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ON NORMAL SUBGROUPS OF PRODUCTS OF NILPOTENT GROUPS

BERNHARD AMBERG, SILVANA FRANCIOSI and FRANCESCO DE GIOVANNI

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Abstract

Let G be a group factorized by finitely many pairwise permutable nilpotent subgroups. The aim of this paper is to find conditions under which at least one of the factors is contained in a proper normal subgroup of G.

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1. Introduction

In [10] Itô proved that if a finite (metabelian) group $G = AB$ is the product of two abelian subgroups A and B with $A \neq B$, then there exists a proper normal subgroup of G containing A or B. This was extended by Kegel in [11] to the case that A and B are finite nilpotent groups (and G is soluble). More generally the first author proved in [1] that the same conclusion holds for soluble products $G = AB$ of two nilpotent subgroups A and B with $A \neq B$, provided that one of them satisfies the maximum or minimum condition on subgroups. In fact a similar argument applies if one of the two factors A and B is minimax (see [5]).

On the other hand Howlett has constructed in [9] a p-group $G = AB$ which is the product of two elementary abelian subgroups A and B with $A \neq B$ and satisfies $A^G = B^G = G$. This example shows that the above results cannot be

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generalized to products of abelian groups of finite torsion-free rank. However the following theorem in particular contains a generalization to soluble products of nilpotent groups with finite abelian section rank.

THEOREM A. *Let $G = AB$ be a hypo-(abelian-or-finite) group factorized by two nilpotent subgroups A and B with $A \neq B$. If at least one of the subgroups A and B has finite abelian section rank, then there exists a proper normal subgroup of G containing A or B.*

We recall that a group G is *hypo-(abelian-or-finite)* if it has a descending normal series whose factors are either abelian or finite. Also, G is *hyper-(abelian-or-finite)* if it has an ascending normal series whose factors are either abelian or finite, and the hypocenter of G is the last term of the lower central series of G (see [25], part 1, page 29).

Our next main result gives a similar condition for products of finitely many nilpotent groups.

THEOREM B. *Let $G = A_1 \dots A_t$ be a group factorized by finitely many pairwise permutable nilpotent subgroups $A_1, \dots, A_t$ with $A_i \neq G$ for some i. If G is hypo-(abelian-or-finite) or hyper-(abelian-or-finite), then one of the subgroups $A_1, \dots, A_t$ is contained in a proper normal subgroup of G provided that at least one of the following conditions holds*:

(a) *The last term of the lower central series of G has finite abelian section rank.*

(b) *Each of the subgroups $A_1, \dots, A_t$ has finite abelian section rank.*

It should be noted that in the above theorems it does not suffice that one of the factors $A_1, \dots, A_t$ is hypercentral and all the others are nilpotent (or even abelian). This is for instance seen by the locally dihedral 2-group G which can be written as $G = AB$ where $A = G$ is hypercentral and B is cyclic of order 2 but $B^G = G$.

However if one considers the stronger hypothesis that A and B are *both* properly contained in $G = AB$, then one can allow one of the factors to be nilpotent and the other locally nilpotent. This is included in the following theorem.

THEOREM C. *Let $G = A_1 \dots A_t$ be a group factorized by finitely many pairwise permutable proper subgroups $A_1, \dots, A_t$, where one of the A_k is locally nilpotent and all the others are nilpotent. If G is hypo-(abelian-or-finite) or hyper-(abelian-or-finite), then one of the subgroups $A_1, \dots, A_t$ is contained in a proper normal*

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subgroup of G provided that at least one of the following conditions holds:

(a) *The last term of the lower central series of G has finite abelian section rank.*

(b) *Each of the subgroups $A_1, \ldots, A_t$ has finite abelian section rank.*

Theorem C becomes false if more than one factor is merely locally nilpotent (see Remark 4.1(b)). Some further similar statements for locally finite groups and hyper-(abelian-or-finite) groups with finite torsion-free rank that are products of nilpotent groups are contained in Section 5.

Finally we note that the dual problem of the existence of a non-trivial normal subgroup of the factorized group $G = AB \neq 1$ contained in A or B has been studied for instance in [19] and [4]; Zaičev shows for example that this condition holds if A and B are abelian and one of them has finite sectional rank. The example of Howlett in [9] mentioned above also shows that this cannot be extended to the case when A and B are abelian with finite torsion-free rank, and it also becomes false for finite products of two nilpotent groups (see [2] or [8]).

The notation is standard and can be found in [15]. In particular we note:

A group G has *finite abelian section rank* if it has no infinite elementary abelian p-sections for any prime p.

A group G has *finite torsion-free rank* if it has a series of finite length whose factors are either torsion groups or else infinite cyclic; the number of infinite cyclic factors in such a series is an invariant of G which is called the *torsion-free rank* $r_0(G)$ of G.

Furthemore, the *factorizer* of a normal subgroup N of the factorized group $G = AB$ is $X(N) = AN \cap BN$; it is easy to see that

$$X(N) = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN).$$

2. Auxiliary results

The following lemma is a slight generalization of the theorem of Kegel and Wielandt (see [11] and [18]). It shows that the groups G in the above theorems are in fact hypoabelian or hyperabelian.

LEMMA 2.1. *Let $G = A_1 \cdots A_t$ be a group factorized by finitely many pairwise permutable locally nilpotent subgroups $A_1, \ldots, A_t$. Then each finite normal subgroup of G is soluble.*

PROOF. If F is a finite normal subgroup of G, then $G/C_G(F)$ is also finite and hence soluble by the theorem of Kegel and Wielandt (see [11] and [18]). It follows that $F/Z(F)$ is soluble, so that also F is soluble.

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The next lemma slightly extends some known results on soluble products of groups of finite rank.

LEMMA 2.2. *Let $G = A_1 \cdots A_t$ be a group factorized by finitely many pairwise permutable subgroups $A_1, \dots, A_t$, at least $t-1$ of which are nilpotent, and let H be a soluble normal subgroup of G.*

(a) *If each of the subgroups $A_1, \dots, A_t$ has finite abelian section rank, then H has finite abelian section rank.*

(b) *If each of the subgroups $A_1, \dots, A_t$ has finite torsion-free rank, then H has finite torsion-free rank and*

$$r_0(H) \leqslant 2\sum_{i=1}^{t} r_0(A_i).$$

PROOF. Let $A_1, \dots, A_{t-1}$ be nilpotent factors, and write $A = A_1$, $B = A_2 \cdots A_t$. If X is the factorizer of H in $G = AB$, then

$$X = HA^* = HB^* = A^*B^* \text{ where } A^* = A \cap BH \text{ and } B^* = B \cap AH.$$

The soluble normal subgroup $K = B \cap H$ of B has finite abelian section rank by induction on t (respectively: K has finite torsion-free rank and $r_0(K) \leqslant 2\Sigma_{i=2}^{t} r_0(A_i)$). Moreover $B^*/K \simeq A^*(A^* \cap H)$ and hence B^* has finite abelian section rank (respectively: B^* has finite torsion-free rank and $r_0(B^*) \leqslant r_0(A_1) + 2\Sigma_{i=2}^{t} r_0(A_i)$). Since X is soluble, it has finite abelian section rank (see [13] or [16]) (respectively: X has finite torsion-free rank and $r_0(X) \leqslant r_0(A^*) + r_0(B^*) \leqslant 2\Sigma_{i=1}^{t} r_0(A_i)$) (see [3] or [16]), and the same is true for H.

3. Proof of Theorems A and B

The following proposition gives some information on factorized groups $G = A_1 \cdots A_t$ for which $A_i^G = G$ for $i = 1, \dots, t$.

PROPOSITION 3.1. *Let $G = A_1 \cdots A_t$ be the product of finitely many pairwise permutable locally nilpotent proper subgroups $A_1, \dots, A_t$ such that A_j is nilpotent for some j. If $A_i^G = G$ for $i = 1, \dots, t$, then the following holds:*

(a) *$A_i Z(G) \neq G$ for $i = 1, \dots, t$.*

(b) *If N is a normal subgroup of G such that G/N is nilpotent, then $G = A_i N$ for $i = 1, \dots, t$.*

(c) *If Γ is the last term of the lower central series of G, then $\Gamma = \gamma_n(G)$ for some positive integer n; in particular $G = A_i\Gamma$ for $i = 1, \dots, t$ and $\Gamma \neq 1$.*

(d) *If M is a normal subgroup of G with $M < \Gamma$, then $A_i M \neq G$ for each nilpotent A_i.*

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(e) *If N is a normal subgroup of G such that G/N is residually finite, then $G = A_iN$ for $i = 1, \dots, t$; in particular G is not residually finite.*

PROOF. (a) and (b) are obvious.

(c) By hypothesis A_j is nilpotent. If c is the nilpotency class of A_j, then $G = A_j\gamma_{c+2}(G)$ and $\gamma_{c+1}(G) = \gamma_{c+2}(G) = \Gamma$.

(d) Assume $A_iM = G$ for some nilpotent A_i. Then also G/M is nilpotent, and this contradicts $M < \Gamma$.

(e) Suppose first that G is finite. Then by Lemma 2.1 G is soluble. Among the counterexamples with a minimal number t of factors choose one $G = A_1 \cdots A_t$ with minimal derived length. If K is the last non-trivial term of the derived series of G, then $A_iK = G$ for all i by minimality. The subgroup $H = (A_1 \cdots A_{t-1}) \cap K$ is normal in G. If $A_iH = G$ for some i, then G/H is nilpotent, so that the proper subgroup $A_1 \cdots A_{t-1}$ of G is subnormal in G, a contradiction. This shows $A_iH \neq G$ for all i. Therefore we may assume $(A_1 \cdots A_{t-1}) \cap K = 1$, so that

$$A_1 \cdots A_{t-1} = A_1K \cap (A_1 \cdots A_{t-1}) = A_1 \quad \text{and} \quad G = A_1A_t.$$

Let $A_1 = A$ and $A_t = B$ and choose a counterexample $G = AB$ of minimal order. Since G is soluble, a minimal normal subgroup N of G has prime exponent p. Clearly $G = AN = BN$, so that $A \cap N$ and $B \cap N$ are normal in G. If $L = A_G$ is the core of A in G, the group G/L is not nilpotent, so that $AL \neq G$ and $BL \neq G$, and hence $L = 1$ by minimality. In particular $C_A(N) = 1$ and $A \cap N = 1$. This implies $C_G(N) = N$. Since p also divides the order of A, the Sylow p-subgroups S/N of G/N is non-trivial. Then S is normal in G and hence $N \cap Z(S)$ is a non-trivial normal subgroup of G, so that $N \leqslant Z(S)$ and $S \leqslant C_G(N) = N$, a contradiction.

Now let $(K_i)_{i \in I}$ be a family of normal subgroups of finite index in G such that $\bigcap_{i \in I} K_i = N$. Then $G = A_nK_i$ for all $i \in I$ and $n = 1, \dots, t$. In particular G/N is nilpotent and therefore $G = A_nN$ for $n = 1, \dots, t$. This proves the proposition.

For the proofs of Theorems A and B one has to consider factorized groups $G = AB$ with $A \neq B$ such that $A^G = B^G = G$. The following proposition gives more information on such groups.

PROPOSITION 3.2. *Let $G = AB$ be a group factorized by two nilpotent subgroups A and B such that $A \neq B$. If $A^G = B^G = G$, then the following conditions hold*:

(a) *If Γ is the last term of the lower central series of G, then $G = A\Gamma = B\Gamma$ and $\Gamma \neq 1$.*

(b) *If Γ is abelian, then for every $a \in Z(A)$ the subgroup $[\Gamma a]$ of Γ is normal in G with $[\Gamma, a] < \Gamma$ and $A[\Gamma a] \neq B[\Gamma, a]$.*

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(c) Γ *is infinite.*

(d) *If* $\Gamma' < \Gamma$, *then* Γ_{ab} *does not have finite abelian section rank.*

(e) *If* $\Gamma' < \Gamma$, *then neither* A *nor* B *does have finite abelian section rank.*

PROOF. (a) follows from Proposition 3.1(c).

(b) Since $G = A\Gamma = B\Gamma$, the subgroup $C = (A \cap \Gamma)(B \cap \Gamma)$ is normal in G and $G/C = (AC/C)(BC/C) = (AC/C)(\Gamma/C) = (BC/C)(\Gamma/C)$ with $(AC/C) \cap (\Gamma/C) = (BC/C) \cap (\Gamma/C) = 1$. Application of Lemma 1.2 of [3] to the factor group G/C yields that the normal subgroup $[\Gamma, a]$ of G is properly contained in Γ for each $a \in Z(A)$.

(c) Assume that Γ is finite. Then the index $|G\colon A \cap B|$ is finite and hence also $G/(A \cap B)_G$ is finite. Therefore we may assume that G is finite. But this contradicts Proposition 3.1(e).

(d) Since $\Gamma' < \Gamma$, we have $A\Gamma' \neq B\Gamma'$ by Proposition 3.1(d). Therefore we may assume that Γ is abelian. Suppose that (d) is false, and among the counterexamples for which the last term of the lower central series has minimal torsion-free rank, choose one $G = AB$ such that also the nilpotency class of A is minimal. Since $G = A\Gamma = B\Gamma$, the subgroups $A \cap \Gamma$ and $B \cap \Gamma$ are normal in G, so that also $C = (A \cap \Gamma)(B \cap \Gamma)$ is normal in G. If $AC = BC$, then $G = AC = A(B \cap \Gamma)$ and hence $G/(B \cap \Gamma)$ is nilpotent, a contradiction. Therefore $AC \neq BC$ and we may assume that $A \cap \Gamma = B \cap \Gamma = 1$.

Suppose first that Γ is a torsion group and let L be the p'-component of Γ for some prime $p \in \pi(\Gamma)$. Since $L < \Gamma$, by Proposition 3.1(d) we have $AL \neq BL$. Therefore we may assume that Γ is an abelian p-group of finite rank. Then there exists a finite characteristic subgroup S of Γ such that Γ/S is radicable. It may be assumed that $S = 1$ and that Γ is radicable. If D is a maximal radicable proper G-invariant subgroup of Γ, the group G/D is also a counterexample; therefore we may assume that $D = 1$. If $a \in Z(A)$, the proper G-invariant subgroup $[\Gamma, a]$ of Γ is radicable, and hence $[\Gamma, a] = 1$ and $a \in Z(G)$. Then $Z(A) \leqslant Z(G)$. But then $G/Z(G) = (AZ(G)/Z(G))(BZ(G)/Z(G))$ is a counterexample where the nilpotency class of $AZ(G)/Z(G)$ is less than that of A. This contradiction shows that Γ is not a torsion group.

If T is the torsion subgroup of Γ, we have $T < \Gamma$ and so $AT \neq BT$, and we can assume that Γ is torsion-free. For any $a \in Z(A)$, $[\Gamma, a]$ is a proper G-invariant subgroup of Γ. If $[\Gamma, a] \neq 1$ for some $a \in Z(A)$, the torsion-free rank of $\Gamma/[\Gamma, a]$ is less than that of Γ and $A[\Gamma, a] \neq B[\Gamma, a]$, and we obtain a contradiction. Therefore $[\Gamma, a] = 1$ for every $a \in Z(A)$, so that $Z(A) \leqslant Z(G)$, and we reach a contradiction as before.

(e) We may assume that $\Gamma' = 1$, so that G is soluble. Suppose that (e) is false, and let $G = AB$ be a counterexample with minimal derived length. If K is the last non-trivial term of the derived series of G, then $G = AK = BK$ by minimality.

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Therefore $A \cap K$ and $B \cap K$ are normal in G and thus $C = (A \cap K)(B \cap K)$ is also normal in G. If $AC = BC$, then $G = AC = A(B \cap K)$ and the group $G/(B \cap K)$ is nilpotent, which is impossible. Therefore $AC \neq BC$, and we may assume that $A \cap K = B \cap K = 1$. Thus A and B are isomorphic and hence they both have finite abelian section rank. By Lemma 2.2 also the soluble group G has finite abelian section rank. This contradicts (d), so that also (e) is proved.

Proof of Theorem A. Assume that Theorem A is false. A counterexample $G = AB$ is hypoabelian by Lemma 2.1, and $\Gamma' < \Gamma$ if $\Gamma \neq 1$. Since A or B has finite abelian section rank, this contradicts Proposition 3.2(e). The theorem is proved.

Proof of Theorem B. Assume that Theorem B is false. A counterexample $G = A_1 \cdots A_t$ is hypoabelian or hyperabelian by Lemma 2.1. If c is the nilpotency class of A_i, we have $G = A_i\gamma_{c+2}(G)$, so that $\gamma_{c+1}(G) = \gamma_{c+2}(G) = \Gamma$ is the last term of the lower central series of G. For every normal subgroup N of G, $\Gamma N/N$ is the last term of the lower central series of G/N.

Let G first be hypoabelian. If $A_iG^{(c+1)} = G$, then the group $G/G^{(c+1)}$ has derived length at most c and so G is soluble. If $A_iG^{(c+1)} \neq G$, we may assume $G^{(c+1)} = 1$. Therefore let G be soluble, and among the counterexamples with a minimal number t of factors, choose one $G = A_1 \cdots A_t$ with minimal derived length. If K is the last non-trivial term of the derived series of G, then $A_jK = G$ for all j by minimality. The subgroup $H = (A_1 \cdots A_{t-1}) \cap K$ is normal in G and $A_jH \neq G$ for all j, since otherwise G/H is nilpotent and the proper subgroup $A_1 \cdots A_{t-1}$ of G is subnormal in G. Therefore we may assume that $(A_1 \cdots A_{t-1}) \cap K = 1$, so that

$$A_1 \cdots A_{t-1} = A_1K \cap (A_1 \cdots A_{t-1}) = A_1 \quad \text{and} \quad G = A_1A_t.$$

This shows that $t = 2$. Application of Proposition 3.2(d) and (e) now leads to a contradiction.

Now let G be hyperabelian. If (b) holds, by Lemma 2.2 every abelian normal section of G has finite sectional rank. Hence Γ has an ascending G-invariant series whose factors are either abelian torsion groups with min-p for every prime p, or else torsion-free abelian groups of finite rank. Then in both cases (a) and (b) Γ has such a series, Σ say. If F is a torsion factor of Σ, then the group $G/C_G(F)$ is residually finite (see [15], Part 1, page 135). By Proposition 3.1(e) $G = A_jC_G(F)$ for all j, and hence $G/C_G(F)$ is nilpotent with bounded class. If F is a torsion-free factor of Γ, then $G/C_G(F)$ is a hyperabelian linear group and hence soluble (see [15], Part 1, page 78), and since the result is true for soluble groups by the first part of the proof, we have $G = A_jC_G(F)$ for all j, so that again

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$G/C_G(F)$ is nilpotent with bounded class. If C denotes the intersection of all $C_G(F)$, the group G/C is nilpotent, so that $\Gamma \leqslant C$ is hypercentral and hence G is hypoabelian. The theorem follows now from the first part of the proof.

COROLLARY 3.3. *Let $G = A_1 \cdots A_t$ be a hypo-(abelian-or-finite) group factorized by finitely many pairwise permutable nilpotent subgroups $A_1, \ldots, A_t$ with $A_i \neq G$ for some i. If G_{ab} has finite abelian section rank, then at least one of the subgroups $A_1, \ldots, A_t$ is contained in a proper normal subgroup of G.*

PROOF. Assume that the corollary is false, and let G be a counterexample. By Lemma 2.1 G is hypoabelian and as before we may assume that G is soluble. Among the counterexamples with a minimal number t of factors choose one $G = A_1 \ldots A_t$ with minimal derived length. If K is the last non-trivial term of the derived series of G, then $G = A_j K$ for all j and hence $C = (A_1 \cap K) \cdots (A_t \cap K)$ is normal in G. If $G = A_1 C = A_1(A_2 \cap K) \cdots (A_t \cap K)$, the group $G/(A_2 \cap K) \cdots (A_t \cap K)$ is nilpotent, which is impossible. Therefore $A_1 C \neq G$ and we may assume that $A_1 \cap K = \cdots = A_t \cap K = 1$. Since G_{ab} has finite abelian section rank, the nilpotent group G/K has finite abelian section rank (see [14]). Hence every A_j has finite abelian section rank. This contradicts Theorem B and proves the corollary.

REMARK 3.4. In Theorem B it is not possible to replace the last term of the lower central series of G by the last non-trivial term of the derived series of G, even if G is soluble. In fact, if $H = A_0 B_0$ is the p-group factorized by two elementary abelian subgroups A_0 and B_0 that is constructed in Section 4 of [9], then $A_0^H = B_0^H = H$. Let K be a finite nilpotent group of derived length at least three, and let $G = H \times K$. Then $G = AB$ where $A = A_0 \times K$, $B = B_0 \times K$, and $A^G = B^G = G$.

4. Proof of Theorem C

Assume that Theorem C is false and let G be a counterexample. By Lemma 2.1 G is hypoabelian or hyperabelian. Let $A_1, \ldots, A_{t-1}$ be the nilpotent factors and let G first be hypoabelian. If $G = A_t \Gamma^{(h)}$ for some positive integer h (where Γ is the last term of the lower central series of G), then the group $G/\Gamma^{(h)}$ is locally nilpotent and $\Gamma/\Gamma^{(h)}$ has finite abelian section rank by Lemma 2.2. If $T/\Gamma^{(h)}$ is the torsion subgroup of $\Gamma/\Gamma^{(h)}$, then the torsion-free group Γ/T is nilpotent by a

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result of Mal′cev (see [15], Part 2, page 35), and hence it has finite rank. It follows that $\Gamma/T \leqslant Z_s(G/T)$ for some positive integer s (see [15], Part 2, page 35). Therefore G/T is nilpotent and $T = \Gamma$, so that $\Gamma/\Gamma^{(h)}$ is a torsion group. For any positive integer m, the group $\Gamma/\Gamma^{(h)}\Gamma^m$ is finite and $G/\Gamma^{(h)}\Gamma^m$ is finite-by-nilpotent and therefore nilpotent. Then $\Gamma^{(h)}\Gamma^m = \Gamma$ and the hypercentral torsion group $\Gamma/\Gamma^{(h)}$ is radicable, so that $\Gamma/\Gamma^{(h)}$ is abelian and hence $h = 1$. In particular $A_t\Gamma^{(2)} \neq G$, and since obviously $A_i\Gamma^{(2)} \neq G$ for all $i \leqslant t-1$, we may assume that $\Gamma^{(2)} = 1$, so that G is soluble.

If (b) holds, the soluble group G has finite abelian section rank by Lemma 2.2. Then in both cases (a) and (b) Γ has a descending G-invariant series Σ whose factors are either torsion abelian groups with min-p for every prime p, or else torsion-free abelian groups of finite rank. If F is a torsion factor of Σ, then $G/C_G(F)$ is residually finite (see [15], Part 1, page 135). By Proposition 3.1 we have $G = A_jC_G(F)$ for all $j \leqslant t$. Therefore $G/C_G(F)$ is nilpotent with bounded class. If F is a torsion-free factor of Σ, the group $G/C_G(F)$ is linear over the field of rational numbers and hence its locally nilpotent subgroup $A_tC_G(F)/C_G(F)$ is nilpotent (see [15], Part 2, page 31). If $A_iC_G(F) \neq G$ for some i, the result follows from Theorem B. Therefore assume that $G = A_iC_G(F)$ for all i, so that $G/C_G(F)$ is nilpotent with bounded class. If C is the intersection of all $C_G(F)$, the group G/C is nilpotent and hence Γ is hypocentral.

If $A_t\Gamma' = G$, then Γ_{ab} is a radicable abelian torsion group and hence the nilpotent group $\Gamma/\gamma_3(\Gamma)$ is abelian (see [15], Part 2, page 125). It follows that $\Gamma' = 1$, since Γ is hypocentral. This implies $G = A_t\Gamma' = A_t$, a contradiction. Therefore $A_i\Gamma' \neq G$ for all i, and we may assume $\Gamma' = 1$. If $A_t\Gamma \neq G$, the result is obvious since G/Γ is nilpotent. If $A_t\Gamma = G$, the subgroup $A_t \cap \Gamma$ is normal in G. If $A_i(A_t \cap \Gamma) = G$ for some $i \leqslant t-1$, then $G/(A_t \cap \Gamma)$ is nilpotent and the assertion follows. On the other hand, if $A_i(A_t \cap \Gamma) \neq G$ for all i, we may assume that $A_t \cap \Gamma = 1$, so that $A_t \simeq G/\Gamma$ is nilpotent and the assertion follows from Theorem B.

Now let G be hyperabelian and assume that (b) holds. If H/K is an abelian normal section of G, then H/K has finite abelian section rank by Lemma 2.2. Therefore Γ has an ascending G-invariant series whose factors are either torsion abelian groups with min-p for every prime p or else torsion-free abelian groups of finite rank. Thus in both cases (a) and (b) Γ has such a series, Σ say. If F is a torsion factor of Σ, the group $G/C_G(F)$ is residually finite (see [15], Part 1, page 135). By Proposition 3.1 $G = A_iC_G(F)$ for all i, so that $G/C_G(F)$ is nilpotent and $\Gamma \leqslant C_G(F)$. If F is a torsion-free factor of Γ, then $G/C_G(F)$ is linear over the field of rational numbers and as above by Theorem B we may assume $G = A_iC_G(F)$ for all i, and hence $G/C_G(F)$ is nilpotent and $\Gamma \leqslant C_G(F)$. It follows that in any case Γ is hypercentral and therefore G is hypoabelian. Application of the first part of the proof leads to a contradiction. Theorem C is proved.

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REMARK 4.1. (a) In Theorem C we cannot assume that $G = AB$ is factorized by two proper hypercentral subgroups A and B, since there exists a metabelian hypercentral 2-group G of finite rank which is factorized by two proper subgroups A and B such that $A^G = B^G = G$. In fact let

$$H = \langle H_0, x \mid H_0 \simeq Z(2^\infty),\ x^2 = 1,\ h^x = h^{-1} \text{ for all } h \text{ in } H_0 \rangle$$

and

$$K = \langle K_0, y \mid K_0 \simeq Z(2^\infty),\ y^2 = 1,\ k^y = k^{-1} \text{ for all } k \text{ in } K_0 \rangle$$

be two copies of the locally dihedral 2-group, and let $G = H \times K$. Then $G = AB$ where $A = H \times \langle y \rangle$ and $B = \langle x \rangle \times K$.

(b) In Theorem C we cannot assume that more than one factor of G is merely locally nilpotent. In fact, if G is the group considered in (a), then $G = ABC$ where $C = \langle x \rangle \times \langle y \rangle$, and so $A^G = B^G = C^G = G$.

5. Some further results

The following lemma is perhaps already known.

LEMMA 5.1. *Let G be a hyperabelian group with finite torsion-free rank. If the periodic normal subgroups of G have finite Prüfer rank, then G has finite Prüfer rank.*

PROOF. Since the maximal periodic normal subgroup of G has finite Prüfer rank, we may assume that G has no non-trivial periodic normal subgroups. Let A be a maximal abelian normal subgroup of G. Then A is a torsion-free abelian group of finite rank and $G/C_G(A)$ is linear over the field of rational numbers. It follows that the periodic subgroups of $G/C_G(A)$ are finite (see [17], page 132). If T/A is the maximal periodic normal subgroup of $C_G(A)/A$, the $A \leqslant Z(T)$ and $T/Z(T)$ is locally finite. Therefore the normal subgroup T' of G is periodic and hence $T' = 1$ and $A = T$. Thus the periodic normal subgroups of G/A are finite. By induction on the torsion-free rank of G, it follows that G/A has finite Prüfer rank. Thus G has finite Prüfer rank.

The following two results are essentially consequences of Theorems B and C.

PROPOSITION 5.2. *Let $G = A_1 \cdots A_t$ be a hyper-(abelian-or-finite) group factorized by finitely many pairwise permutable subgroups $A_1, \ldots, A_t$ with finite torsion-free rank, one of which is locally nilpotent and all the others are nilpotent. If*

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the last term Γ *of the lower central series of* G *is torsion-free, then* Γ *is soluble with finite Prüfer rank and hence there exists a proper normal subgroup of* G *containing one of the subgroups* A_i *in the following two cases*:

(a) *Each of the subgroups* $A_1, \dots, A_t$ *is nilpotent and one of them is properly contained in* G.

(b) *Each of the subgroups* $A_1, \dots, A_t$ *is properly contained in* G.

PROOF. By Lemma 2.1 the group G is hyperabelian. Hence Γ has an ascending G-invariant series

$$1 = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_\sigma = \Gamma$$

with abelian factors. Assume that Γ is not soluble and let α be the first ordinal such that Γ_α is not soluble; then $\Gamma_\alpha = \bigcup_{\beta < \alpha} \Gamma_\beta$. For each ordinal $\beta < \alpha$ the subgroup Γ_β is soluble, and hence has bounded finite torsion-free rank by Lemma 2.2. It follows that Γ_α has finite torsion-free rank, and hence it has finite Prüfer rank by Lemma 5.1. Since Γ_α is also torsion-free, it is soluble (see [15], Part 2, pages 176–178), but this is a contradiction. Therefore Γ is soluble and hence has finite torsion-free rank by Lemma 2.2; then Γ has finite Prüfer rank by Lemma 5.1. Application of Theorems B and C concludes the proof.

A locally finite group G has finite abelian section rank if and only if it satisfies min-p for every prime p ([12]).

PROPOSITION 5.3. *Let* $G = A_1 \cdots A_t$ *be a locally finite group factorized by finitely many pairwise permutable subgroups* $A_1, \dots, A_t$, *one of which is locally nilpotent and all the others are nilpotent. If all the* A_i *are proper subgroups of* G *or if all the* A_i *are nilpotent and one of them is a proper subgroup of* G, *then one of the subgroups* A_i *is contained in a proper normal subgroup of* G *provided that at least one of the following conditions holds*:

(a) *The last term* Γ *of the lower central series of* G *satisfies min-p for every prime* p.

(b) *Each of the subgroups* $A_1, \dots, A_t$ *satisfies min-p for every prime* p.

PROOF. If (b) holds, the group G satisfies min-p for every prime p by a result of N. S. Černikov [7]. Therefore only case (a) has to be considered. By Proposition 3.1 $\Gamma = \gamma_n(G)$ for some positive integer n. Since Γ is a locally finite group with min-p for every prime p, it is (locally soluble)-by-finite by a result of Belyaev-Pavlyuk-Šunkov (see [6]). Thus the finite residual R of Γ is abelian and the maximal p-subgroups of Γ/R are finite, so that, for every prime p, the subgroup $K_p/R = 0_{p'}(\Gamma/R)$ has finite index in Γ/R (see [12], pages 94–95). The group G/K_p is finite-by-nilpotent and is factorized by the nilpotent subgroups $A_1K_p/K_p, \dots, A_tK_p/K_p$ and hence it is soluble by Lemma 2.1. If $A_iK_p \neq G$ for

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some p and some i, then the result follows from Theorem B. Otherwise $G = A_i K_p$ for every p, so that G/K_p is nilpotent with bounded class and hence G/R is nilpotent. Therefore G is soluble and the result follows from Theorem C.

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Fachbereich Mathematik
Universität Mainz
Saarstrasse 21
D-6500 Mainz
West Germany

Dipartimento di Matematica
Università di Napoli
Via Mezzocannone 8
I-80134 Napoli
Italy

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