Canad. Math. Bull. Vol. 58 (3), 2015 pp. 538–547
http://dx.doi.org/10.4153/CMB-2015-007-9

Minimal Non Self Dual Groups

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Abstract. A group G is *self dual* if every subgroup of G is isomorphic to a quotient of G and every quotient of G is isomorphic to a subgroup of G. It is *minimal non self dual* if every proper subgroup of G is self dual but G is not self dual. In this paper, the structure of minimal non self dual groups is determined.

1 Introduction

Let G be a finite group. It is *s-self dual* if every subgroup of G is isomorphic to a quotient of G. It is *self dual* if it is s-self dual and every quotient of G is isomorphic to a subgroup of G. The study of finite self dual groups was initiated by Armond E. Spencer in [6]. He obtained the following results.

Theorem 1.1 *A finite group G is self dual if and only if G is nilpotent and all Sylow subgroups of G are self dual.*

The structure of finite self dual p-groups was determined by L. An, J. Ding and Q. Zhang in [1]. They obtained the following result.

Theorem 1.2 *If G is a finite p-group, then G is self dual if and only if G is abelian or $G = \langle a, b \mid a^p = b^p = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle \times M$, where $p > 2$, M is abelian and* $\exp(M) \le p$.

By Theorem 1.1 and Theorem 1.2, the structure of finite self dual groups is determined completely.

It is clear that s-self duality and self duality are inherited by subgroups. Hence, we define that a group G is *minimal non s-self dual* if proper subgroups of G are all s-self dual but G is not s-self dual. Likewise, a group G is *minimal non self dual* if proper subgroups of G are all self dual but G is not self dual. In this article, we study non s-self dual groups with "large" s-self duality. Obviously, minimal non s-self dual groups play an important role in the study of the s-self duality of a group. Based on this observation, we classified minimal non s-self dual groups, and as a byproduct, the classification of minimal non self dual groups is given.

Received by the editors June 12, 2013.
Published electronically May 29, 2015.
This work was supported by NSFC (no. 11271301, no. 11471266 and no. U1204101), Program of Young Creative Talents in Higher Education of Guangdong (Natural Science, no. 2014KQNCX189) and Natural Science Foundation Project of CQCSTC (no. cstc2014jcyjA00010).
AMS subject classification: **20D15**.
Keywords: minimal non self dual group, finite group, metacyclic group, metabelian group.

Let G be a finite p-group. We use $c(G)$, $d(G)$ and $\exp(G)$ to denote the nilpotency class, the minimal number of generators, and the exponent of G respectively. We use C_n to denote the cyclic group of order n, and C_n^m to denote the direct product of m copies of C_n. We use $a \in G\backslash H$ to denote $a \in G$ but $a \notin H$.

In this paper, p is always a prime. For other notation and terminology, the reader is referred to [3].

2 Preliminaries

Lemma 2.1 ([1]) *If G is a finite s-self dual p-group, then G is one of the following groups:*

(i) *an abelian p-group;*
(ii) *$M_p(1,1,1) \times M$, where M is abelian and* $\exp(M) \leq p$;
(iii) *$M_p(n,n) \times M$, where $n \geq 2$, M is abelian and* $\exp(M) < p^n$.

Lemma 2.2 ([1]) *If G is a finite self dual p-group, then G is one of the following groups:*

(i) *an abelian p-group;*
(ii) *$M_p(1,1,1) \times M$, where M is abelian and* $\exp(M) \leq p$.

A finite group G is said to be *minimal nonabelian* if proper subgroups of G are all abelian, but G is not abelian. For minimal nonabelian p-groups, the following lemmas are well known.

Lemma 2.3 ([5]) *Let G be a minimal nonabelian p-group. Then G is one of the following non-isomorphic groups:*

(i) $Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, [a,b] = a^2 \rangle$;
(ii) $M_p(n,m) = \langle a, b \mid a^{p^n} = b^{p^m} = 1, [a,b] = a^{p^{n-1}} \rangle$, *where* $n \geq 2$;
(iii) $M_p(n,m,1) = \langle a, b \mid a^{p^n} = b^{p^m} = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1 \rangle$, *where $n \geq m$, and if $p = 2$, then $n + m \geq 3$.*

Lemma 2.4 ([7, Lemma 2.2]) *Let G be a finite p-group. Then the following conditions are equivalent:*

(i) *G is a minimal nonabelian p-group;*
(ii) $d(G) = 2$ *and* $|G'| = p$;
(iii) $d(G) = 2$ *and* $Z(G) = \Phi(G)$.

Lemma 2.5 ([2]) *Let G be a finite group of order p^4, for p odd. If G has a nonabelian maximal subgroup, then nonabelian maximal subgroups of G are all isomorphic to $M_p(1,1,1)$ if and only if G is one of the following non-isomorphic groups:*

(i) $M_p(1,1,1) \times C_p$, *where* $p > 2$;
(ii) $\langle a, b \mid a^p = b^p = c^p = d^p = 1, [a,b] = c, [c,b] = d, [c,a] = [d,a] = [d,b] = 1 \rangle$, *where* $p > 3$;
(iii) $\langle a, b \mid a^9 = b^3 = c^3 = 1, [a,b] = c, [c,b] = a^3 \rangle$.

A finite group G is said to be metacyclic if there is a cyclic normal subgroup N such that G/N is cyclic. We need the following characteristic of metacyclic p-groups.

Lemma 2.6 ([9]) *Let G be a finite p-group. If G has a nonabelian maximal subgroup and all nonabelian maximal subgroups of G are isomorphic to $M_p(n,n)$, then G is metacyclic.*

A nonabelian group G is said to be metabelian if G' is abelian. Since minimal non self dual groups are metabelian, the following commutator formula is useful in this paper.

Lemma 2.7 ([8]) *Let G be a metabelian group and $a, b \in G$. For any positive integers i and j, let*

$$[ia, jb] = [a, b, \underbrace{a, \dots, a}_{i-1}, \underbrace{b, \dots, b}_{j-1}].$$

(i) *For any positive integers m and n,*

$$[a^m, b^n] = \prod_{i=1}^{m} \prod_{j=1}^{n} [ia, jb]^{\binom{m}{i}\binom{n}{j}}.$$

(ii) *For any positive integer n,*

$$(ab^{-1})^n = a^n \prod_{i+j \le n} [ia, jb]^{\binom{n}{i+j}} b^{-n}.$$

Lemma 2.8 ([4]) *Assume that maximal subgroups of a finite group G are all nilpotent but G is not nilpotent.*

(i) *$|G| = p^m q^n$, where p and q are unequal primes and there is a unique Sylow p-subgroup $P \lhd G$ and a Sylow q-subgroup Q is cyclic. Hence, $G = QP$.*
(ii) *If $p > 2$, then $\exp(P) = p$; if $p = 2$, then $\exp(P) \le 4$.*

3 Finite p-groups whose Nonabelian Maximal Subgroups are all Isomorphic to $M_p(1,1,1) \times C_p^n$ with $p > 2$

Lemma 3.1 *Let G be a finite p-group. If G possesses a nonabelian maximal subgroup and all nonabelian maximal subgroups of G are of exponent p, then $p \ge 3$ and one of the following holds:*

(i) *G is a p-group of exponent p;*
(ii) *$G = \langle a, b \mid a^{p^2} = b^p = c_i^p = 1, [a, b] = c_1, [c_i, a] = 1, [c_i, b] = c_{i+1}, c_{p-1} = a^{-p}, [a^{-p}, b] = 1\rangle$, where $i = 1, 2, \dots, p-2$.*

(When $|G| \ge p^5$, $c(\langle ab, c_1\rangle) \ge 3$.)

Proof We claim that $p \ge 3$. (Otherwise, $p = 2$, so G doesn't have nonabelian maximal subgroup, since a finite 2-group of exponent 2 is abelian, contradicting the assumption of the lemma). Let H be a nonabelian maximal group of G. Then $\exp(H) = p$. We may assume that $\exp(G) \ge p^2$. Since $|G/H| = p$ and $\exp(H) = p$, we have $\exp(G) = p^2$ and $o(a) = p^2$ for any $a \in G \backslash H$.

We claim that $d(G) = 2$. Otherwise, $d(G) \geq 3$. Since $o(a) = p^2$, $a \in Z(G)$. Since H is not abelian and $\exp(H) = p$, H possesses a nonabelian subgroup $\langle x, y\rangle$, where $\exp(\langle x, y\rangle) = p$, then G possesses a nonabelian proper subgroup $\langle ax, y\rangle$, where $\exp(\langle ax, y\rangle) = p^2$, a contradiction.

Since $d(G) = 2$, there exists $b \in H$ such that $G = \langle a, b\rangle$. It follows from $o(a) = p^2$ that $\big[a, [a, b]\big] = 1$. Since $\exp(G') = p$ and $|G| \geq p^4$, $o\big(\big[b, [a, b]\big]\big) = p$. If $a^p \notin G'$, then $(ab^{-1})^p \equiv a^p (\operatorname{mod} G')$, we have $o(ab^{-1}) = p^2$, and hence $\langle ab^{-1}, [a, b]\rangle$ is abelian. But $\big[a, [a, b]\big] = 1$ and $o\big(\big[b, [a, b]\big]\big) = p$, a contradiction. Hence $a^p \in G'$. Since $[a^p, b] = [a, b]^p = 1$, $a^p \in Z(G)$. Since again $[a, d] = 1$ for any $d \in G'$, $|G_k/G_{k+1}| = p$, where $1 \leq k \leq c(G) - 1$ and $G_{c(G)-1} = \langle a^p\rangle$. Note that $\langle ab^{-1}, [a, b]\rangle$ is a nonabelian proper subgroup of G. Then $o(ab^{-1}) = p$. On the other hand, we compute:

$$(ab^{-1})^p = a^p [a, b]^{\binom{p}{2}} [a, 2b]^{\binom{p}{3}} \cdots [a, (p-1)b].$$

Then $[a, (p-1)b] = a^{-p}$. Thus we get the group of type (iii) in the statement of the lemma. ∎

Theorem 3.2 *Let G be a minimal non s-self dual p-group. If G possesses a nonabelian maximal subgroup and the nonabelian maximal subgroups of G are all isomorphic to $M_p(1,1,1) \times C_p^n$ with $p > 2$, then G is one of the following non isomorphic groups:*

(i) $M_p(1,1,1) \times C_p$, *where* $p > 2$;
(ii) $\langle a, b \mid a^p = b^p = c^p = d^p = 1, [a, b] = c, [c, b] = d, [c, a] = [d, a] = [d, b] = 1\rangle$, *where* $p > 3$;
(iii) $\langle a, b \mid a^9 = b^3 = c^3 = 1, [a, b] = c, [c, b] = a^3, [c, a] = 1\rangle$;
(iv) $\langle a, b | a^p = b^p = c^p = d^p = e^p = 1, [a, b] = c, [b, c] = d, [a, c] = e, [d, a] = [d, b] = [e, a] = [e, b] = 1\rangle$, *where* $p > 3$;
(v) $\langle a, b, c | a^p = b^p = c^p = d^p = e^p = 1, [b, c] = d, [a, b] = e, [a, c] = [d, a] = [d, b] = [d, c] = [e, a] = [e, b] = [e, c] = 1\rangle$, *where* $p > 2$;
(vi) $\langle a, b, c | a^p = b^p = c^p = h_i^p = 1, [b, c] = h_1, [a, b] = h_2, [a, c] = h_3, [h_i, a] = [h_i, b] = [h_i, c] = 1\rangle$, *where* $p > 2$, $i = 1, 2, 3$;
(vii) $M_p(1,1,1) * M_p(1,1,1)$, *where* $p > 2$.

Proof Let H be a maximal subgroup of G, and $H = \langle b, c | b^p = c^p = d^p = 1, [b, c] = d, [d, b] = [d, c] = 1\rangle \times M$, where $M \cong C_p^n$ and $p > 2$. We know that $|G| \geq p^4$.

If $|G| = p^4$, then we get groups of types (i), (ii), and (iii) in the statement of the theorem by Lemma 2.5.

If $|G| \geq p^5$, then $a \notin Z(G)$ for any $a \in G \backslash H$ (otherwise, there exists $a \in G \backslash H$ such that $a \in Z(G)$. If $o(a) = p$, then $G \cong M_p(1,1,1) \times C_p^{n+1}$, a contradiction. If $o(a) = p^2$, then $G = \langle a, b, c\rangle$ since $\exp(M_p(1,1,1) \times C_p^n) = p$, so $|G| = p^4$, a contradiction). We have $\exp(G) = p$ by Lemma 3.1. Thus $G = H \rtimes \langle a\rangle$, where $o(a) = p$ and $a \notin Z(G)$. It is easy to see that $\langle x, y\rangle$ is abelian or isomorphic to $M_p(1,1,1)$ for any $\langle x, y\rangle < G$. Since $\Phi(H) = \langle d\rangle \leq Z(G)$, there exists $x \in H \backslash \Phi(H)$ such that $[a, x] \neq 1$.

Case 1: $x \in \langle b, c\rangle \backslash \langle d\rangle$. Without loss of generality, we assume that $x = b$. Since $p^2 \leq |\langle a, b, Z(H)\rangle / Z(H)| \leq p^3$ and $|G/Z(H)| = p^3$, $|G/\langle a, b, Z(H)\rangle| \leq p$.

https://doi.org/10.4153/CMB-2015-007-9 Published online by Cambridge University Press

Subcase 1: $G = \langle a, b, Z(H)\rangle$. Since $H = \langle b, [a,b], Z(H)\rangle$, $\big[b,[a,b]\big] \neq 1$. Without loss of generality we assume that $c = [a,b]$. Since $\langle a, c\rangle < G$ and $\langle b, c\rangle < G$, $\big[a,[a,c]\big] = \big[b,[b,c]\big] = 1$, and so $G_3 \le Z(H)$. Then $\big[b,[a,c]\big] = \big[b,[b,c]\big] = 1$. It follows that $c(G) = 3$. Let $[a,c] = e$ and $[b,c] = d$. We have $G = \langle a, b\rangle$ with the following relations:

$$a^p = b^p = c^p = d^p = e^p = 1, [a,b] = c, [a,c] = e \in Z(G), [b,c] = d \in Z(G)\backslash\{1\}.$$

Since $|G| \ge p^5$, $e \notin \langle d\rangle$. If $p = 3$, $\langle ba, c\rangle \cong M_3(2,1)$, a contradiction. Thus we get a group of type (iv) in the statement of the theorem.

Subcase 2: $|G/\langle a, b, Z(H)\rangle| = p$. Since $p^2 \le |\langle a, c, Z(H)\rangle/Z(H)| \le p^3$ and $|G/Z(H)| = p^3$, $|G/\langle a, c, Z(H)\rangle| \le p$. If $G = \langle a, c, Z(H)\rangle$, replace c and b with b and c respectively, it turns into Subcase 1. If $|G/\langle a, c, Z(H)\rangle| = p$, then $G = \langle a, b, c\rangle$. Since $\langle [a,b]\rangle$ char $\langle a, b, Z(H)\rangle \trianglelefteq G$ and $o([a,b]) = p$, $[a,b] \in Z(G)$. Using the same argument, we can also demonstrate that $[a,c] \in Z(G)$. Hence, $G = \langle a, b, c\rangle$ with the following relations:

$$[a,b] = h_1 \in Z(G)\backslash\{1\}, [a,c] = h_2 \in Z(G)\backslash\{1\}, [b,c] = d \in Z(G)\backslash\{1\}.$$

If $h_1 \notin \langle d\rangle$ and $h_2 \in \langle d, h_1\rangle$, let $h_2 = d^s h_1^t$. Replacing a, c and h_1 with ab^{-s}, cb^{-t} and e respectively, we get a group of type (v) in the statement of the theorem. If $h_1 \notin \langle d\rangle$ and $h_2 \notin \langle d, h_1\rangle$, replace h_2, h_1 and d with h_3, h_2 and h_1 respectively. We get a group of type (vi) in the statement of the theorem. If $h_1 \in \langle d\rangle$, then $h_2 \notin \langle d\rangle$ since $|G| \ge p^5$. Replace b and c with c and b^{-1} respectively. It turns into $[a,b] \notin \langle d\rangle$ and $[a,c] \in \langle d\rangle$, *i.e.*, $h_1 \notin \langle d\rangle$ and $h_2 \in \langle d\rangle$.

Case 2: $x \in M$. For any $y \in \langle b, c\rangle$ we have $[a,y] = 1$. Note that $\langle a, x, b, c\rangle \ncong M_p(1,1,1) \times C_p^n$. Then $G = \langle a, x, b, c\rangle$ with the following relations:

$$[b,c] = d \in Z(G)\backslash\{1\}, \quad [a,x] = h \in Z(G)\backslash\{1\},$$
$$[a,b] = [a,c] = [x,b] = [x,c] = 1.$$

If $h \notin \langle d\rangle$, then G possesses a proper subgroup $\langle ac, x, b\rangle$ which is not s-self dual, contradicting the assumption of the theorem. Hence, $h = d^i$, where $(p,i) = 1$. Replacing a with a^s, where $is \equiv 1 (\mathrm{mod}\, p)$, we get a group of type (vii) in the statement of the theorem.

Case 3: $x = sk$**, where** $s \in \langle b, c\rangle\backslash\{1\}, k \in M\backslash\{1\}$. Assume that $M = \langle k\rangle \times \langle k_1\rangle \times \cdots \times \langle k_n\rangle$. If $s \in \langle b, c\rangle\backslash\Phi(\langle b, c\rangle)$, there exists $f \in \langle b, c\rangle\backslash\Phi(\langle b, c\rangle)$ such that $H = \langle sk, f\rangle \times M$. We can replace b and c with sk and f respectively. Then $b = x$, it turns into Case 1. If $s \in \Phi(\langle b, c\rangle)\backslash\{1\}$, then $H = \langle b, c\rangle \times \langle sk\rangle \times \cdots \times \langle k_n\rangle$. We can replace M with $\langle sk\rangle \times \langle k_1\rangle \times \cdots \times \langle k_n\rangle$. Then $b \in M$, it turns into Case 2.

Next, we need to prove that the groups of types (i)–(vii) are not isomorphic to each other. By Lemma 2.5, we know that the groups of types (i), (ii) and (iii) are not mutually isomorphic. Since $|G| = p^4$ for types (i)–(iii), $|G| = p^5$ for types (iv), (v), and (vii), and $|G| = p^6$ for type (vi), we only need to prove that the groups of type (iv), (v), and (vii) are not isomorphic to each other. Since $d(G) = 2$ for type (iv), $d(G) = 3$

for type (v) and $d(G) = 4$ for type (vii), the groups of types (iv), (v), and (vii) are not isomorphic to each other.

It is trivial to check that the groups in the theorem satisfy the conditions in the theorem. ■

4 Finite p-groups whose Nonabelian Maximal Subgroups are all Isomorphic to $M_p(n,n) \times M$, where $n \geq 2, M$ is Abelian and $\exp(M) < p^n$

Lemma 4.1 *Assume that nonabelian maximal subgroups of a minimal non s-self dual p-group G are all isomorphic to $M_p(n,n) \times M$, where M is abelian and $\exp(M) < p^n$. If H is a nonabelian maximal subgroup of G and $G' \leq Z(H)$, then $o(a) > p$ and $a^p \in M \backslash \Phi(M)$ for any $a \in G \backslash H$.*

Proof Let $H = \langle x, y | x^{p^n} = y^{p^n} = 1, [x,y] = x^{p^{n-1}} \rangle \times M$, where M is abelian, $\exp(M) = p^m$ and $m < n$. For any $a \in G \backslash H$, we consider three cases:

(1) $a^p \in \langle x, y \rangle \backslash \langle x^p, y^p \rangle$;
(2) $a^p \in \langle x^p, y^p \rangle$;
(3) $a^p = de$, where $d \in \langle x, y \rangle$ and $e \in M \backslash \{1\}$.

Case 1. By $\langle x, y \rangle \cong M_p(n,n)$, we know that $o(a) = p^{n+1}$ and there exists $b \in H \backslash Z(H)$ such that $\langle a^p, b \rangle = \langle x, y \rangle$. Hence $G = \langle a, b \rangle$. Since $G' \leq Z(H)$ and $\exp\big(Z(H)\big) = p^{n-1}$, $\exp(G') \leq p^{n-1}$. Note that $o(a) = p^{n+1}$ and $o(b) = p^n$, we have that $\langle [a,b], a \rangle$, $\langle [a,b], b \rangle$ and $\langle a, b^p \rangle$ are abelian, and hence $[a,b] \in Z(G)$ and $[a, b^p] = 1$, which implies $[a^p, b] = [a, b^p] = 1$, contradicting $\langle a^p, b \rangle \cong M_p(n,n)$.

Case 2. Without loss of generality, we assume that $a^p = x^p$ or $a^p = y^p$. If $a^p = x^p$, $[a,x] = 1$ by $\langle a, x \rangle < G$, then $o(ax^{-1}) = p$, and so $G \cong M_p(n,n) \times M \times C_p$, contradicting the assumption of the theorem. Thus $a^p \neq x^p$. Using the same argument, we have $a^p \neq y^p$.

Case 3. If $e \in \Phi(M)$, let $e = h^p$, where $h \in M$, then $[a,h] = 1$ since $\langle a, h \rangle < G$ and $\langle a, h \rangle \ncong M_p(n,n)$. Replace a with ah^{-1}, then $a^p \in \langle x, y \rangle$. It turns into Case 1 or Case 2. If $e \in M \backslash \Phi(M)$ and $d \in \langle x, y \rangle \backslash \langle x^p, y^p \rangle$. There exists $s \in \langle x, y \rangle$ such that $\langle d, s \rangle = \langle x, y \rangle$. We have $\langle de, s \rangle \cong \langle x, y \rangle$ and $H = \langle de, s \rangle \times M$. Replace $\{x, y\}$ with $\{de, s\}$, we have $a^p \in \langle x, y \rangle \backslash \langle x^p, y^p \rangle$, it turns into Case 1. If $e \in M \backslash \Phi(M)$, $d \in \langle x^p, y^p \rangle \backslash \{1\}$ and $o(d) \leq o(e)$, we may replace e with $d^{-1}e$, we get $a^p = e \in M \backslash \Phi(M)$. If $e \in M \backslash \Phi(M)$, $d \in \langle x^p, y^p \rangle \backslash \{1\}$ and $o(d) > o(e)$, let $d = x^{ip} y^{jp}$, we have $a^p = x^{ip} y^{jp} e$. Since $G' \leq Z(H)$ and $a^p \notin M \backslash \Phi(M)$, $d(G) \geq 3$. We have $o([a, x^i]) \leq p$ and $o([a, y^j]) \leq p$. Replace a and e with $a(x^i y^j)^{-1}$ and $[a, x^i]^{\binom{p}{2}} [a, y^j]^{\binom{p}{2}} [x,y]^{ij\binom{p}{2}} e$ respectively, we get $a^p = e \in M \backslash \Phi(M)$. Hence $a^p \in M \backslash \Phi(M)$ for any $a \in G \backslash H$. ■

Theorem 4.2 *Let G be a minimal non s-self dual p-group. If G possesses a nonabelian maximal subgroup and nonabelian maximal subgroups of G are all isomorphic* *to* $M_p(n,n) \times M$, *where* $n \geq 2$, M *is abelian and* $\exp(M) < p^n$, *then* G *is one of the following non isomorphic groups:*

(i) $\langle a, b | a^8 = b^4 = 1, [a,b] = a^2 \rangle$;
(ii) $\langle a, b | a^8 = b^4 = 1, [a,b] = a^{-2} \rangle$;
(iii) $\langle a, b | a^4 = b^4 = c^4 = 1, [a,b] = c, [a,c] = c^2, [b,c] = c^2 \rangle$;
(iv) $\langle a, b, c | a^{p^n} = b^{p^n} = c^{p^n} = 1, [a,b] = 1, [a,c] = a^{p^{n-1}}, [b,c] = b^{p^{n-1}} \rangle$, *where* $n \geq 2$, *when* $p = 2$, $n \geq 3$;
(v) $M_p(n,n) \times C_{p^2}$, *where* $n \geq 2$.

Proof Let H be a maximal subgroup of G, and $H = \langle x, y | x^{p^n} = y^{p^n} = 1, [x,y] = x^{p^{n-1}} \rangle \times M$, where M is abelian and $\exp(M) < p^n$. We have $G = \langle a, H \rangle$ for any $a \in G \backslash H$.

Since $Z(H) = \langle x^p, y^p \rangle \times M$, we have $\langle a, c \rangle < G$ and $\langle a, c \rangle$ is abelian for any $c \in Z(H)$. Thus $c \in Z(G)$, which implies $\langle x^p, y^p \rangle \times M \leq Z(G)$. Since $|G/Z(H)| = p^3$, we consider two cases:

(1) $G' \nleq Z(H)$,
(2) $G' \leq Z(H)$.

Case 1: $G' \nleq Z(H)$. By $|G/Z(H)| = p^3$ and $Z(H) \leq Z(G)$, we know that $Z(H) = Z(G)$ and there exists $b, c \in H$ such that $[a,b] = c$ and $\langle b, c \rangle = \langle x, y \rangle$. Hence, $G = \langle a, b \rangle$ with the following relations:

$$a^{p^t} = b^{p^n} = c^{p^n} = 1, a^p \in H, \quad [a,b] = c, [b,c] = b^{ip^{n-1}} c^{jp^{n-1}} \in Z(G),$$
$$[a,c] = a^{kp^{t-1}} c^{sp^{n-1}} \in Z(G), \quad o(b^{ip^{n-1}} c^{jp^{n-1}}) = p, \quad o(a^{kp^{t-1}} c^{sp^{n-1}}) \leq p.$$

It follows from $a^{p^2} \in Z(G)$ that $1 = [a^{p^2}, b] = c^{p^2} [a,c]^{\binom{p^2}{2}} = c^{p^2}$, then $n = 2$ and $\exp(Z(G)) = p$. By $a^{p^2} \in Z(G)$, we have $t = 2$ or 3.

It follows from $b^{ip} c^{jp} \in Z(G)$ and $o(b^{ip} c^{jp}) = p$ that $1 = [a, b^{ip} c^{jp}] = c^{ip}$, *i.e.*, $p | i$, then $[b,c] = c^{jp}$, where $(j,p) = 1$. Replace b and c with b^s and $[a, b^s]$ respectively, where $sj \equiv 1 (\mathrm{mod}\, p)$, then $[b,c] = c^p$. Since $b^p \in Z(G)$, $[a, b^p] = c^{p(1-\binom{p}{2})} = 1$, then $p = 2$. Thus $G = \langle a, b \rangle$ with the following relations:

$$o(a) = 4 \text{ or } 8, \quad a^2 \in H, \quad b^4 = c^4 = 1,$$
$$[a,b] = c, \quad [b,c] = c^2, \quad [a,c] = a^{k2^{t-1}} c^{2s}.$$

If $o(a) = 8$, then $[a,c] = 1$ and $a^2 \in \langle b, c \rangle$. By $[a^2, b] \neq 1$, we have $\langle a^2, b \rangle = \langle b, c \rangle$, then $c = a^2, a^{-2}, a^2 b^2$ or $a^{-2} b^2$. When $c = a^2$, we get a group of type (i) in the statement of the theorem. When $c = a^{-2}$, we get a group of type (ii) in the statement of the theorem. When $c = a^2 b^2$, $o(ab) = 4$. Replace a with ab; then $o(a) = 4$. Thus we get a group of type (iii) in the statement of the theorem. When $c = a^{-2} b^2$, $o(ab) = 2$, then $ab \in G \backslash H$ and $ab \in Z(G)$. We have $G \cong M_p(n,n) \times M \times C_p$, contradicting the assumption of the theorem.

If $o(a) = 4$, we have $[a,c] = c^2$ by $1 = [a^2, b] = c^2 [c,a]$. Since G is not metacyclic, we have $|M| \geq p$ by Lemma 2.6, so $a^2 \notin \langle b, c \rangle$. Thus we get a group of type (iii) in the statement of the theorem.

Case 2: $G' \leq Z(H)$**.** By Lemma 4.1, we have $o(a) > p$ and $a^p \in M \backslash \Phi(M)$, then $d(G) \geq 3$. We claim that $d(G) = 3$. (Otherwise, $d(G) \geq 4$, then $\langle x, y, a \rangle < G$. But $\langle x, y, a \rangle \ncong M_p(n,n) \times M$, where $\exp(M) < p^n$, a contradiction). We have $M = \langle a^p \rangle$, and hence $G = \langle a, x, y \rangle$.

Subcase 1: $\langle a, x \rangle \cong M_p(n,n)$**.** Since $\langle a, y \rangle < G$, $\langle a, y \rangle$ is abelian or isomorphic to $M_p(n,n)$. Let $[a,x] = a^{ip^{n-1}} x^{jp^{n-1}}$, $[a,y] = a^{sp^{n-1}} y^{tp^{n-1}}$.

If $p = n = 2$, we have $[a,y] = y^2$, a^2 or 1 (otherwise, $[a,y] = a^2 y^2$, then $o(ay) = 2$, contradicting Lemma 4.1). Since $\langle a, xy \rangle \cong M_2(2,2)$, $[a,xy] = a^{2i} x^{2j} [a,y] \in \langle a^2, y^2 \rangle$, then $2 | j$. It follows that $[a,x] = a^2$. Replace x and y with b and c respectively, we have:

(i) $[a,b] = a^2$, $[a,c] = c^2$, $[b,c] = b^2$;
(ii) $[a,b] = a^2$, $[a,c] = 1$, $[b,c] = b^2$;
(iii) $[a,b] = a^2$, $[a,c] = a^2$, $[b,c] = b^2$.

We know that $o(abc) = 2$ for type (i), contradicting Lemma 4.1. We replace c with bc for type (iii), it turns into type (ii). But $\langle ac, b \rangle < G$ and $\langle ac, b \rangle \cong M_2(2,2,1)$ for type (ii), a contradiction.

Except for $p = n = 2$. We claim that $[a,x] = x^{p^{n-1}}$ and $[a,y] = a^{p^{n-1}} y^{p^{n-1}}$. Note that $\langle ay, x \rangle < G$, then $[ay, x] = a^{ip^{n-1}} x^{(j-1)p^{n-1}} \in \langle (ay)^{p^{n-1}}, x^{p^{n-1}} \rangle$, so $i = 0$. Replace a with a^s, where $sj \equiv 1 (\text{mod}\, p)$, then $[a,x] = x^{p^{n-1}}$. Note that $\langle a, xy \rangle \cong M_p(n,n)$ and $\langle ax, y \rangle \cong M_p(n,n)$, then $[a, xy] = a^{sp^{n-1}} x^{p^{n-1}} y^{tp^{n-1}} \in \langle a^{p^{n-1}}, (xy)^{p^{n-1}} \rangle$ and $[ax, y] = a^{sp^{n-1}} x^{p^{n-1}} y^{tp^{n-1}} \in \langle (ax)^{p^{n-1}}, y^{p^{n-1}} \rangle$, we have $s = t = 1$, *i.e.*, $[a,y] = a^{p^{n-1}} y^{p^{n-1}}$. Replace ay, x and y with a, b and c respectively, we have $[a,b] = 1$ and $[a,c] = a^{p^{n-1}}$. Thus we get a group of type (iv) in the statement of the theorem.

Subcase 2: $\langle a, x \rangle$ **is abelian.** If $[a,y] = 1$, we get a group of type (v) in the statement of the theorem. If $[a,y] \neq 1$, we have $\langle a, y \rangle \cong M_p(n,n)$ by $\langle a, y \rangle < G$. Let $[a,y] = a^{ip^{n-1}} y^{jp^{n-1}}$. Since $\langle ax, y \rangle < G$, $[ax, y] = a^{ip^{n-1}} x^{p^{n-1}} y^{jp^{n-1}} \in \langle a^{p^{n-1}} x^{p^{n-1}}, y^{p^{n-1}} \rangle$, then $i = 1$. When $p = n = 2$, we have $ay \in G \backslash H$ and $(ay)^2 \notin M \backslash \Phi(M)$, contradicting Theorem 4.1. Except for $p = n = 2$, we have $[a, xy] = a^{p^{n-1}} y^{jp^{n-1}} \in \langle a^{p^{n-1}}, x^{p^{n-1}} y^{p^{n-1}} \rangle$ by $\langle a, xy \rangle \cong M_p(n,n)$, then $p \mid j$. Replace x and y with b and c respectively. Thus we get a group of type (iv) in the statement of the theorem.

In the following, we prove that the groups of types (i)–(iv) are not isomorphic to each other. Since $d(G) = 2$ and G is metacyclic for type (i) and (ii), $d(G) = 2$ and G is not metacyclic for type (iii), $d(G) = 3$ and $G' \cong C_p^2$ for type (iv), and $d(G) = 3$ and $G' \cong C_p$ for type (v), we only need to prove that the groups of type (i) and (ii) are not isomorphic to each other. If $G_1 = \langle a_1, b_1 \rangle$ for type (i) is isomorphic to $G_2 = \langle a_2, b_2 \rangle$ for type (ii). Let the map $\sigma\colon a_1 \longmapsto b_2^j a_2^i, b_1 \longmapsto b_2^l a_2^k$ be an isomorphism from G_1 onto G_2. Since $G_2' = (a_1^2)^\sigma = (b_2^j a_2^i)^2$, $2 \mid j$ and $2 \nmid l$. Since again $(b_1^2)^\sigma = b_2^{2l}$, the map σ is an isomorphism from $\langle b_1^2 \rangle$ onto $\langle b_2^2 \rangle$. By $\langle b_1^2 \rangle \leq Z(G_1)$ and $\langle b_2^2 \rangle \leq Z(G_2)$, we know that $G_1/\langle b_1^2 \rangle \cong G_2/\langle b_2^2 \rangle$. But $G_1/\langle b_1^2 \rangle \cong SD_{16}$ and $G_2/\langle b_2^2 \rangle \cong D_{16}$, a contradiction. Hence, the groups of type (i) and (ii) are not isomorphic to each other.

https://doi.org/10.4153/CMB-2015-007-9 Published online by Cambridge University Press

It is trivial to check that the groups in the theorem satisfy the conditions in the theorem. ■

5 Minimal Non s-self Dual Groups and Minimal Non Self Dual Groups

Lemma 5.1 *Let G be a minimal nonabelian p-group.*

(i) *If G is not s-self dual, then G is one of the following non-isomorphic groups:*
  (a) Q_8;
  (b) $M_p(n,m)$, *where* $n \neq m$;
  (c) $M_p(n,m,1)$,*where* $n+m \geq 3$.

(ii) *If G is not self dual, then G is one of the following non-isomorphic groups:*
  (a) Q_8;
  (b) $M_p(n,m)$;
  (c) $M_p(n,m,1)$, *where* $n+m \geq 3$.

Proof We can get the groups in the statement of the lemma by Theorem 1.2 and by Lemmas 2.3 and 2.1. ■

Lemma 5.2 *Let G be a finite p-group. If there exists a maximal subgroup H of G isomorphic to* $M_p(1,1,1) \times C_p^m$, *then no maximal subgroup K of G can be isomorphic to* $M_p(n,n) \times M$, *where* $n \geq 2$, *M is abelian and* $\exp(M) < p^n$.

Proof Suppose that there exists a maximal subgroup K of G isomorphic to $M_p(n,n) \times M$, where M is abelian and $\exp M < p^n$. It follows from $G = HK$ and $HK/H \cong K/H \cap K$ that $|K/H \cap K| = |G/H| = p$. Using the same argument, we can also demonstrate that $|H/H \cap K| = p$. But there exists no $H \cap K$ such that $|K/H \cap K| = |H/H \cap K| = p$, a contradiction. ■

Theorem 5.3 *Let G be a minimal non s-self dual group.*

(i) *If G is nilpotent, then G is determined by Lemma 5.1* (i)*, Theorem 3.2, and Theorem 4.2.*
(ii) *If G is not nilpotent, then* $G \cong P \rtimes \langle c \rangle$, *where P is a s-self dual p-group,* $\langle c \rangle \cong C_{q^m}$ *is an automorphism of P of order q and p, q are unequal primes.*

Proof If G is a nilpotent group, then G is a p-group. By Lemmas 2.1 and 5.2, we can get case (i). If G is not a nilpotent group, it follows from Lemmas 2.1 and 2.8 and the assumption that we can get case (ii). ■

Theorem 5.4 *Let G be a minimal non self dual group.*

(i) *If G is nilpotent, then G is determined by Lemma 5.1* (ii) *and Theorem 3.2.*
(ii) *If G is not nilpotent, then* $G \cong P \rtimes \langle c \rangle$, *where P is a self dual p-group,* $\langle c \rangle \cong C_{q^m}$ *is an automorphism of P of order q and p, q are unequal primes.*

Proof If G is a nilpotent group, then G is a p-group. By Lemmas 2.2 and 5.2, we can get case (i). If G is not a nilpotent group, it follows from Lemmas 2.2 and 2.8 and the assumption that we can get case (ii). ■

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