \frac{(\theta_{2m}-4.0015)(|\varphi_{2m}|+0.938)}{4(\theta_{2n}|\varphi_{2m}|+1)} > \frac{(44.9785)(44.968)}{4[(48.98)(44.03)+1]} > k,$$

$$M_{2m}^{(4)} > \frac{(\theta_{2m}-2.002)(|\varphi_{2m}|-0.939)}{4(\theta_{2m}|\varphi_{2m}|+1)} > \frac{(46.978)(41.081)}{4[(48.98)(42.02)+1]} > k.$$

Thus, even when $a_{2m+1} = -44$, $M_{2m} > k$ for $m \geqq 3$.

(ii) *Proof that* $M_{2m+1} > k, m \geqq 3$. Allowing again $a_{2m+1}$ to take the additional value $-44$, we obtain the result analogously. Let $r = 2m+1$, then

$$(7.6)\quad \begin{cases} M_r^{(1)} = \dfrac{(|\theta_r|-1+|\lambda_r|)(\varphi_r+1-|\mu_r|)}{4(|\theta_r|\varphi_r+1)}, \\ M_r^{(2)} = \dfrac{(|\theta_r|+1+|\lambda_r|)(\varphi_r-1+|\mu_r|)}{4(|\theta_r|\varphi_r+1)}, \\ M_r^{(3)} = \dfrac{(|\theta_r|+1-|\lambda_r|)(\varphi_r-1-|\mu_r|)}{4(|\theta_r|\varphi_r+1)}, \\ M_r^{(4)} = \dfrac{(|\theta_r|-1-|\lambda_r|)(\varphi_r+1+|\mu_r|)}{4(|\theta_r|\varphi_r+1)}. \end{cases}$$

We easily obtain the following bounds;

$$0.0604 < |\lambda_7| \leqq |\lambda_r| < \tfrac{1}{49}\left(3+\frac{0.07}{42}\right) < 0.0613,$$

$$3.001 < |\mu_r| = 3+\frac{3}{49|\varphi_{r+1}|} < 3.0015,$$

$$42.02 < |\theta_r| < 44.03 \text{ and } 49.02 < \varphi_r < 49.024.$$

Clearly $M_r^{(2)} > \frac{1}{4}$, and

$$M_r^{(1)} > \frac{(|\theta_r|-0.94)(\varphi_r-2.002)}{4(|\theta_r|\varphi_r+1)} > \frac{(41.08)(47.018)}{4[(42.02)(49.02)+1]} > k,$$

$$M_r^{(3)} > \frac{(|\theta_r|+0.938)(\varphi_r-4.002)}{4(|\theta_r|\varphi_r+1)} > \frac{(44.968)(45.018)}{4[(44.03)(49.02)+1]} > k,$$

$$M_r^{(4)} > \frac{(|\theta_r|-1.062)(\varphi_r+4)}{4(|\theta_r|\varphi_r+1)} > \frac{(40.958)(53.03)}{4[(42.02)(49.03)+1]} > k.$$

The result follows.

We treat the left hand chain in a slightly different way.

(iii) *Proof that* $M_m^{(1)} > k$, $m \leqq -2$. Using (7.2),

$$[\overline{5,\ 5,\ 2}] = \frac{2(4+\sqrt{10})}{3} = 4.77485\cdots$$

$$[\overline{5,\ 2,\ 5}] = \frac{7+2\sqrt{10}}{3} = 4.44151\cdots$$

Whenever $a_{m+1} = 2, 3, 4$, we have $\varphi_m < 4$, hence from (2.9), (6.2),

$$M_m^{(1)} > \frac{\theta_m}{\theta_m\varphi_m-1} > \frac{1}{\varphi_m} > \tfrac{1}{4}.$$

If, however, $a_{m+1} = 5$, then either $\varphi_m < [5, 2, 5] < 4.45$ and $\theta_m < 4.45$ or $\varphi_m < [5, 5, 2] < 4.78$ and $\theta_m < 1.8$. In both cases $M_m^{(1)} > k$.

(iv) *Proof that* $M_m^{(2)} > k$, $m \leqq -2$. By Theorem 3.2, for $m \leqq -2$, $|\mu_m| = \varphi_m - 1 - \alpha_m$, where

$$\alpha_m = \frac{\varphi_{-1}-1-|\mu_{-1}|}{\varphi_{m+1}\varphi_{m+2}\cdots\varphi_{-1}}.$$

The sequence $\{\alpha_m\}$ is monotone decreasing as $m \to -\infty$,

$$\alpha_{-2} < \frac{\varphi_{-1}-1}{\varphi_{-1}} < \frac{1.1}{2.1} < 0.6,$$

and for $m \leqq -4$,

$$\alpha_m \leqq \alpha_{-4} < \frac{\varphi_{-1}-1}{\varphi_{-3}\varphi_{-2}\varphi_{-1}} = \frac{\varphi_{-1}-1}{15\varphi_{-1}-4} < 0.04.$$

Now when $m = -2, -3$, then $\theta_m > 2$, $\varphi_m > 3$; thus

$$M_m^{(2)} = \frac{(\theta_m-1)(\varphi_m-1-\alpha_m/2)}{(\theta_m\varphi_m-1)} > \frac{1.7}{5} > k.$$

When $m \leqq -4$, $\theta_m > [\overline{2,\ 5,\ 5}] > 1.79$, $\varphi_m > [2, 3, 3] > 1.62$ and hence

$$M_m^{(2)} > \frac{(\theta_m-1)(\varphi_m-1.02)}{(\theta_m\varphi_m-1)} > \frac{(0.79)(0.6)}{1.9} > k.$$

(v) *Proof that* $M_m^{(3)} > k$, $m \leqq -2$. *When* $m = -2, -3$, *then*

$$|\mu_m| > 2,\quad \theta_m < 4,\quad \text{and}\quad \varphi_m < [4, 4] = 3.75;$$

thus

$$M_m^{(3)} > \frac{\varphi_m+3}{2(\theta_m\varphi_m-1)} > \frac{6.75}{28} > k.$$

When $m \leqq -4$, we have $\alpha_m < 0.04$ and

$$M_m^{(3)} > \frac{\varphi_m - 0.02}{\theta_m \varphi_m - 1}.$$

When $a_m = 5$, $a_{m+1} = 5$, then $\varphi_m < 5$, $\theta_m = [\overline{5, 2, 5}] < 4.45$, and

$$M_m^{(3)} > \frac{4.98}{21.25} > k.$$

When $a_m = 5, a_{m+1} = 2$, then $\varphi_m < [2, 5, 5] < 1.792$, $\theta_m = [\overline{5, 5, 2}] < 4.775$ and

$$M_m^{(3)} > \frac{1.772}{7.557} > k.$$

When $a_m = 2$, then $\theta_m < 2$ and

$$M_m^{(3)} > \frac{1}{\theta_m} > k.$$

(vi) *Proof that* $M_1 = k$. It is clear that $M_1^{(3)} > k$, and

$$M_1^{(4)} > M_1^{(1)} > M_1^{(2)}$$

whenever

$$\frac{\varphi_1+1}{|\mu_1|} > \frac{|\theta_1|-1}{|\lambda_1|},$$

and

$$|\theta_1| - |\lambda_1| > \frac{\varphi_1}{1+|\mu_1|}.$$

Table 1 implies that these conditions are satisfied. Now since $|\lambda_1| = 1/\theta_0$, then

$$M_1^{(2)} = \frac{3(\varphi_1 - 2 + |\tau_2|)}{|\theta_1|\varphi_1 + 1}. \tag{7.7}$$

We have

$$\varphi_1 = [21, 461, -17, 50, \varphi_5] = \frac{8238730\, \varphi_5 - 164581}{392361\, \varphi_5 - 7838},$$

and

$$|\tau_2| = \frac{24727\, \varphi_5 - 494 + |\mu_5|}{392361\, \varphi_5 - 7838}.$$

Thus from (7.2), (7.4), (7.7) and Lemma 7.1,

$$M_1^{(2)} = \frac{\frac{3}{49}(366458018\, \varphi_5 - 7320551)}{(8238730|\theta_1| + 392361)\varphi_5 - (164581|\theta_1| + 7838)} = k.$$

The result now follows.

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## 8. Isolation of the value of $a_0$

We now continue to restrict the possible values that can occur in a chain $\{a_{n+1}, \varepsilon_n\}$ for which $M \geqq k$. Now by Lemma 5.1, and (5.1), since

$$|\lambda_1| = |\varepsilon_0| + \frac{1-\theta_0}{\theta_0}$$

and $|a_1| - |\varepsilon_0| + 1 > 0$, we have

$$0.0668 > \left|\frac{\lambda_1}{\theta_1}\right| = \frac{\theta_0(|\varepsilon_0|-1)+1}{\theta_0|a_1|+1} > \frac{|\varepsilon_0|-1}{|a_1|} > \frac{|\mu_0|-1.0668}{|\varphi_0|+1/|\varphi_1|}.$$

Hence

$$(8.1) \qquad |\mu_0| < 0.0668|\varphi_0| + 1.14$$

whenever $\theta_0 > 2.3$, (6.1) implies

$$M_0^{(3)} < \frac{1.0668|\varphi_0|+0.14}{2(2.3|\varphi_0|+1)} < \frac{1.0668}{4.6} < k.$$

Thus $\theta_0 < 2.3$, and if $a_0 = 3$, then

$$\theta_{-1} = \frac{1}{3-\theta_0} < 1.43.$$

Now as $\varepsilon_0 \leqq -1$, we have by Lemma 5.1,

$$|\varphi_0| \geqq 3 + \frac{1}{\varphi_1} > 2.9.$$

Consequently by (8.1),

$$|\mu_{-1}| = 1 + \left|\frac{\mu_0}{\varphi_0}\right| < 1.0668 + \frac{1.14}{|\varphi_0|} < 1.5.$$

Since $\varphi_{-1} > 3$, we have from (6.2)

$$M_{-1}^{(2)} < \frac{(\theta_{-1}-1)(\varphi_{-1}+0.5)}{2(\theta_{-1}\varphi_{-1}-1)} < \frac{(0.43)(3.5)}{6.58} < k.$$

Hence we may enunciate the following result.

THEOREM 8.1 *Any critical chain has* $a_0 = 2$.

## 9. Isolation of the value of $a_1$

In this section we will use the following temporary notation: $a = |a_1|$, $\varepsilon = |\varepsilon_0|$, $c = a - \varepsilon$. By (2.6) and (3.8) it follows that $c$ is even, and furthermore $c \geqq 2$. The following series of lemmas provide bounds on the value of $c$.

LEMMA 9.1. $$\frac{c(c+2)}{c+1} \geqq 4k\left(|\varphi_0|+\frac{1}{\theta_0}\right).$$

LEMMA 9.2. $$c < |\varphi_0|-|\mu_0|+0.063.$$

LEMMA 9.3. *If* $$v = 4k\left(|\varphi_0|+\frac{1}{\theta_0}\right),$$

*then* $$c \geqq xv,$$

*where $x$ is the positive root of $vx^2-(v-2)x-1 = 0$ i.e.*

$$x = \frac{v-2+\sqrt{v^2+4}}{2v}. \tag{9.1}$$

NOTE. Since $dx/dv > 0$, $x$ is an increasing function of $v$, and hence we may replace $v$ in (9.1) by some lower bound of $4k(|\varphi_0|+1/\theta_0)$.

PROOF OF LEMMA 9.1. Using the basic recurrence relations between the variables at consecutive values of their index, we obtain:

$$\begin{aligned} M_1 &\leqq \min\{M_1^{(1)}, M_1^{(2)}\} \\ &= \frac{1}{4(\theta_0|\varphi_0|+1)}\min\{\theta_0 c|(|\varphi_0|-c-|\mu_0|+1)|,\ \theta_0(c+2)|(1-|\varphi_0|+c+|\mu_0|)|\} \end{aligned}$$

Now from (2.9), and since we are supposing $M_1^{(1)} \geqq k$, $M_1^{(2)} \geqq k$, then, by addition,

$$\frac{1}{\theta_0 c}+\frac{1}{\theta_0(c+2)} \leqq \frac{|(|\varphi_0|-c-|\mu_0|+1)+(1-|\varphi_0|+c+|\mu_0|)|}{4k(\theta_0|\varphi_0|+1)}$$

and the result follows.

PROOF OF LEMMA 9.2. From the basic relations it follows that

$$c = |\varphi_0|-|\mu_0|+\frac{\mu_1-1}{\varphi_1}.$$

When $\varphi_1 > 0$, the result follows from Lemma 5.1. If $\varphi_1 < 0$, and $\mu_1 > 0$, the result again follows from Lemma 5.1, since $\theta_1\varphi_1 > 0$.

In the final case when $\varphi_1 < 0$ and $\mu_1 < 0$, then the result holds if

$$\frac{1+|\mu_1|}{|\varphi_1|} \leqq 1-4k < 0.063.$$

If, however

$$\frac{1+|\mu_1|}{|\varphi_1|} > 1-4k,$$

then

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$$M_1^{(1)} < \frac{(|\theta_1|-1)(|\varphi_1|-1-|\mu_1|)}{4(|\theta_1\varphi_1|-1)} < \frac{(|\theta_1|-1)4k|\varphi_1|}{4(|\theta_1\varphi_1|-1)} < k.$$

PROOF OF LEMMA 9.3. Lemma 9.1 implies that

$$c \geqq \frac{c+1}{c+2}\cdot v,$$

which inequality holds if and only if

$$c^2+(v-2)c-v \geqq 0.$$

Since $c > 0$, then we have

$$c \geqq \frac{v-2+\sqrt{v^2+4}}{2v}\cdot v = xv.$$

The following three lemmas enable us to restrict the range of values taken by $a$.

LEMMA 9.4. $\theta_0 < 1.7382$ *whenever* $|\varphi_0| > 20$.

LEMMA 9.5. $a \leqq 22$.

LEMMA 9.6. $a$ *is odd, and satisfies* $11 \leqq a \leqq 21$; *furthermore if* $\varphi_1 < 0$, *then* $17 \leqq a \leqq 21$.

PROOF OF LEMMA 9.4. Using $\theta_0 < 2$, we find that $v > 19.2$, and by Lemma 9.3, and the subsequent note,

$$c > (0.95)v > (0.89)(|\varphi_0|+\tfrac{1}{2}).$$

Together with Lemma 9.2, this implies that

$$|\mu_{-1}| = |\tau_0| < 0.11.$$

Thus whenever $\theta_{-1} > 3.8191$, since $\varphi_{-1} > 2$, (6.2) implies

$$M_{-1}^{(3)} < \frac{\varphi_{-1}+1.11}{2(\theta_{-1}\varphi_{-1}-1)} < \frac{3.11}{13.2764} < k,$$

whence

$$\theta_{-1} < 3.8191, \quad \text{or} \quad \theta_0 < 1.7382.$$

PROOF OF LEMMA 9.5. When $a \geqq 23$, Lemmas 5.1, 9.4 imply $v > 4k(22.937+0.5753) > 22.0314$, and so using Lemma 9.3,

$$c > (0.8964)\left(|\varphi_0| + \frac{1}{\theta_0}\right).$$

Together with Lemma 9.2 this gives

$$|\mu_0| < 0.1036|\varphi_0|-0.45. \tag{9.2}$$

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When $\theta_{-1} > 3.8123$, since $\varphi_{-1} > 2$, (6.2) implies

$$M_{-1}^{(3)} < \frac{\varphi_{-1}+1.1036}{2(\theta_{-1}\varphi_{-1}-1)} < \frac{3.1036}{13.2492} < k,$$

and when $\theta_{-1} < 3.8123$, then $\theta_0 < 1.7377$, and by (6.1)

$$M_0^{(2)} < \frac{(0.7377)(1.1036|\varphi_0|+0.55)}{2(1.7377|\varphi_0|+1)}$$
$$< \frac{(0.7377)(1.1036)}{3.4754} < k.$$

Proof of Lemma 9.6. From Lemma 9.1 we have

$$|\varphi_0| < \frac{1}{4k}\frac{c(c+2)}{c+1} - \tfrac{1}{2} < U,$$

where $U$ is some convenient upper bound. Tabulating these results for $c = 2, 4, \cdots, 20$, we obtain

Table 2

| $c$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| $U$ | 2.35 | 4.63 | 6.82 | 8.99 | 11.15 | 13.30 | 15.44 | 17.59 | 19.73 | 21.87 |

If $a$ is even, from § 6 we know that $\varepsilon \geqq 2$, and so $a \geqq c+2$. Hence $|\varphi_0| > a-0.063$, and

$$c+1.937 < |\varphi_0| < U.$$

Inserting the values from Table 2, we obtain a contradiction in each case, thus excluding the possibility of $a$ being even.

Similarly when $a$ is odd $a \geqq c+1$, and so

$$c+0.937 < |\varphi_0| < U,$$

which provides a contradiction from Table 2, for all $c \leqq 6$. Hence $a$ is odd, and $9 \leqq a \leqq 21$. If $\varphi_1 > 0$, then Table 2 implies that $a \geqq 11$. If $\varphi_1 < 0$, then $\theta_1 < 0$, and Lemma 5.1 implies that $|\theta_1| > 15.8$, and it is easily checked that if $a = 15$, then $M_0^{(2)} < k$. This completes the proof.

Lemma 9.7.

$$M < \frac{(3|\varphi_0|+1+|\mu_0|)(|\varphi_0|+1+|\mu_0|)}{2\{7|\varphi_0|^2+(3|\mu_0|+7)|\varphi_0|+2|\mu_0|+2\}},$$

*the right hand side increases with $|\mu_0|$, and decreases with $|\varphi_0|$.*

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PROOF. The method is similar to that of Lemma 9.1. Using the basic formulae of Part 1, we see that

$$M \leqq \min \{M_{-1}^{(3)}, M_0^{(2)}\}$$
$$= \frac{1}{2(\theta_0|\varphi_0|+1)} \min \{(2-\theta_0)(3|\varphi_0|+1+|\mu_0|), (\theta_0-1)(|\varphi_0|+1+|\mu_0|)\}.$$

Now since $M_{-1}^{(3)}$ decreases, and $M_0^{(2)}$ increases as a function of $\theta_0$, then their minimum cannot exceed their common value, which occurs at

$$\theta_0 = \frac{7|\varphi_0|+3+3|\mu_0|}{4|\varphi_0|+2+2|\mu_0|}.$$

The result follows by substitution. The function clearly increases in $|\mu_0|$. Since $|\varepsilon_0| \geqq 1$, the derivative with respect to $|\varphi_0|$ is seen to be negative, whereby the function decreases in $|\varphi_0|$.

THEOREM 9.1. *Any critical chain has*

$$a_1 = -11, \quad \varepsilon_0 = -1 \quad \text{and} \quad a_2 > 0.$$

PROOF. From Table 2, it is clear that for all $a$ that remain, $\varepsilon = 1$, else a contradiction is obtained as in Lemma 9.6.

Hence by (3.8)

$$|\lambda_1| = 1 + \frac{1-\theta_0}{\theta_0} = \frac{1}{\theta_0}. \tag{9.3}$$

Suppose that $13 \leqq a \leqq 21$. If either $\varepsilon_1 = 0$ or $\mu_1/\varphi_1 < 0$, then by Lemma 5.1,

$$|\mu_0| = 1 + \frac{\mu_1}{\varphi_1} < 1.04.$$

Now $|\varphi_0| > 13$, (since if $a = 13$, we may suppose by Lemma 9.6 that $\varphi_1 > 0$), and so substituting these bounds in Lemma 9.7,

$$M < \frac{(41.04)(15.04)}{2637} < k.$$

If, however $\mu_1/\varphi_1 > 0$ and $\varepsilon_1 \neq 0$, then we may consider the three cases:

(i) $\varphi_1 < 0$; then (9.3) implies

$$M_1^{(1)} < \frac{(a-1)(|\varphi_1|-1-|\mu_1|)}{4(|\theta_1\varphi_1|-1)} < \frac{a-1}{4(a+0.5)} < \frac{5}{21.5} < k.$$

(ii) $0 < \varphi_1 < 10$: then by Lemma 5.1,

$$|\mu_1| < 0.07|\varphi_1| < 0.7,$$

contradicting $|\varepsilon_1| \geqq 1$.

(iii) $\varphi_1 > 10$: since $\varepsilon_1 \neq 0$,

$$M_1^{(1)} < \frac{(a-1)(|\varphi_1|+1-|\mu_1|)}{4(|\theta_1\varphi_1|+1)}$$

$$< \frac{(20)(10.07)}{4((21.5)(10)+1)} < k.$$

Thus we conclude that $a = 11$, $\varepsilon = 1$, and hence $\varphi_1 > 0$.

## 10. Isolation of the value of $a_2$

Lemma 10.1.
$1.7165 < \theta_0 < 1.73251$, *whence* $0.5771 < 1/\theta_0 < 0.5826$; *also* $\varepsilon_1 < 0$.

Proof. Now since by Theorem 9.1, $11 < |\varphi_0| < 12$, then $\varphi_{-1} > 2.0833$, and

$$|\mu_{-1}| < \frac{1.0668}{11} < 0.097.$$

Thus if $\theta_{-1} > 3.7384$,

$$M_{-1}^{(3)} < \frac{2.0833+1.097}{2(6.7882)} < k.$$

Hence we have $\theta_0 < 1.73251$, and $1/\theta_0 > 0.5771$.

We have $c = 10$ and Lemma 9.1 implies

$$|\varphi_0| < 11.6424-0.5771 = 11.0653.$$

Consequently $\varphi_1 > 15$, and if $\varepsilon_1 \geqq 0$, by (9.3),

$$M_1^{(1)} < \frac{10(\varphi_1+1.0668)}{4(|\theta_1|\varphi_1+1)} < \frac{160.668}{4\{(15)(11.577)+1\}} < k.$$

Thus $\varepsilon_1 < 0$, and hence $|\mu_0| = 1+\mu_1/\varphi_1 < 1$.

If $\theta_0 < 1.7165$, since $|\varphi_0| > 11$,

$$M_0^{(2)} < \frac{(\theta_0-1)(|\varphi_0|+2)}{2(\theta_0|\varphi_0|+1)} < \frac{(0.7165)(13)}{2(19.8815)} < k.$$

The complete lemma now follows.

Lemma 10.2. *In any critical chain*

$$\varepsilon_1 = -1.$$

Proof. Suppose $\varepsilon_1 \leqq -2$, then $|\mu_1| > 1.933$, and

$$M_1^{(4)} < \frac{(a-1+2/\theta_0)(\varphi_1-0.933)}{4\{(a+1/\theta_0)\varphi_1+1\}}.$$

This is an increasing function of $1/\theta_0$, and so by Lemma 10.1, when $\varphi_1 < 36$,

$$M_1^{(4)} < \frac{(11.166)(\varphi_1-0.933)}{4(11.5826\,\varphi_1+1)} < \frac{(11.166)(35.067)}{1671.8} < k,$$

and when $\varphi_1 > 36$, by Theorem 2.4,

$$M_1 < \frac{(|\theta_1|-1)(\varphi_1+1)}{4(|\theta_1|\varphi_1+1)} < \frac{(10.59)(37)}{4\{(11.59)(36)+1\}} < k.$$

The result now follows from the previous lemma.

THEOREM 10.1. *Any critical chain has*

$$a_2 = 21,\ |\mu_1| < 1,\ a_3 > 0 \text{ and } \varepsilon_2 < 0.$$

PROOF. Consider the following two cases.

(i) $|\mu_1| > 1$. Since Lemma 5.1 implies that $|\mu_1| < 1.0668$, Lemma 10.1 and (9.3) give when $\varphi_1 > 23.5$,

$$M_1^{(1)} < \frac{10(\varphi_1+2.0668)}{4(11.577\,\varphi_1+1)} < \frac{255.668}{1092.2} < k,$$

and when $\varphi_1 < 21.5$,

$$M_1^{(2)} < \frac{12(\varphi_1-2)}{4(11.577\,\varphi_1+1)} < \frac{58.9}{249.9} < k.$$

Since $|\varepsilon_1| = 1$, then $a_2 = 23$. Now $|\mu_1| > 1$ implies $\mu_2/\varphi_2 > 0$, and since $|\lambda_2| > 1$, when $\varphi_2 > 0$,

$$M_2^{(3)} < \frac{(\theta_2-1-|\lambda_2|)(\varphi_2-1)}{4(\theta_2\varphi_2-1)} < \frac{(\theta_2-2)}{4\theta_2} < \frac{21.1}{92.4} < k,$$

and when $\varphi_2 < 0$, $\varepsilon_2 \neq 0$, then $|\varphi_2| > 2$ implies

$$M_2^{(3)} < \frac{(\theta_2-2)(|\varphi_2|+0.067)}{4(\theta_2|\varphi_2|+1)} < \frac{(21.1)(2.067)}{188.8} < k.$$

If, however, $\varphi_2 < 0$, $\varepsilon_2 = 0$, then $\varphi_1 > 23$ and $|\mu_1| < 1+|\tau_3/\varphi_2| < 1.04$; hence

$$M_1^{(1)} < \frac{10(\varphi_1+2.04)}{4(11.577\,\varphi_1+1)} < \frac{250.4}{1069} < k.$$

It follows that for critical chains we must have the case:

*(ii)* $|\mu_1| \leqq 1$. *Since Lemma 5.1 implies that* $|\mu_1| > 0.933$ *when* $\varphi_1 > 22.6$,

$$M_1^{(1)} < \frac{10(\varphi_1+2)}{4(11.577\,\varphi_1+1)} < \frac{246}{1050.5} < k,$$

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and when $\varphi_1 < 20.5$,

$$M_1^{(2)} < \frac{12(\varphi_1 - 1.933)}{4(11.577\,\varphi_1 + 1)} < \frac{222.9}{953.3} < k.$$

Thus we conclude that $a_2 = 21$, and if $a_3 < 0$, then $\mu_2 \geqq 0$. Now

$$|\lambda_2| = 1 + \frac{1}{11\,\theta_0 + 1} > 1.049,$$

and $\theta_2 < 21.1$; thus when $|\mu_2| + 1 \geqq 0.061|\varphi_2|$,

$$\begin{aligned} M_2^{(4)} &< \frac{(\theta_2 - 0.049)(0.939)|\varphi_2|}{4(\theta_2|\varphi_2| + 1)} \\ &< \frac{(21.051)(0.939)}{84.4} < k, \end{aligned}$$

and when $|\mu_2| + 1 < 0.061|\varphi_2|$,

$$\begin{aligned} M_1^{(2)} &\leqq \frac{12\left(19 + \dfrac{|\mu_2| + 1}{|\varphi_2|}\right)}{4\left\{\left(21 + \dfrac{1}{|\varphi_2|}\right)|\theta_1| + 1\right\}} \\ &< \frac{3(19.061)}{244.11} < k. \end{aligned}$$

Thus we have that $a_3 > 0$ and $\mu_2 \leqq 0$. If $\varepsilon_2 = 0$, then $|\mu_2/\varphi_2| < 0.04$, and since $\varphi_1 < 21$,

$$M_1^{(2)} < \frac{3(\varphi_1 - 1.96)}{|\theta_1|\varphi_1 + 1} < \frac{3(19.04)}{244} < k.$$

## 11. The maximal chain for $\theta_0$

We will now examine possible $a$-chains as $n \to -\infty$.

Lemma 11.1. $\theta_0 < 1.73134$.

Proof: We have

$$|\mu_{-1}| = \frac{\varphi_1 - |\mu_1|}{11\,\varphi_1 + 1} < \frac{21 - 0.9332}{232} < 0.086495,$$

and

$$\varphi_{-1} = 2 + \frac{\varphi_1}{11\,\varphi_1 + 1} > 2 + \frac{20.9}{230.9} > 2.090515,$$

since $20.9 < \varphi_1 < 21$.

If $\theta_0 > 1.73134$, then $\theta_{-1} > 3.72217$, and so

$$M_{-1}^{(3)} < \frac{3.17701}{2\{(2.090515)(3.72217)-1\}} < k.$$

The following two lemmas will enable us to determine the chain which gives the maximal allowable value for $\theta_0$.

LEMMA 11.2. $\theta_0 \leqq [2, 4, 4, 3, 2, \theta_{-5}]$.

LEMMA 11.3. $a_n \leqq 5$ *for* $n \leqq -1$.

PROOF OF LEMMA 11.2. This results from the fact that a semi-regular expansion to the integer above is an increasing function of $a_n$, if $a_1, \cdots, a_{n-1}$ remain fixed, and $a_{n+1}, a_{n+2}, \cdots$ take arbitrary positive integral values [3]. We note that

$$1.73134 = [2, 4, 4, 3, 2, 19, \cdots],$$

and so the result follows from Lemma 11.1.

PROOF OF LEMMA 11.3. If, for some $n \leqq -1$, $1/\theta_n > 0.766$, then by (2.9),

$$M_n^{(2)} < \frac{(\theta_n-1)(\varphi_n-1)}{\theta_n\varphi_n-1} < \frac{\theta_n-1}{\theta_n} < k.$$

Thus we have, by symmetry, that both $1/\theta_n$ and $1/\varphi_n$ are less than 0.766.

Now, if for any $n \leqq -2$, we have $a_n \geqq 6$, then

$$M_n^{(3)} < \frac{\varphi_n}{\theta_n\varphi_n-1} = \frac{1}{a_n-\left(\dfrac{1}{\theta_{n-1}}+\dfrac{1}{\varphi_n}\right)} < \frac{1}{4.468} < k.$$

THEOREM 11.1. *The maximal chain for* $\theta_0$ *is*

$$[2, 4, 4, 3, \overline{2, 5, 5}] = |\theta_0|_c,$$

*where the subscript c refers to the value of the variable for the chain* (*c*).

PROOF. The previous lemma shows that $a_{-5}$ and $a_{-6}$ cannot exceed 5, and by the argument of Lemma 11.2, we make $\theta_0$ largest by taking each partial quotient as large as possible. Put $a_{-5} = a_{-6} = 5$.

Now $\varphi_{-6} > [5, 2, 3, 3] = \frac{57}{13}$, and if $\theta_{-6} > 4.5$,

$$M_{-6}^{(3)} < \frac{\varphi_{-6}}{\theta_{-6}\varphi_{-6}-1} < \tfrac{114}{487} < k.$$

But if $\theta_{-6} < 4.5$, then whenever $a_{-6} = 5$, we have $\theta_{-7} < 2$.

The result follows by a simple inductive process.

The importance of this result will become evident later, when we show that the minimum of the critical chain is taken at $M_1^{(2)}$ which is an increasing function of $\theta_0$.

LEMMA 11.4. $\theta_0 \leqq \dfrac{5195-2\sqrt{10}}{2997} = 1.7312897\cdots,$

*which implies* $1/\theta_0 > 0.57760405\cdots$.

This is a corollary of Theorem 11.1, and follows as in (7.3).

## 12. Isolation of the value of $a_3$

LEMMA 12.1. *$M_2^{(2)}$ and $M_2^{(4)}$ decrease and increase, respectively, when $\theta_0$ increases.*

PROOF. We may write $M_2^{(2)}$ and $M_2^{(4)}$ as functions of $\theta_0$ by using the basic relations and sign pattern of the chain already known, together with $|\lambda_1| = 1/\theta_0$. For example

$$M_2^{(2)} = \frac{\left(21+\dfrac{\theta_0+1}{11\ \theta_0+1}\right)(\varphi_2-1-|\mu_2|)}{\dfrac{\theta_0\varphi_2}{11\ \theta_0+1}+21\ \varphi_2-1},$$

and the result follows by the comment (5.1). Similarly for $M_2^{(4)}$.

LEMMA 12.2. $436 < \varphi_2 < 470$.

LEMMA 12.3. $0.063 < |\mu_2/\varphi_2| < 0.06316$.

PROOF OF LEMMA 12.2. If $\varphi_2 > 470$, then $\varphi_1 = 2-1/\varphi_2 > 20.99787$. When $|\mu_2|-1 \leqq 0.06088\ \varphi_2$, we have by Lemma 11.4,

$$M_1^{(2)} = \frac{3\left(19+\dfrac{|\mu_2|-1}{\varphi_2}\right)}{|\theta_1|\varphi_1+1} < \frac{3(19.06088)}{(11.57760405)(20.99787)+1} < k$$

and when $|\mu_2|-1 > 0.06088\ \varphi_2$, then Lemmas 11.4 and 12.1 imply

$$M_2^{(4)} < \frac{(21.036484)(0.93912)(470)}{4\{(21.08637)(470)-1\}} < k.$$

If $\varphi_2 < 436$, then $\varphi_1 < 20.99771$. Thus when $|\mu_2| \leqq 0.06303\ \varphi_2$,

$$M_1^{(2)} = \frac{3(\varphi_1-2+|\tau_2|)}{|\theta_1|\varphi_1+1} < \frac{3(19.06074)}{(11.57760405)(20.99771)+1} < k$$

and when $|\mu_2| > 0.06303\ \varphi_2$, Lemmas 10.1 and 12.1 imply

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$$M_2^{(2)} < \frac{(21.1367)(0.93697\,\varphi_2-1)}{4(21.08633\,\varphi_2-1)} < k.$$

Hence

$$436 < \varphi_2 < 470.$$

Proof of Lemma 12.3. Since we now have $\varphi_1 < 20.997873$, the inequalities $|\mu_2/\varphi_2| \leqq 0.063$ and $|\mu_2/\varphi_2| \geqq 0.06316$ imply that $M_1^{(2)}$ and $M_2^{(4)}$, respectively, do not exceed $k$, by the method of the previous lemma.

Lemma 12.4. *For any critical chain* $\varepsilon_2 = -29$.

Proof. If $|\varepsilon_2| \leqq 27$, then

$$\left|\frac{\mu_2}{\varphi_2}\right| < \frac{27.0668}{436} < 0.063,$$

and if $|\varepsilon_2| \geqq 30$, then

$$\left|\frac{\mu_2}{\varphi_2}\right| > \frac{29.93}{470} > 0.06316,$$

and both these cases contradict Lemma 12.3. There remains only to exclude the possibility $\varepsilon_2 = -28$.

In this case, if $a_3 \geqq 446$, then

$$\left|\frac{\mu_2}{\varphi_2}\right| < \frac{28.0668}{445.93} < 0.063,$$

and if $a_3 \leqq 442$, we obtain

$$\left|\frac{\mu_2}{\varphi_2}\right| > \frac{27.93}{442.1} > 0.06316,$$

which are again contradictions of Lemma 12.3. Thus we have $a_3 = 444$, whenever $\varepsilon_2 = -28$, (since $\varepsilon_2$ and $a_3$ have the same parity).

Now $\varphi_2 < 444.1$, and so $\varphi_1 < 20.9977483$. We have $|\mu_2/\varphi_2| < 0.06307$, for if not, as in the previous lemmas,

$$M_2^{(2)} < \frac{(21.1367)(0.93693\,\varphi_2-1)}{4\{21.08633\,\varphi_2-1\}} < k.$$

When $|\theta_1| > 11.57763$,

$$M_1^{(2)} < \frac{3(19.0608183)}{(11.57763)(20.9977483)+1} < k,$$

and when $|\theta_1| < 11.57763$, then $\theta_0 > 1.73121$; now if $|\mu_2/\varphi_2| \geqq 0.06305$, by Lemma 12.1,

$$M_2^{(2)} < \frac{(21.13627)(0.93695\,\varphi_2-1)}{4(21.08637\,\varphi_2-1)} < k.$$

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Thus we have, in this case,

$$\left|\frac{\mu_2}{\varphi_2}\right| < 0.06305. \tag{12.1}$$

As in the proof of Lemma 5.1, since $\theta_3 > 440$, we have $|\varphi_3| > 12$. Thus $\varphi_2 < 444.09$, and whenever $|\mu_2| \geqq 28$, we have

$$\left|\frac{\mu_2}{\varphi_2}\right| > \frac{28}{444.09} > 0.06305,$$

contrary to (12.1).

Hence we have $|\mu_2| < 28$, which implies $\mu_3/\varphi_3 < 0$. We have two cases.

(i) $\varphi_3 < 0$, $(\mu_3 > 0)$. Now by (9.3), let

$$\alpha = \frac{|\lambda_2|-1}{\theta_2} = \frac{1}{|\theta_0\theta_1\theta_2|}.$$

Then $\alpha < 0.003$. When $|\mu_3|+1 \geqq 0.003|\varphi_3|$,

$$M_3^{(4)} \leqq \frac{(417+\alpha)(0.997)}{4\theta_3} < \frac{(417.003)(0.997)}{4(443.95)} < k,$$

and when $|\mu_3|+1 < 0.003|\varphi_3|$;

$$\left|\frac{\mu_2}{\varphi_2}\right| > \frac{27.997|\varphi_3|+1}{444|\varphi_3|+1} > \frac{27.997}{444} > 0.06305,$$

contradicting (12.1).

(ii) $\varphi_3 > 0$, $(\mu_3 < 0)$. If $|\mu_3| \leqq 1.03$, since we may suppose by Lemma 5.1 that $\varphi_3 > 15$, by the above method,

$$M_3^{(3)} < \frac{(415.003)(15.03)}{4\{(443.95)(15)-1\}} < k.$$

Thus if $|\varepsilon_3| = 1$, we have $1.03 < |\mu_3| < 1.07$. Consequently when $\varphi_3 < 31$,

$$\left|\frac{\mu_2}{\varphi_2}\right| < \frac{28\,\varphi_3-1}{444\,\varphi_3-1} < \tfrac{867}{13763} < 0.063,$$

and when $\varphi_3 > 31$,

$$M_3^{(3)} < \frac{(415.003)(31.07)}{4\{(443.95)(31)-1\}} < k.$$

Suppose that $|\varepsilon_3| \geqq 2$, then $|\mu_3| > 1.9332$. Thus when $\varphi_3 < 380$,

$$M_3^{(4)} < \frac{(417.003)(\varphi_3-0.93)}{4(443.95\,\varphi_3-1)} < k,$$

and when $\varphi_3 > 380$, we have the following two subcases:

(a) If $|\mu_3|-1 \geqq 0.003\,\varphi_3$, then

$$M_3^{(4)} < \frac{(417.003)(0.997)\varphi_3}{4(443.95\,\varphi_3-1)} < \frac{(415.752)(380)}{4\{(443.95)(380)-1\}} < k.$$

(b) If $|\mu_3|-1 < 0.003\varphi_3$, then $|\mu_3/\varphi_3| < 0.003+1/\varphi_3 < 0.0057$, and so

$$\left|\frac{\mu_2}{\varphi_2}\right| > \frac{27.9943}{444} > 0.06305,$$

contradicting (12.1). The result is now complete, since $\varepsilon_2$ cannot have the value $-28$.

THEOREM 12.1. *Any critical chain has*

$$\varepsilon_2 = -29, \quad a_3 = 461, \quad a_4 < 0, \quad \text{and} \quad \mu_3 < 0.$$

PROOF. If $a_3 \geqq 463$, then

$$\left|\frac{\mu_2}{\varphi_2}\right| < \frac{29.07}{462.9} < 0.063,$$

and if $a_3 \leqq 459$ and $|\mu_2| \geqq 29$, then

$$\left|\frac{\mu_2}{\varphi_2}\right| > \frac{29}{459.1} > 0.06316;$$

both cases contradict Lemma 12.3. Whenever $|\mu_2| < 29$, then we have $\mu_3/\varphi_3 < 0$. If $a_3 \leqq 457$, then

$$\left|\frac{\mu_2}{\varphi_2}\right| > \frac{28.93}{457.1} > 0.06316,$$

again a contradiction.

Further, when $a_3 = 459$, ($|\mu_2| < 29$), if $|\mu_2/\varphi_2| \geqq 0.063043$, then we have, after Lemma 12.1,

$$M_2^{(4)} < \frac{(21.036484)\{(458.93)(0.936957)+1\}}{4\{(21.0863736)(458.93)-1\}} < k.$$

If, however

$$\left|\frac{\mu_2}{\varphi_2}\right| < 0.063043, \tag{12.2}$$

then we may consider the two cases.

(i) $\varphi_3 < 0$, ($\mu_3 > 0$). When $|\mu_3/\varphi_3| \geqq 0.005$, then since $|\lambda_3| > 28.93$, and $\theta_3 < 460$,

$$M_3^{(4)} < \frac{(\theta_3+1-|\lambda_3|)(0.995|\varphi_3|-1)}{4(\theta_3|\varphi_3|+1)} < \frac{(432.07)(0.995)}{4(460)} < k;$$

when $|\mu_3/\varphi_3| < 0.005$, then

$$\left|\frac{\mu_2}{\varphi_2}\right| > \frac{28.995}{459.1} > 0.0631,$$

contrary to (12.2).

(ii) $\varphi_3 > 0$, $(\mu_3 < 0)$. As in Lemma 5.1, we have since $\theta_3 > 458$, that $\varphi_3 > 15$, and hence $\theta_3\varphi_3 > 6870$. Thus, whenever $|\mu_3/\varphi_3| > 0.06315$, we have from Theorem 2.4,

$$M_3 < \frac{|\theta_3\varphi_3|(0.93685)}{4(|\theta_3\varphi_3|-1)} < \frac{(6870)(0.93685)}{4(6869)} < k.$$

If, however, $|\mu_3/\varphi_3| \leqq 0.06315$, then

$$\left|\frac{\mu_2}{\varphi_2}\right| > \frac{28.93685}{459} > 0.063043,$$

contradicting (12.2).

We therefore conclude that $a_3 = 461$. If $|\mu_2| \leqq 29$, then

$$\left|\frac{\mu_2}{\varphi_2}\right| < \frac{29}{460.93} < 0.063,$$

a contradiction. When $|\mu_2| > 29$, then $\mu_3/\varphi_3 > 0$, and if $\varphi_3 > 0$,

$$M_3^{(3)} < \frac{\theta_3-1-|\lambda_3|}{4\theta_3} < \frac{431.003}{4(460.95)} < k.$$

The complete theorem now follows.

## 13. Isolation of the value of $a_4$

We can immediately show the following result.

THEOREM 13.1. *Any critical chain has*

$$a_4 = -17, \quad \varepsilon_3 = -1, \quad a_5 > 0, \quad \textit{and} \quad \mu_4 < 0.$$

PROOF. Suppose that $|\mu_3| \geqq 1$, then if again $\alpha = 1/|\theta_0\theta_1\theta_2|$,

$$M_3^{(3)} = \frac{(431+\alpha)(|\varphi_3|+1-|\mu_3|)}{4(\theta_3|\varphi_3|+1)} < \frac{431.003}{4(460.95)} < k.$$

Now if $\varepsilon_3 = 0$, by Lemma 5.1, we have

$$\left|\frac{\mu_3}{\varphi_3}\right| < \frac{0.07}{10} = 0.007,$$

and hence

$$\left|\frac{\mu_2}{\varphi_2}\right| < \frac{27.007}{461} < 0.063,$$

contradicting Lemma 12.3. Thus

$$\varepsilon_3 = -1, \text{ and } \mu_4/\varphi_4 < 0. \tag{13.1}$$

As in Lemma 5.1, if $|\varphi_3| < 15.34$, Theorem 2.4 implies

$$M_3 < \frac{(\theta_3+1)(|\varphi_3|-1)}{4(\theta_3|\varphi_3|+1)} < \frac{(461)(14.34)}{4\{(460)(15.34)+1\}} < k.$$

It easily follows that $a_4 \neq -15$, since $|\varphi_4| > 3$. Hence $|a_4| \geqq 17$.

Now since $\varphi_2 < 461.07$, then $\varphi_1 < 20.9978312$, and if $|\mu_2/\varphi_2| \leqq 0.0630211$, then as usual,

$$M_1^{(2)} < \frac{3(19.0608523)}{(11.57760405)(20.9978312)+1} < k.$$

When $|\mu_2/\varphi_2| > 0.0630211$, then

$$29 + \left|\frac{\mu_3}{\varphi_3}\right| > (0.0630211)\left(461 + \frac{1}{|\varphi_3|}\right);$$

if $|\varphi_3| > 17.1$, then it follows that $|\mu_3| > 0.964$, and so

$$M_3^{(3)} < \frac{(431.003)(17.1+0.036)}{4\{(460.95)(17.1)+1\}} < k.$$

Consequently $|a_4| \leqq 17$, and hence $a_4 = -17$. If $a_5 < 0$, then (13.1) implies $\mu_4 > 0$, and we can distinguish two cases.

(i) When $|\mu_4/\varphi_4| \geqq 0.0405$,

$$M_3^{(4)} < \frac{(433.003)(|\varphi_3|-|\tau_4|)}{4(460.95|\varphi_3|+1)} < \frac{(433.003)(16.9595)}{4\{(460.95)(17)+1\}} < k.$$

(ii) When $|\mu_4/\varphi_4| < 0.0405$, since $|\theta_4| < 17.1$,

$$M_4^{(1)} = \frac{(|\theta_4|-1-|\lambda_4|)(|\varphi_4|-1+|\mu_4|)}{4(|\theta_4\varphi_4|-1)} < \frac{(17.1-1.93)(1.0405)}{4(17.1)} < k.$$

Thus we have $a_5 > 0$, and the theorem is complete.

## 14. Isolation of the value of $a_5$

We prove the following succession of lemmas.

LEMMA 14.1. $0.03838 < |\mu_4/\varphi_4| < 0.040406$, *and* $\varepsilon_4 \neq 0$.

LEMMA 14.2. $\varphi_4 < 55$.

LEMMA 14.3. $\varepsilon_4 = -2$.

LEMMA 14.4. $a_5 = 48, 50,$ or $52$.

PROOF OF LEMMA 14.1. Now $\theta_3 = 461 - 1/\theta_2 > 460.9525$, and $|\varphi_3| > 17$; hence if $|\mu_4/\varphi_4| \leqq 0.03838$,

$$M_3^{(3)} < \frac{(436.0024)(17.03838)}{31348.77} < k.$$

Thus $|\mu_4/\varphi_4| > 0.03838$.

If $\varepsilon_4 = 0$, then $|\mu_4/\varphi_4| = |\tau_5/\varphi_4| < 0.034$, contradicting this result. Thus we have $|\mu_4| > 0.93$, and if $|\varphi_4| < 29$, we obtain

$$M_4^{(2)} < \frac{(|\theta_4|+0.07)(\varphi_4-1.93)}{4(|\theta_4|\varphi_4+1)} < \frac{(17.07)(27.07)}{4\{(17)(29)+1\}} < k.$$

Supposing $|\varphi_4| > 29$, then if $|\mu_4/\varphi_4| \geqq 0.040406$,

$$M_3^{(4)} < \frac{(433.0024)(17.035-0.040406)}{4\{(460.9525)(17.035)+1\}} < k.$$

PROOF OF LEMMA 14.2. Since $|\lambda_2| > 1$,

$$|\sigma_3| < \frac{29\theta_3-1}{461\theta_3-1} < 0.063,$$

and $|\theta_4| < 17.003$. If $\varphi_4 > 55$, then by the previous result,

$$\begin{aligned} M_4^{(1)} &< \frac{(|\theta_4|-2+|\sigma_3|)(1.04041\varphi_4+1)}{4(|\theta_4|\varphi_4+1)} \\ &< \frac{(15.066)\{(1.04041)(55)+1\}}{4\{(17.003)(55)+1\}} \\ &< k. \end{aligned}$$

PROOF OF LEMMA 14.3. After Lemma 14.1, $|\varepsilon_4| \geqq 1$. We may suppose (as in the proof of Lemma 14.1) that $|\varphi_4| > 29$, and consequently

$$1.1 < (0.03838)(29) < |\mu_4| < (0.04041)(55) < 2.3.$$

The result follows by Lemma 5.1.

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PROOF OF LEMMA 14.4. If $a_5 \leqq 46$, then $\varphi_4 < 46.1$, and

$$\left|\frac{\mu_4}{\varphi_4}\right| > \frac{1.93}{46.1} > 0.041.$$

Similarly, if $a_5 = 54$,

$$\left|\frac{\mu_4}{\varphi_4}\right| < \frac{2.0668}{53.933} < 0.03833.$$

Both these results contradict Lemma 14.1, and the lemma follows since $a_5$ must be even.

THEOREM 14.1. *Any critical chain has*

$$a_5 = 50, \quad |\mu_4| < 2, \quad \textit{and} \quad \varepsilon_5 \neq 0.$$

PROOF. Suppose that $|\mu_4/\varphi_4| \geqq 0.03945$. By the previous lemma $|\varphi_3| > 17.01919$, and so

$$\left|\frac{\mu_2}{\varphi_2}\right| = \frac{29|\varphi_3|+|\mu_3|}{461|\varphi_3|+1} < 0.06302112.$$

Also $\varphi_2 < 461.05876$, implying $\varphi_1 < 20.99783108$. Thus by Lemma 11.4,

$$M_1^{(2)} < \frac{3(19.0608522)}{20.99783108|\theta_1|+1} < k.$$

(14.1) Hence
$$\left|\frac{\mu_4}{\varphi_4}\right| < 0.03945.$$

Suppose that $a_5 = 52$. Now $\theta_4 < 17.0022$, and

$$|\sigma_3| < \frac{29\theta_2-1}{461\theta_2-1} < 0.062815,$$

since $\theta_2 < 22$. If $0 < \varphi_5 < 20$, then Theorem 2.4 implies

$$M_5 < \frac{(\theta_5-1)(\varphi_5-1)}{4(\theta_5\varphi_5-1)} < \frac{(51.1)(19)}{4\{(52.1)(20)-1\}} < k.$$

Thus whatever the sign of $\varphi_5$, we have $\varphi_4 > 51.95$, and by (14.1)

$$\begin{aligned} M_4^{(1)} &< \frac{(|\theta_4|-2+|\sigma_3|)(1.03945\varphi_4+1)}{4(|\theta_4|\varphi_4+1)} \\ &< \frac{(15.06502)\{(1.03945)(51.95)+1\}}{4\{(17.0022)(51.95)+1\}} \\ &< k. \end{aligned}$$

Thus $a_5 \neq 52$.

(i) If $a_5 = 48$, then

$$\left|\frac{\mu_4}{\varphi_4}\right| > \frac{1.93}{48.1} > 0.04.$$

(ii) If $a_5 = 50$, and either $|\mu_4| > 2$, or $\varepsilon_5 = 0$, then by Lemma 5.1,

$$\left|\frac{\mu_4}{\varphi_4}\right| > \frac{1.993}{50.1} > 0.0397.$$

In both cases (i) and (ii) (14.1) is contradicted, and so the theorem holds.

## 15. Structure of a critical chain for $n \geqq 6$

We will show in this section that if $M_n \geqq k$, for $n \geqq 6$, then the chain must have a certain periodic character. Throughout the section we will consider the chain for $\theta_0$ to be held constant.

LEMMA 15.1. *If $|\mu_2|$ and $|\tau_2|$ both increase, then $M_1^{(2)}$ increases.*

LEMMA 15.2. *We have $a_6 > 0$.*

LEMMA 15.3. *If $|\mu_5|$ and $|\tau_5|$ both increase, then $M_1^{(2)}$ increases.*

PROOF OF LEMMA 15.1. We may write

$$M_1^{(2)} = \frac{3(\varphi_1-2+|\tau_2|)}{|\theta_1|\varphi_1+1} = \frac{3(19\varphi_2+|\mu_2|-1)}{(21|\theta_1|+1)\varphi_2-|\theta_1|},$$

from which the result follows for the cases when $\varphi_1$ increases and $\varphi_2$ decreases, respectively.

PROOF OF LEMMA 15.2. Suppose to the contrary that $a_6 < 0$. Then, since $\mu_5/\varphi_5 < 0$, we have $\mu_5 > 0$. Clearly $|\lambda_5| > 2.05$, and $|\theta_5| < 50.1$. If $|\mu_5/\varphi_5| \geqq 0.043$, then

$$M_5^{(4)} = \frac{(\theta_5+1-|\lambda_5|)(|\varphi_5|-1-|\mu_5|)}{4(\theta_5|\varphi_5|+1)} < \frac{(49.05)(0.957)}{4(50.1)} < k.$$

Then we have $|\mu_5/\varphi_5| < 0.043$, and so

$$\left|\frac{\mu_4}{\varphi_4}\right| > \frac{1.957}{50.1} > 0.039.$$

Since $|\varphi_3| > 17.0199$, then we see, as on previous occasions,

$$\left|\frac{\mu_2}{\varphi_2}\right| = \frac{29|\varphi_3|+1-|\tau_4|}{461|\varphi_3|+1} < 0.063021173.$$

Similarly, $|\mu_2| = 29+|\mu_3/\varphi_3| < 29.05647$. If we calculate to sufficient accuracy the corresponding values for the chain (c) of § 7, we obtain

$$(15.1) \qquad \left|\frac{\mu_2}{\varphi_2}\right|_c = 0.0630211983\cdots, \quad |\mu_2|_c = 29.056475\cdots,$$

where the subscript $c$, will denote, as before, the value of the particular variable for the chain ($c$). Thus in the case when $a_6 < 0$, we have

$$|\mu_2| < |\mu_2|_c, \quad \left|\frac{\mu_2}{\varphi_2}\right| < \left|\frac{\mu_2}{\varphi_2}\right|_c,$$

which by Lemma 15.1 implies

$$M_1^{(2)} < (M_1^{(2)})_c = k.$$

PROOF OF LEMMA 15.3. By the previous lemma, and the information already known about the chain,

$$|\mu_2| = 29 + \frac{48\varphi_5+|\mu_5|-1}{851\varphi_5-17}.$$

Since $|\mu_5| > 0.9$, the remark (5.1) implies that $|\mu_2|$ will increase if $\varphi_5$ decreases. If $\varphi_5$ decreases, then $\varphi_2$ will decrease, and so $|\mu_2/\varphi_2|$ will increase.

If, however, $\varphi_5$ increases, then since

$$|\mu_2| = 29 + \frac{(48+|\tau_5|)\varphi_5-1}{851\varphi_5-17},$$

which increases in $\varphi_5$, as $851-17(48+|\tau_5|) > 0$, we have again that $|\mu_2|$ increases. It is easily checked that when $|\tau_5|$ and $\varphi_5$ increase, $|\mu_3|$ increases and $|\varphi_3|$ decreases, implying that

$$\left|\frac{\mu_2}{\varphi_2}\right| = \frac{29|\varphi_3|+|\mu_3|}{461|\varphi_3|+1}$$

will increase.

The complete result now follows from Lemma 15.1.

LEMMA 15.4. *We have that* $|\mu_5/\varphi_5| < 0.062$.

PROOF. Assume that $|\mu_5/\varphi_5| \geqq 0.062$. Now since $50.0588 < \theta_5 < 50.0589$, and

$$0.0551 < \frac{460-29}{(17)(460)+1} < |\sigma_4| = \frac{\theta_3-|\lambda_3|}{17\theta_3+1} < 0.0552,$$

we have, when $\varphi_5 > 53.5$,

$$M_5^{(4)} \leqq \frac{(\theta_5-1-|\sigma_4|)(0.938\varphi_5+1)}{4(\theta_5\varphi_5-1)} < \frac{(49.0038)(51.183)}{10708} < k,$$

and when $\varphi_5 < 48.1$,

$$M_5^{(2)} \leqq \frac{(\theta_5+1+|\sigma_4|)(0.938\varphi_5-1)}{4(\theta_5\varphi_5-1)} < \frac{(51.114)(44.1178)}{9627} < k.$$

Thus we have that $49 \leqq a_6 \leqq 53$. Hence we may deduce from the inequalities

$$0.062 \leqq \left|\frac{\mu_5}{\varphi_5}\right| < 0.0668,$$

that

(15.2) $$3.034 < |\mu_5| < 3.6,$$

whence $\varepsilon_5 = -3$, $a_6$ is odd, and $\mu_6/\varphi_6 > 0$.

If $a_6 \geqq 51$, then $|\mu_5| > (50.9)(0.062) > 3.1$, contradicting Lemma 5.1. Clearly $\varepsilon_6 \neq 0$, else $|\mu_5| = 3+|\mu_6/\varphi_6| < 3.01$, contradicting (15.2). Thus $|\mu_6| > 0.93$, and since $\theta_6 < 49$, and $|\varphi_6| > 10$ (by Lemma 5.1), we have whatever the sign of $\varphi_6$,

$$M_6^{(3)} < \frac{(\theta_6-1-|\lambda_6|)(|\varphi_6|+0.07)}{4(\theta_6|\varphi_6|-1)} < \frac{(45.07)(10.07)}{1956} < k.$$

This completes the lemma, and we may now isolate the values of $a_6$, $\varepsilon_5$, and $\varepsilon_6$.

Theorem 15.1. *Any critical chain has*

$$a_6 = 49, \quad \varepsilon_5 = -3, \quad \varepsilon_6 = 0, \quad \mu_6 < 0, \quad \textit{and} \quad a_7 < 0.$$

Proof. Suppose that $|\mu_5/\varphi_5| \leqq 0.06$, then

$$\left|\frac{\mu_4}{\varphi_4}\right| > \frac{1.94}{50} > 0.0388,$$

and so $|\mu_3| < 0.9612$.

Now $|\varphi_3| > 17.02$, implying that

$$\left|\frac{\mu_3}{\varphi_3}\right| < \frac{0.9612}{17.02} < 0.056475.$$

We also have

$$\left|\frac{\mu_2}{\varphi_2}\right| < \frac{29(17.02)+0.9612}{461(17.02)+1} < 0.063021198,$$

and so, from (15.1), $|\tau_2| < |\tau_2|_c$, $|\mu_2| < |\tau_2|_c$, and thus $M_1^{(2)} < k$.

We therefore have

(15.3) $$0.06 < \left|\frac{\mu_5}{\varphi_5}\right| < 0.062.$$

We use the method of the previous lemma to determine bounds on $a_6$. When $\varphi_5 > 59.5$,

$$M_5^{(4)} < \frac{(49.0038)(0.94\varphi_5+1)}{4(50.0589\varphi_5-1)} < \frac{2789.8}{11910} < k,$$

and when $\varphi_5 < 43.5$,

$$M_5^{(2)} < \frac{(51.114)(0.94\varphi_5-1)}{4(50.0588\varphi_5-1)} < \frac{2039}{8706} < k.$$

Thus we have $44 \leqq a_6 \leqq 59$, and by (15.3) it follows that $2.6 < |\mu_5| < 3.7$, whence $\varepsilon_5 = -3$. By Lemma 5.1, $2.933 < |\mu_5| < 3.067$; thus if $a_6 \geqq 53$ or $a_6 \leqq 47$, then

$$\left|\frac{\mu_5}{\varphi_5}\right| < \frac{3.067}{52.9} < 0.06 \quad \text{or} \quad \left|\frac{\mu_5}{\varphi_5}\right| > \frac{2.933}{47.1} > 0.062,$$

respectively, and both contradict (15.3). Thus $a_6 = 49$ or $51$. Consider the following two cases.

(i) $|\mu_5| \leqq 3$, $(\tau_6 \leqq 0)$. If $\varphi_5 > 49$, then $|\mu_5/\varphi_5| < \frac{3}{49} < |\mu_5/\varphi_5|_c$ and Lemma 15.1 implies $M_1^{(2)} < k$. If $\varphi_5 < 49$, then $\varphi_6 > 0$, and so $\mu_6 \leqq 0$. Hence when $|\mu_6| \leqq 0.018\varphi_6$,

$$M_6^{(3)} < \frac{(\theta_6-4+|\sigma_5|)(1.018\varphi_5-1)}{4(\theta_6\varphi_6-1)} < \frac{(45.1)(1.018)}{4(49)} < k,$$

and when $|\mu_6| > 0.018\varphi_6$, we have

$$\left|\frac{\mu_5}{\varphi_5}\right| < \frac{3-0.018}{48.93} < 0.061 < \left|\frac{\mu_5}{\varphi_5}\right|_c,$$

which, together with $|\mu_5| \leqq 3$, again implies that $M_1^{(2)} < k$.

(ii) $|\mu_5| > 3$, $(\tau_6 > 0)$. Now if $\varepsilon_6 \neq 0$, then $|\mu_6| > 0.93$, and since we have $\theta_6 < 51$ and by Lemma 5.1, $|\varphi_6| > 10$,

$$M_6^{(3)} < \frac{(\theta_6-4+|\sigma_5|)(|\varphi_6|+1-|\mu_6|)}{4(\theta_6|\varphi_6|-1)} < \frac{(47.07)(10.07)}{4\{(51)(10)-1\}} < k.$$

Hence $\varepsilon_6 = 0$. Thus when $a_6 = 51$, we have, as usual,

$$|\tau_6| < 0.007, \quad \text{and} \quad |\tau_5| < \frac{3.007}{50.93} < 0.06,$$

contradicting (15.3).

Thus we have $a_6 = 49$, $\varepsilon_6 = 0$, and $|\mu_5| > 3$. If $a_7 > 0$, then $\mu_6 > 0$, and

$$M_6^{(3)} < \frac{\theta_6 - 4 + |\sigma_5|}{4\theta_6} < \frac{45.1}{196} < k.$$

The theorem now follows.

We will now prove the main result of this section. It will fix the structure of any critical chain pair for $n \geqq 5$. The proof that the chain (c) of § 7 is in fact *the* critical chain, will be a corollary of this result.

THEOREM 15.2. *In any critical chain, we have for* $n \geqq 3$,

$$a_{2n} = 49, \quad \varepsilon_{2n-1} = (-1)^n 3, \quad \varepsilon_{2n} = 0, \quad \textit{and}$$
$$a_{2n+1} = -42, -44, \textit{ or } -46.$$

REMARK. One may also easily exclude $a_{2n+1} = -46$, in this result, but it is not necessary for the proof of the corollary.

PROOF. Suppose that $|\mu_7/\varphi_7| \leqq 0.06$. Now since $\theta_4 > 17$ and $|\sigma_3| > 0.05$, we have

$$|\sigma_5| = \frac{2|\theta_4| + |\lambda_4|}{50|\theta_4| + 1} < \frac{(2)(17) + 0.95}{(50)(17) + 1} < 0.04107.$$

Also

$$|\theta_6| < [49, 50, -17] < 48.98003.$$

When $|\varphi_6| < 40.085$, since $|\mu_6| = |\tau_7| \leqq 0.06$,

$$M_6^{(4)} = \frac{(\theta_6 - 2 + |\sigma_5|)(|\varphi_5| - 1 + |\mu_6|)}{4(\theta_6|\varphi_6| + 1)} < \frac{(47.0211)(39.145)}{4\{(48.98003)(40.085) + 1\}} < k.$$

If $|\varphi_6| > 40.085$, and $a_7 = -40$, then $0 < \varphi_7 < 11.8$, implying by Theorem 2.4, that $M_7 < k$. Thus we have $|\varphi_6| > 41.9$, and so

$$|\mu_5| = 3 + \left|\frac{\tau_7}{\varphi_6}\right| < 3 + \frac{0.06}{41.9} < 3.00144 < |\mu_5|_c,$$

and

$$\left|\frac{\mu_5}{\varphi_5}\right| \leqq \frac{3|\varphi_6| + 0.06}{49|\varphi_6| + 1} < \tfrac{3}{49} = \left|\frac{\mu_5}{\varphi_5}\right|_c.$$

Consequently, by Lemma 15.3, we have

$$0.06 < \left|\frac{\mu_7}{\varphi_7}\right| < 0.067.$$

Using the above bounds on $\theta_6$ and $|\sigma_5|$, we have when $|\varphi_6| > 47.9$,

$$M_6^{(3)} < \frac{(45.0211)(47.9 + 0.94)}{4\{(48.98003)(47.9) + 1\}} < k$$

and when $|\varphi_6| < 39$,

$$M_6^{(4)} < \frac{(47.0211)(39-0.933)}{4\{(48.98003)(39)+1\}} < k.$$

Since $\varepsilon_6 = 0$, then $|a_7|$ is even, and $40 \leqq |a_7| \leqq 46$.

It is more difficult to exclude $|a_7| = 40$ in this case, than when $n > 3$, because of the sign of $a_5$. However, if we can prove that

$$(15.4) \qquad \left|\frac{\mu_7}{\varphi_7}\right| < 0.06135, \quad \text{and} \quad |\varphi_6| < 40.03,$$

then

$$M_6^{(4)} < \frac{(47.0211)(39.09135)}{4\{(48.98003)(40.03)+1\}} < k.$$

Let us consider the following inductive hypothesis:

(15.5) *For all* $3 \leqq n \leqq m$, *with* $m, n$ *integral, suppose* $a_{2n} = 49$, $\varepsilon_{2n-1} = (-1)^n 3$, $\varepsilon_{2n} = 0$ *with* $(-1)^n \mu_{2n} > 0$, $|\tau_{2n+1}| > 0.06$, *and* $a_{2n+1} = -42, -44$ *or* $-46$.

As in § 7, it follows that for all such $n$ the products are given by (7.5) and (7.6). We will observe the same notations. The hypothesis holds for $n = 3$, by Theorem 15.1, and the above observations, with the proviso that $a_7$ may also take the value $-40$. Thus, wherever appropriate, we will use the less stringent bound $|a_{2n+1}| \geqq 40$, until we have shown that $|a_7| \geqq 42$ (that is, until the condition (15.4) is satisfied).

The proof follows the method of the previous theorem. Suppose that $a_{2m+2} < 0$, then since $|\theta_{2m+1}| < 46.1$ and $(\lambda_{2m+1})(\mu_{2m+1}) < 0$, we have

$$M_{2m+1} < \frac{(|\theta_{2m+1}|-1+|\lambda_{2m+1}|)(|\varphi_{2m+1}|-1-|\mu_{2m+1}|)}{4(|\theta_{2m+1}\varphi_{2m+1}|-1)}$$

$$< \frac{(46.1-0.93)(0.94)}{4(46.1)} < k.$$

Hence we have

$$(15.6) \qquad a_{2m+2} > 0, \quad \text{and thus} \quad \mu_{2m+1} = (-1)^{m+1}|\mu_{2m+1}|.$$

Now

$$(15.7) \qquad \begin{aligned} 0.06 < |\lambda_7| \leqq |\lambda_{2m+1}| < \frac{3.007}{49} < 0.0614 \\ 40.02 < |\theta_{2m+1}| < 46.03. \end{aligned}$$

Using the notation (7.6), we therefore have when $\varphi_{2m+1} > 59.5$,

$$M_{2m+1}^{(1)} < \frac{(45.0914)(0.94\varphi_{2m+1}+1)}{4(46.03\,\varphi_{2m+1}+1)} < \frac{(45.0914)(56.93)}{10959} < k,$$

and when $\varphi_{2m+1} < 41.5$,

$$M^{(3)}_{2m+1} < \frac{(|\theta_{2m+1}|+1-|\sigma_{2m}|)(0.94\,\varphi_{2m+1}-1)}{4(|\theta_{2m+1}|\varphi_{2m+1}+1)} < \frac{(40.96)(38.01)}{6647} < k.$$

Hence we have $42 \leqq a_{2m+2} \leqq 59$. Now by Lemma 5.1, and the inductive hypothesis

$$0.06 < \left|\frac{\mu_{2m+1}}{\varphi_{2m+1}}\right| < 0.063,$$

implying $2.5 < |\mu_{2m+1}| < 3.8$, whence $\varepsilon_{2m+1} = (-1)^{m+1}3$. We consider the following cases, which eventually isolate the value of $a_{2m+2}$.

(i) If $a_{2m+2} \geqq 53$, then

$$\left|\frac{\mu_{2m+1}}{\varphi_{2m+1}}\right| < \frac{3.067}{52.9} < 0.06.$$

(ii) If $a_{2m+2} = 51$, and either $|\mu_{2m+1}| < 3$, or $\varepsilon_{2m+2} = 0$, then

$$\left|\frac{\mu_{2m+1}}{\varphi_{2m+1}}\right| < \frac{3.007}{50.93} < 0.06.$$

Both (i) and (ii) contradict (15.5).

(iii) If $a_{2m+2} = 51$ or $49$, and $|\mu_{2m+1}| > 3$, with $\varepsilon_{2m+2} \neq 0$, then $(-1)^m \mu_{2m+2}/\varphi_{2m+2} > 0$, by (15.6). Thus whatever the sign of $\varphi_{2m+2}$, or parity of $m$, we have

$$M_{2m+2} < \frac{(\theta_{2m+2}-1-|\lambda_{2m+2}|)(|\varphi_{2m+2}|+1-|\mu_{2m+2}|)}{4(\theta_{2m+2}|\varphi_{2m+2}|-1)}.$$

Now since $|\mu_{2m+2}| > 0.93$, $|\varphi_{2m+2}| > 10$, and $|\lambda_{2m+2}| > 3$, then

$$M_{2m+2} < \frac{(51.1-4)(10.07)}{2040} < k.$$

We have therefore excluded $a_{2m+2} \geqq 51$, and also $a_{2m+2} = 49$ when $|\mu_{2m+1}| > 3$ and $|\varepsilon_{2m+2}| \geqq 1$.

In what follows we will supose, as in § 7, that without loss of generality, $m$ is odd.

(iv) If $a_{2m+2} \leqq 45$, then since $|\mu_{2m+1}| > 2.93$, and by the bounds (15.7), we have

$$M^{(3)}_{2m+1} < \frac{(40.96)(45.1-3.93)}{4\{(40.02)(45.1)+1\}} < k.$$

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(v) Suppose $a_{2m+2} = 47$, and $\varphi_{2m+2} > 0$. Then if $|\mu_{2m+1}| \geqq 2.948$, we have

$$M^{(3)}_{2m+1} < \frac{(40.96)(47-3.948)}{4\{(40.02)(47)+1\}} < k.$$

When $|\mu_{2m+1}| < 2.948$, we have $|\tau_{2m+2}| > 0.052$, and $(-1)^{m+1}\mu_{2m+2} > 0$, by (15.6). In fact, by Lemma 5.1, $\varphi_{2m+2} > 15$, and $|\lambda_{2m+2}| > 3$. Now when $|\mu_{2m+2}| < 1.07$,

$$M^{(2)}_{2m+2} < \frac{(\theta_{2m+2}-1-|\lambda_{2m+2}|)(\varphi_{2m+2}+0.07)}{4(\theta_{2m+2}\varphi_{2m+2}-1)}$$
$$< \frac{(43.1)(15.07)}{4\{(47.1)(15)-1\}} < k.$$

If $|\varepsilon_{2m+2}| \geqq 2$, then $|\mu_{2m+2}| > 1.93$, and when $\varphi_{2m+2} < 40$,

$$M^{(1)}_{2m+2} < \frac{(\theta_{2m+2}+1-|\lambda_{2m+2}|)(\varphi_{2m+2}+1-|\mu_{2m+2}|)}{4(\theta_{2m+2}\varphi_{2m+2}-1)}$$
$$< \frac{(45.1)(41-1.93)}{4\{(47.1)(40)-1\}} < k,$$

and when $\varphi_{2m+2} > 40$, then since $|\mu_{2m+2}/\varphi_{2m+2}| > 0.052$, we have

$$M^{(1)}_{2m+2} < \frac{(45.1)\{(0.948)(40)+1\}}{4\{(47.1)(40)-1\}} < k.$$

This excludes the case $\varphi_{2m+1} < 47$.

(vi) Suppose $a_{2m+2} = 47$, and $\varphi_{2m+2} < 0$, then as in (v), if $|\mu_{2m+1}| \geqq 2.978$, since $\varphi_{2m+1} < 47.1$, we obtain $M^{(3)}_{2m+1} < k$. When $|\mu_{2m+1}| < 2.978$, then $(-1)^m\mu_{2m+2} > 0$, and $|\tau_{2m+2}| > 0.022$. Hence

$$M^{(1)}_{2m+2} = \frac{(\theta_{2m+2}+1-|\lambda_{2m+2}|)(|\varphi_{2m+2}|-1-|\mu_{2m+2}|)}{4(\theta_{2m+2}|\varphi_{2m+2}|+1)}$$
$$< \frac{(45.1)(0.978)}{4(47.1)} < k.$$

(vii) We conclude from (i) to (vi) that $a_{2m+2} = 49$. We consider the two subcases.

(a) When $|\mu_{2m+1}| \leqq 3$, if $\varphi_{2m+1} > 49$, then $|\mu_{2m+1}/\varphi_{2m+1}| < \frac{3}{49}$. If $\varphi_{2m+1} < 49$, $(\varphi_{2m+1} > 0)$, then as before $(-1)^{m+1}\mu_{2m+2} \geqq 0$, and when $|\tau_{2m+2}| \leqq 0.02$,

$$M^{(2)}_{2m+2} < \frac{(45.1)(1.02\varphi_{2m+2}-1)}{4(49.1\,\varphi_{2m+2}-1)} < k,$$

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and when $|\tau_{2m+2}| > 0.02$,

$$\left|\frac{\mu_{2m+1}}{\varphi_{2m+1}}\right| < \frac{2.98}{48.93} < \tfrac{3}{49}.$$

Thus whenever $a_{2m+2} = 49$, and $|\mu_{2m+1}| \leqq 3$,

$$\left|\frac{\mu_{2m+1}}{\varphi_{2m+1}}\right| < \tfrac{3}{49}. \tag{15.8}$$

(b) When $|\mu_{2m+1}| > 3$, we have by (15.6), $(-1)^m \mu_{2m+2}/\varphi_{2m+2} > 0$. If $\varphi_{2m+2} > 0$, then exactly as before we have

$$M^{(2)}_{2m+2} < \frac{\theta_{2m+2}-1-|\lambda_{2m+2}|}{4\theta_{2m+2}} < \frac{45.1}{4(49.1)} < k.$$

Thus after (iii), we have, if $|\mu_{2m+1}| > 3$,

$$\varphi_{2m+2} < 0,\ (-1)^{m+1}\mu_{2m+2} > 0, \quad \text{and} \quad \varepsilon_{2m+2} = 0. \tag{15.9}$$

We have seen earlier in this proof that if $|\theta_{2m+2}| > 41$, then $|\varphi_{2m+2}| > 11.8$. Thus in the case where $|\mu_{2m+1}| > 3$, we have

$$|\tau_{2m+2}| < \frac{0.0668}{11.8} < 0.0057,$$

and so

$$\left|\frac{\mu_{2m+1}}{\varphi_{2m+1}}\right| < \frac{3.0057}{49} < 0.06135.$$

Thus by (15.8), in both cases (a) and (b), $|\tau_{2m+1}| < 0.06135$. When $m = 3$, if $a_7 = -40$, then $|\varphi_6| < 40.03$, and so by (15.4) we may take $42 \leqq |a_7| \leqq 46$ in the inductive hypothesis (15.5).

Now if for some $n$ $(3 \leqq n \leqq m)$ we have $|\tau_{2n+1}| < \frac{3}{49}$, then

$$|\tau_{2n-1}| = \frac{3|\varphi_{2n}|+|\tau_{2n+1}|}{49|\varphi_{2n}|+1} < \tfrac{3}{49}.$$

Thus if any $|\tau_{n+1}| < \frac{3}{49}$, by induction $|\tau_5| < \frac{3}{49} = |\tau_5|_c$. By (15.8), this is true for $|\mu_{2m+1}| \leqq 3$.

The semi-regular continued fraction for $|\varphi_6|$,

$$|\varphi_6| = [|a_7|, -49, |a_9|, -49, \cdots, |a_{2m+1}|, -49, \cdots],$$

is an increasing function of $|a_{2n+1}|$, if $|a_{2r+1}|$ remains fixed for $3 \leqq r < n$. Since the inductive hypothesis now implies $|a_{2n+1}| \geqq 42$, $3 \leqq n \leqq m$, then if for some such $n$, $|a_{2n+1}| > 42$, we have $|\varphi_6| > |\varphi_6|_c = [\overline{42, -49}]$. Thus if $|\tau_{2m+1}| < \frac{3}{49}$, we have $|\tau_7| < \frac{3}{49}$, and

$$|\mu_5| = 3 + \left|\frac{\tau_7}{\varphi_6}\right| < 3 + \frac{3}{49|\varphi_6|_c} = |\mu_5|_c.$$

Thus by Lemma 15.3, $M_1^{(2)} < k$. Consequently, when $|\tau_{2m+1}| < \frac{3}{49}$, we have for $3 \leqq n \leqq m$,

$$|a_{2n+1}| = 42. \tag{15.10}$$

We now show that if $|\tau_{2n+1}|$ and $|\mu_{2n+1}|$ both increase, then so too does $|\mu_{2n-1}|$, whenever $3 \leqq n \leqq m$. This is easily seen to be true when $|\tau_{2m+1}| < \frac{3}{49}$, since

$$|\mu_{2n-1}| = 3 + \frac{|\mu_{2n+1}| \cdot |\tau_{2n+1}|}{42|\mu_{2n+1}| + |\tau_{2n+1}|},$$

and the right hand side increases in both $|\mu_{2n+1}|$ and $|\tau_{2n+1}|$.

Thus whenever $|\mu_{2m+1}| \leqq 3$, we have both $|\tau_5| < |\tau_5|_c$ and $|\mu_5| < |\mu_5|_c$, and so again by Lemma 15.3, $M_1^{(2)} < k$. Hence we have $|\mu_{2m+1}| > 3$, and all its consequences (15.9). There remains only to prove $|\tau_{2m+3}| > 0.06$, and $42 \leqq |a_{2m+3}| \leqq 46$.

When $|a_{2m+3}| \leqq 40$, we have $|\varphi_{2m+2}| < 40.1$, and since $|\mu_{2m+2}| = |\tau_{2m+3}| < 0.07$, $\theta_{2m+2} < 49.1$, and $|\lambda_{2m+2}| > 3$,

$$\begin{aligned} M_{2m+2}^{(1)} &< \frac{(\theta_{2m+2}+1-|\lambda_{2m+2}|)(|\varphi_{2m+2}|-1+|\mu_{2m+2}|)}{4(\theta_{2m+2}|\varphi_{2m+2}|+1)} \\ &< \frac{(47.1)(39.17)}{4\{(49.1)(40.1)+1\}} < k. \end{aligned}$$

Thus we have $|a_{2m+3}| \geqq 42$. Suppose $|\tau_{2m+3}| \leqq 0.06$, then

$$|\mu_{2m+1}| \leqq 3 + \frac{0.06}{|\varphi_{2m+2}|} < 3.00144 < |\mu_{2m+1}|_c,$$

and

$$|\tau_{2m+1}| < \frac{3|\varphi_{2m+2}|+0.06}{49|\varphi_{2m+2}|+1} < \frac{3}{49} = |\tau_{2m+1}|_c.$$

It then follows by an identical argument to that above, that $M_1^{(2)} < k$. Hence $|\tau_{2m+3}| > 0.06$.

When $|\varphi_{2m+2}| > 47.9$,

$$M_{2m+2}^{(3)} < \frac{(45.1)(47.9+0.94)}{4\{(49.1)(47.9)+1\}} < k.$$

Hence $a_{2m+3} = -42$, $-44$, or $-46$. The inductive hypothesis is now shown to hold at the $(m+1)$th step, and so the theorem follows.

Corollary. *The chain* $(c)$ *is the critical chain.*

Proof. So far, in this section, we have been holding the chain for $\theta_0$ constant. Now,

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$$M_1^{(2)} = \frac{3(\varphi_1 - 1 - |\mu_1|)}{|\theta_1|\varphi_1 + 1}$$

is clearly an increasing function of $\theta_0$, and so takes its maximum value at the largest allowable value for $\theta_0$, which by Theorem 11.1, is

$$\theta_0 = [2, 4, 4, 3, \overline{2, 5, 5}].$$

By Lemma 7.1 we have $|\tau_5| = \frac{3}{49} = |\tau_5|_c$. If $|a_{2n+1}| > 42$ for any $n \geqq 3$, we have as in Theorem 15.2, $|\mu_5| < |\mu_5|_c$, and consequently $M_1^{(2)} < k$. Thus the chain (c) gives the maximum possible value for $M_1^{(2)}$, for chains in which $M_n \geqq k$, for all $n$. The result follows from this.

## 16. Critical forms

From § 7 and (4.2) we may exhibit the critical form

$$\pm \frac{(\theta x + y)(x + \varphi y + \alpha)}{\theta\varphi - 1},$$

with

$$\theta = \frac{2\sqrt{10} - 5195}{2997},$$

$$\varphi = \frac{91018391\,\varphi_5 - 1818229}{8238730\,\varphi_5 - 164581},$$

and

$$\alpha = -\tfrac{1}{2}\left(\varphi + \frac{18014063\,\varphi_5 - 359856}{49(8238730\,\varphi_5 - 164581)}\right),$$

where

$$\varphi_5 = \frac{147 + \sqrt{21651}}{6}.$$

The critical value is attained by this inhomogeneous form at $(x, y) = (-6, 0)$, and also by all other forms equivalent to it by a unimodular integral transformation.

## PART III: SUBSIDIARY RESULTS

## 17. Further questions

In part II we evaluated the best possible constant $k$, for the mixed form problem. We immediately wonder whether $k$ is an isolated constant, and what values $M(f; \alpha)$ may take in the range $[0, k]$.

We may readily imagine from the structure of the critical chain (c)

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that $k$ is a point of accumulation of infima of certain mixed forms, since we have already seen in § 7, that the local products for large $n$ still exceed $k$, even if $a_{2n+1} = -44$. We will make use of the following result.

THEOREM 17.1. *Suppose we are considering chain pairs with the property that, for all $n$,*

$$|\theta_n| > A > 1, \quad |\varphi_n| > B > 1.$$

*Suppose two such chains have a common segment, that agrees for at least $2r+2$ consecutive values of the chain pair. Then if $F$ is any (fixed) one of the alternatives in* (2.10), (6.1) *or* (6.2) *at the central step of the common segment, we have*

$$F = F' + O(1/r),$$

*where the prime is used to distinguish between the two chains, and the constant implied by the order notation is a function of $A$ and $B$ only.*

PROOF. We may suppose, without loss of generality, that the common chain segment is

$$(17.1) \qquad (a_{-r+i}, \varepsilon_{-r+i-1}), \qquad i = 0, 1, \cdots, 2r+1.$$

Then, in the notation of § 2, (see [3]),

$$\varphi_0 = [a_1, a_2, \cdots, a_r, \varphi_r] = \frac{p_r\varphi_r - p_{r-1}}{q_r\varphi_r - q_{r-1}},$$

$$\varphi_0' = [a_1, a_2, \cdots, a_r, \varphi_r'] = \frac{p_r\varphi_r' - p_{r-1}}{q_r\varphi_r' - q_{r-1}},$$

from which it follows that

$$(17.2) \qquad |\varphi_0 - \varphi_0'| = \frac{|\varphi_r - \varphi_r'|}{|(q_r\varphi_r - q_{r-1})(q_r\varphi_r' - q_{r-1})|} = O\left(\frac{1}{r^2}\right),$$

since from [3], $p_n q_{n-1} - q_n p_{n-1} = 1$ and $|q_r| > |q_{r-1}| > r-1$. Similarly

$$(17.3) \qquad \theta_0 = \theta_0' + O\left(\frac{1}{r^2}\right).$$

If $n = [\frac{1}{2}r]$, where $[x]$ is the integral part of $x$, then since

$$\frac{|\mu_{n+1}|}{|\varphi_1\varphi_2\cdots\varphi_{n+1}|} < \frac{1}{|\varphi_1\varphi_2\cdots\varphi_n|} = \frac{1}{|q_n\varphi_n - q_{n-1}|} = O\left(\frac{1}{n}\right),$$

we have

$$|\mu_0 - \mu_0'| \leqq |\varepsilon_1| \left|\frac{1}{\varphi_1} - \frac{1}{\varphi_1'}\right| + |\varepsilon_2| \left|\frac{1}{\varphi_1\varphi_2} - \frac{1}{\varphi_1'\varphi_2'}\right| + \cdots$$
$$\cdots + |\varepsilon_n| \left|\frac{1}{\varphi_1\cdots\varphi_n} - \frac{1}{\varphi_1'\cdots\varphi_n'}\right| + O\left(\frac{1}{n}\right).$$

Now since $\varphi_1$ and $\varphi'_1$ have at least the first $n$ partial quotients identical, we have by (17.2),

$$|\varepsilon_1| \left| \frac{1}{\varphi_1} - \frac{1}{\varphi'_1} \right| = \frac{|\varepsilon_1||\varphi_1 - \varphi'_1|}{\varphi_1 \varphi'_1} < |\varphi_1 - \varphi'_1| = O\left(\frac{1}{n^2}\right).$$

By an inductive argument it follows that, for $j \leqq n$,

$$|\varepsilon_j| \left| \frac{1}{\varphi_1 \cdots \varphi_j} - \frac{1}{\varphi'_1 \cdots \varphi'_j} \right| = O\left(\frac{1}{n^2}\right).$$

Thus by (17.4),

$$\mu_0 = \mu'_0 + O\left(\frac{1}{n}\right) = \mu'_0 + O\left(\frac{1}{r}\right). \tag{17.5}$$

Similarly

$$\lambda_0 = \lambda'_0 + O\left(\frac{1}{r}\right). \tag{17.6}$$

Now $F$ is of the form $F = x_0 y_0 / |\theta_0 \varphi_0 - 1|$, where by (17.2), (17.3), (17.5), and (17.6)

$$x_0 = x'_0 + O\left(\frac{1}{r}\right)$$

$$y_0 = y'_0 + O\left(\frac{1}{r}\right).$$

Using (2.9), we obtain

$$|F - F'| = O\left(\frac{|\theta_0 \varphi_0 - \theta'_0 \varphi'_0|}{|\theta_0 \varphi_0 - 1|}\right) + O\left(\frac{1}{r}\right) = O\left(\frac{1}{r}\right).$$

Note that at each step of the argument the constant implied by the O notation depends only on $A$ and $B$.

This result will enable us to prove

THEOREM 17.2. *For every $k'$, such that $0 \leqq k' < k$, there exist uncountably many binary quadratic forms $f$, to each of which there corresponds at least one real non-zero number $\alpha$, with*

$$M(f; \alpha) = k'.$$

NOTE. It will become apparent that the following is really a straight forward extension. There exist uncountably many $\theta$, each for which there correspond uncountably many pairs $(\varphi, \alpha)$ such that

$$\inf_{(x,y) \neq (0,0)} \left| \frac{(\theta x + y)(x + \varphi y + \alpha)}{\theta \varphi - 1} \right| = k'.$$

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## 18. Construction of the chains $(c^*)$

For convenience we will use the usual notation that any repeated chain segment in a continued fraction expansion may be denoted by enclosing it within brackets, subscripted by the number of repetitions.

We first note that Theorem 17.2 holds in the case $k' = 0$. Given any integer $s > 0$, we can find an integer $r_s$, such that for all $r \geqq r_s$,

$$[(2)_r, x] < 1 + \frac{1}{s}. \tag{18.1}$$

Consider any chain of the type (3.8), which has $\varepsilon_n = 0$, for all $n \geqq 0$, and

$$\varphi_0 = [(2)_{2r_1}, 4, (2)_{2r_2}, 4, \cdots, (2)_{2r_s}, 4, \cdots].$$

Now at the central step of the block $(2)_{2r_s}$, we have for the corresponding $m$, by Theorem 2.4 and (18.1),

$$M_m < \frac{(\theta_m - 1)(\varphi_m - 1)}{4(\theta_m \varphi_m - 1)} < \frac{(1/s)^2}{4\{(1+1/s)^2 - 1\}} < \frac{1}{s}.$$

The infimum of such a chain is consequently O, and there are uncountably many sequences $\{r'_s\}$, with $r'_s \geqq r_s$, for all $s$.

When $k' > 0$, we will construct a chain which is a modification of the critical chain (c). Theorem 17.1 ensures us that, since all the partial quotients of (c) are bounded below, there exists an integer $N$, such that no matter how we change the chain (c), for $n \geqq N$, we will always have $M_1^{(2)} > k'$.

Define $\omega = [\overline{100}] = 50 + 7\sqrt{51}$, and an irrational (in general) number $\alpha$ by

$$\frac{(\omega - 1)(1 - \alpha)}{4\omega} = k'. \tag{18.2}$$

Since $k' < \frac{1}{4}$, then $0 < \alpha < 1$. If $\alpha$ is irrational, then expand $1/\alpha$ as a semi-regular continued fraction to the integer above, and compute the sequence of convergents $\{p_n/q_n\}$, given by (2.7); if $\alpha$ is rational put $p_n/q_n = 1/\alpha$ for all $n$. It is known [3], that $\{p_n/q_n\}$ converges to $1/\alpha$ from above, and hence

$$\frac{p_n}{q_n} \geqq \frac{1}{\alpha}, \qquad \text{for all } n. \tag{18.3}$$

Now let $\{r_n\}$ be a strictly monotone increasing sequence of positive integers. Consider the chain denoted $(c^*)$, which is identical to $c$ for all $n \leqq N$ (defined above), and for $n > N$ has the form:

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$$(18.4)\quad \begin{matrix} \cdots & -42, & 49, \\ \cdots & 0, & 3, \end{matrix} \left| \begin{matrix} -42, & 100, & 1000, \\ 0, & -5, & 0, \end{matrix} \begin{pmatrix} 100 \\ 0 \end{pmatrix}_{2r_1-1} \begin{matrix} -2mp_1 \\ 2mq_1 \end{matrix} \begin{pmatrix} 100 \\ 0 \end{pmatrix}_{2r_2} \begin{matrix} -2mp_2 \cdots \\ 2mq_2 \cdots \end{matrix} \right. $$
$$\begin{matrix} \cdots \\ \cdots \end{matrix} \begin{pmatrix} 100 \\ 0 \end{pmatrix}_{2r_s} \begin{matrix} -2mp_s \cdots \\ 2mq_s \cdots \end{matrix}$$

where $m$ is an arbitrary positive integer, and the vertical line signifies the point after which the chain differs from (c).

## 19. Evaluation of the infimum for the chain ($c^*$)

(i) Without loss of generality, we can take $a_N = 49$, $\varepsilon_{N-1} = 3$. All the bounds on the variables for $n < N$ conform to the requirements of § 7, implying that $M_n > k'$, for $n < N$. Now

$$|\varphi_N|_c > |\varphi_N| > 42,\ 0.0499 < |\mu_N| < |\mu_N|_c,$$

and $\theta_N > 49.023$, implying that $M_N^{(i)}$ exceeds $k$, for $i = 2, 3, 4$. Also

$$M_N^{(1)} > \frac{(\theta_N - 2.0015)(|\varphi_N| - 0.9501)}{4(\theta_N|\varphi_N|+1)} > \frac{(47.0215)(41.0499)}{4\{(49.023)(42)+1\}} > k.$$

(ii) At the next step, clearly $M_{N+1}^{(3)} > k$, and

$$M_{N+1}^{(1)} > \frac{(|\theta_{N+1}| - 1.1)(\varphi_{N+1}+5.9)}{4(|\theta_{N+1}|\varphi_{N+1}+1)} > \frac{(40.9)(105.9)}{4(4201)} > k.$$

Now since $\varphi_{N+1} > (\varphi_{N+1})_c$, and $|\mu_{N+1}| < |\mu_{N+1}|_c$, then we have $M_{N+1}^{(2)} > (M_{N+1}^{(2)})_c > k$. Also $|\theta_{N+1}| > 42$, $\varphi_{N+1} > 99$, $|\lambda_{N+1}| > 0.06$, and $|\mu_{N+1}| < 5.01$, implying

$$M_{N+1}^{(4)} > \frac{(|\theta_{N+1}| - 0.94)(\varphi_{N+1} - 4.01)}{4(|\theta_{N+1}|\varphi_{N+1}+1)} > \frac{(41.06)(94.99)}{4\{(42)(99)+1\}} > k.$$

(iii) Now

$$\mu_{N+2} = \frac{(-1)^{2r_1}}{\varphi_{N+3}\cdots\varphi_{N+1+2r_1}} \left(\frac{\mu_{N+2+2r_1}}{\varphi_{N+2+2r_1}}\right) < 0,$$

and $|\mu_{N+2}| < 0.01$, $5 < |\lambda_{N+2}| < 5.002$, $\theta_{N+2} > 100$, $\varphi_{N+2} > 999$. Clearly $M_{N+2}^{(1)} > k$.

$$M_{N+2}^{(2)} > \frac{\varphi_{N+2} - 1.01}{4\varphi_{N+2}} > k,$$

$$M_{N+2}^{(4)} > M_{N+2}^{(3)} > \frac{(\theta_{N+2} - 6.002)(\varphi_{N+2} - 1)}{4(\theta_{N+2}\varphi_{N+2} - 1)}$$
$$> \frac{(93.998)(998)}{4\{(100)(999) - 1\}} > k.$$

(iv) Suppose that we examine a step in the chain for which $a_n \geqq 100$, $a_{n+1} \geqq 100$, and $\varepsilon_{n-1} = \varepsilon_n = 0$; then $|\lambda_n| < 1$, and $|\mu_n| < 1$, implying

$$M_n \geqq \frac{(\theta_n - 1 - |\lambda_n|)(\varphi_n - 1 - |\mu_n|)}{4(\theta_n\varphi_n - 1)} > \frac{(\theta_n - 2)(\varphi_n - 2)}{4(\theta_n\varphi_n - 1)}$$
$$> \frac{97^2}{4(99^2 - 1)} > k,$$

Hence (i) to (iv) imply that the only steps in the chain that we still need to examine are $M_n$ and $M_{n+1}$, where $a_{n+1} = -2mp_s$, for some $s$. Let $n$ and $s$ denote such a position in the chain; then by the argument of (iii) we have $\lambda_n < 0$ and $\mu_{n+1} < 0$. We also have that

$$\theta_n > \omega, \ \varphi_{n+1} > \omega. \tag{19.1}$$

Clearly $M_n^{(1)} > k$. Using the methods of § 7, we obtain

$$M_n^{(3)} = \frac{(\theta_n - 1 - |\lambda_n|)(|\varphi_n| + 1 + |\mu_n|)}{4(\theta_n\varphi_n + 1)} > \frac{(\theta_n - 1 + |\lambda_n|)(|\varphi_n| + 1 - |\mu_n|)}{4(\theta_n|\varphi_n| + 1)} = M_n^{(2)},$$

if and only if

$$\frac{\theta_n - 1}{|\lambda_n|} > \frac{|\varphi_n| + 1}{|\mu_n|}.$$

Now by the form of $a_{n+1}$ and $\varepsilon_n$,

$$\frac{|\varphi_n| + 1}{|\mu_n|}$$

is uniformly bounded for the particular $n$ under consideration (the bound being a function of $k'$). Since $|\lambda_n|$ may be made arbitrarily small by choosing $r_s$ suitably large, a suitable choice of $r_1$ enables the above condition to be satisfied.

Similarly, we have $M_n^{(4)} > M_n^{(2)}$ if and only if

$$|\varphi_n| - |\mu_n| > \frac{|\theta_n|}{1 - |\lambda_n|}.$$

Now
$$\frac{|\theta_n|}{1 - |\lambda_n|}$$

is uniformly bounded, and by (18.3)

$$|\varphi_n| - |\mu_n| > (2mp_s + 0.01) - (2mq_s + 0.001)$$
$$> 2m(p_s - q_s)$$
$$\geqq 2mq_s\left(\frac{1}{\alpha} - 1\right),$$

and hence may be made large enough by a suitable choice of $m$. Thus we may suppose $M_n^{(4)} > M_n^{(2)}$, and so $M_n = M_n^{(2)}$, provided $r_1$ and $m$ are chosen (as functions of $k'$) to be large enough. Now

$$\frac{|\mu_n|}{|\varphi_n|} < \frac{2mq_s+0.001}{2mp_s+1/\varphi_{n+1}} < \frac{q_s}{p_s} + \frac{0.001}{|\varphi_n|}.$$

Thus by (19.1), (18.2) and (18.3),

$$\begin{aligned} M_n &> \frac{(\omega-1)\left\{|\varphi_n|\left(1-\dfrac{q_s}{p_s}\right)+0.999\right\}}{4(\omega|\varphi_n|+1)} \\ &> \frac{(\omega-1)\left(1-\dfrac{q_s}{p_s}\right)}{4\omega} \\ &\geqq \frac{(\omega-1)(1-\alpha)}{4\omega} = k'. \end{aligned}$$

Now at the step $M_{n+1}$, the roles of $|\theta_n|$, $|\varphi_n|$ and $|\mu_n|$, $|\lambda_n|$ are interchanged, and the same bounds apply for the corresponding variables. Thus, under the same conditions on $r_1$ and $m$, we have

$$M_{n+1} = M_{n+1}^{(3)} > k'.$$

It therefore follows for the chain $(c^*)$ that

$$M_n > k', \qquad \text{for all } n.$$

Let

$$S = \{n;\, a_{n+1} = -2mp_s, \text{ for some } s\},$$

then since $r_s \to \infty$ as $s \to \infty$, we have

$$\lim_{\substack{n\to\infty \\ n\in S}} |\lambda_n| = 0, \quad \lim_{\substack{n\to\infty \\ n\in S}} \left|\frac{\mu_n}{\varphi_n}\right| = \lim_{i\to\infty} \frac{q_i}{p_i} = \alpha,$$

$$\lim_{\substack{n\to\infty \\ n\in S}} \theta_n = \omega, \quad \lim_{\substack{n\to\infty \\ n\in S}} |\varphi_n| = \infty.\ [1]$$

Hence

$$\lim_{\substack{n\to\infty \\ n\in S}} M_n = \frac{(\omega-1)(1-\alpha)}{4\omega} = k'.$$

Consequently the infimum of the mixed form corresponding to $(c^*)$ is $k'$. There are uncountably many forms since $\{r_n\}$ is an arbitrary increasing sequence (except for $r_1$), of which there are uncountably many to choose from.

[1] In the case when $1/\alpha$ is rational, e.g. $1/\alpha = p/q$, take $p_n = np$ and $q_n = nq$.

Uncountably many $\theta$ may be obtained in a similar way, from Theorem 17.1, by replacing $\theta_{-l} = [\overline{5, 5, 2}]$ with $[4, (3)_{s_1} 4, (3)_{s_2}, \cdots]$, for an arbitrary increasing integer sequence $\{s_n\}$, provided $l$ is large enough. I do not give the proof of this but it follows by straight forward calculations of the type just given.

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