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THE FUNDAMENTAL AND NUMERICAL SOLUTIONS OF THE RIESZ SPACE-FRACTIONAL REACTION–DISPERSION EQUATION

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Abstract

A Riesz space-fractional reaction–dispersion equation (RSFRDE) is obtained from the classical reaction–dispersion equation (RDE) by replacing the second-order space derivative with a Riesz derivative of order $\beta \in (1, 2]$. In this paper, using Laplace and Fourier transforms, we obtain the fundamental solution for a RSFRDE. We propose an explicit finite-difference approximation for a RSFRDE in a bounded spatial domain, and analyse its stability and convergence. Some numerical examples are presented.

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1. Introduction

Space-fractional diffusion equations have been shown to be useful as models of anomalous transport in many diverse disciplines, including finance, semiconductor research, biology and hydrogeology [9, 19]. For example, they have been used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium [1, 22] or in financial markets to model high-frequency price dynamics [21, 25]. Feller [4] provided a basic analytic theory for the space-fractional diffusion processes via inversion of the Riesz potential. Mainardi *et al.* [17] presented an explicit representation of the Green function for the space-fractional diffusion equation, and provided a general representation of the Green functions for which the fundamental solution can be interpreted as a spatial probability density. Gorenflo and

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Mainardi [6] considered random walk models for space-fractional diffusion processes. Gorenflo *et al.* [5] used the method of Laplace transform to obtain the Wright function as the scale-invariant solution of the diffusion-wave equation. Benson *et al.* [2, 3] considered the space-fractional advection–dispersion equation and gave an analytic solution in terms of the α-stable process. Mainardi [16] obtained the fundamental solutions for the basic Cauchy and signalling problems for a time-fractional diffusion-wave equation. Liu *et al.* [13] derived the complete solution of the time-fractional advection–dispersion equation.

However, numerical methods and theoretical analyses of fractional differential equations are still at an early stage of development. Lin and Liu [10] proposed higher-order (2–6) approximations of a nonlinear fractional-order ordinary differential equation with initial value and proved the consistency, convergence and stability of the fractional higher-order methods. Lynch *et al.* [15] presented two different discretization methods for fractional-order equations, but stability and convergence was not presented. Shen and Liu [23] estimated the discretization error of the space-fractional diffusion equation. Liu *et al.* [12] presented the numerical solution of a space-fractional Fokker–Planck equation. Meerschaert *et al.* [18] considered the finite-difference approximations for two-sided space-fractional partial differential equations and discussed their stability, consistency and convergence.

Henry and Wearne [8] considered a two-species fractional reaction–diffusion system. Fractional reaction–diffusion equations can be used to model activator–inhibitor dynamics with anomalous diffusion, which occurs in spatially inhomogeneous media [8]. To the best of the authors' knowledge, this area has not been explored vigorously.

In this paper we define a Riesz space-fractional reaction–dispersion equation (RSFRDE). Using the method of Laplace and Fourier transforms, we obtain their fundamental solutions. We then propose an explicit finite-difference approximation (EFDA) scheme for these equations in a bounded spatial domain, and analyse its stability and convergence. Some numerical examples will be presented to show the application of the technique.

2. The fundamental solution of the RSFRDE

The following RSFRDE with initial and boundary conditions is considered:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -u(x,t) + {}_xD_0^{\beta} u(x,t), & x \in \mathbf{R},\ t \in \mathbf{R}^+, \\ u(x,0) = g(x), & x \in \mathbf{R}, \\ u(\pm\infty, t) = 0, & t \in \mathbf{R}^+, \end{cases} \tag{2.1}$$

where ${}_xD_0^{\beta}$ is the Riesz fractional derivative of order β for $1 < \beta \le 2$, defined by analytic continuation in the whole range $0 < \beta \le 2$, $\beta \ne 1$ (see [7]) as

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$$ {}_xD_0^{\beta} := -c(I_+^{-\beta} + I_-^{-\beta}), \tag{2.2} $$

$$ \text{where} \quad c = \frac{1}{2\cos(\beta\pi/2)}, \quad I_{\pm}^{-\beta} = \frac{d^2x}{dx^2} I_{\pm}^{2-\beta}, $$

and the Weyl integrals $I_{\pm}^{\beta}$ defined in [20] are as follows:

$$ \begin{cases} (I_+^{\beta}\phi)(x) = \dfrac{1}{\Gamma(\beta)} \displaystyle\int_{-\infty}^{x} (x-\xi)^{\beta-1}\phi(\xi)\, d\xi, & \beta > 0, \\ (I_-^{\beta}\phi)(x) = \dfrac{1}{\Gamma(\beta)} \displaystyle\int_{x}^{+\infty} (\xi-x)^{\beta-1}\phi(\xi)\, d\xi, & \beta > 0, \end{cases} $$

where $\phi(x) \in L_1(-\infty, +\infty)$.

Note that ${}_xD_0^{\beta}$ is a pseudo-differential operator with the symbol ${}_x\widehat{D}_0^{\beta}(k) = -|k|^{\beta}$. In particular, we have ${}_xD_0^2 = d^2/dx^2$, but ${}_xD_0^1 \neq d/dx$. Throughout the remainder of this section we derive the fundamental solution to (2.1) by applying Laplace and Fourier transforms to (2.1) with an initial condition with respect to the variables t and x. Recall the following formulae proved in [17]:

$$ L\{f'(t); s\} = s\tilde{f}(s) - f(0^+), \quad \mathcal{F}\{{}_xD_0^{\beta} f(x); k\} = -|k|^{\beta}\widehat{f}(k). $$

Applying the Laplace transform to (2.1) produces the following nonhomogeneous differential equation:

$$ s\tilde{u}(x, s) - g(x) = -\tilde{u}(x, s) + {}_xD_0^{\beta}\tilde{u}(x, s). \tag{2.3} $$

Next, application of the Fourier transform to (2.3) with respect to the variable x taking into account the Fourier transform of the Riesz fractional derivative, yields

$$ s\widehat{\tilde{u}}(k, s) - \widehat{g}(k) = -\widehat{\tilde{u}}(k, s) - |k|^{\beta}\,\widehat{\tilde{u}}(k, s). \tag{2.4} $$

From (2.4) we obtain

$$ \widehat{\tilde{u}}(k, s) = \frac{\widehat{g}(k)}{s - (-1 - |k|^{\beta})}. $$

By using the known Laplace transform

$$ e^{ct} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-c}, \quad \mathrm{Re}(s) > |c|, $$

where $c \in \mathbf{R}$, we have that

$$ e^{(-1-|k|^{\beta})t}\,\widehat{g}(k) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\widehat{g}(k)}{s - (-1 - |k|^{\beta})} = \widehat{\tilde{u}}(k, s). \tag{2.5} $$

Inverting the Laplace transform in (2.5) gives

$$ \widehat{u}(k, t) = e^{(-1-|k|^{\beta})t}\,\widehat{g}(k). \tag{2.6} $$

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To invert the Fourier transform in (2.6), we recall the formulae

$$f(x) = \mathcal{F}^{-1}\{\widehat{f}(k); x\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}\widehat{f}(k)\,dk, \quad x \in \mathbf{R},$$
$$\widehat{f}(k) = \int_{-\infty}^{+\infty} e^{ikx} f(x)\,dx, \quad k \in \mathbf{R}.$$

Then,

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx} e^{(-1-|k|^{\beta})t}\,\widehat{g}(k)\,dk = \int_{-\infty}^{+\infty} G_{\beta}(x-y,t)g(y)\,dy,$$

where

$$G_{\beta}(x-y,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(-1-|k|^{\beta})t-ik(x-y)}\,dk,$$

the Green's function of (2.1) obtained when $g(x) = \delta(x)$ (the Dirac delta function).

3. An EFDA for RSFRDE

In this section, we obtain the numerical solution of a RSFRDE in a bounded spatial domain

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -u(x,t) + {}_xD_0^{\beta}u(x,t), & 0 < x < b,\ 0 \le t \le T, \\ u(x,0) = g(x), & 0 \le x \le b, \\ \dfrac{\partial u(0,t)}{\partial x} = 0, \quad u(b,t) = 0, & 0 \le t \le T, \end{cases} \tag{3.1}$$

where $1 < \beta \le 2$, and we assume that both $u(x,t)$ and $g(x)$ are real-valued and sufficiently well-behaved functions. We discretize the Riesz derivative to derive a numerical solution for the RSFRDE.

From (2.2), using the boundary conditions $\partial u(0,t)/\partial x = 0$ and $u(b,t) = 0$,

$$\begin{aligned} I_+^{-\beta} &= \frac{d^2x}{dx^2} I_+^{2-\beta} = \frac{d^2}{dx^2}\left[\frac{1}{\Gamma(2-\beta)}\int_0^x (x-\xi)^{1-\beta}u(\xi,t)\,d\xi\right] \\ &= \frac{u(0,t)x^{-\beta}}{\Gamma(1-\beta)} + \frac{1}{\Gamma(2-\beta)}\int_0^x \frac{\partial^2 u(\xi,t)}{\partial \xi^2}(x-\xi)^{1-\beta}\,d\xi, \end{aligned} \tag{3.2}$$

$$\begin{aligned} I_-^{-\beta} &= \frac{d^2x}{dx^2} I_-^{2-\beta} = \frac{d^2}{dx^2}\left[\frac{1}{\Gamma(2-\beta)}\int_x^b (\xi-x)^{1-\beta}u(\xi,t)\,d\xi\right] \\ &= \frac{1}{\Gamma(2-\beta)}\left[-(b-x)^{1-\beta}\frac{\partial u(b,t)}{\partial x} - \int_b^x (\xi-x)^{1-\beta}u''_{\xi}(\xi,t)\,d\xi\right]. \end{aligned} \tag{3.3}$$

Define $\Delta t = \tau$ as the grid step in time, $t_n = n\tau$, $0 \le t_n \le T$, as the integration time, $\Delta x = h > 0$ as the grid size in the spatial variable x, $h = b/L$, L being a

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positive integer, $u(x_l, t_n) = u(lh, n\tau)$, $u_L^n = u(b, n\tau)$ and u_l^n denotes the numerical solution at point (x_l, t_n). Using a second-order difference approximation, the resulting discretization on $I_+^{-\beta}$ and $I_-^{-\beta}$ takes the following form:

$$I_+^{-\beta} u(x_l, t_n) \approx \frac{h^{-\beta}}{\Gamma(3-\beta)}\left[\frac{u_0^n(1-\beta)(2-\beta)}{l^\beta} + \sum_{j=0}^{l-1} c_j\big(u_{l-j+1}^n - 2u_{l-j}^n + u_{l-j-1}^n\big)\right] \quad \text{and}$$

$$I_-^{-\beta} u(x_l, t_n) \approx \frac{h^{-\beta}}{\Gamma(3-\beta)}\left[-\frac{(2-\beta)(u_L^n - u_{L-1}^n)}{(L-l)^{\beta-1}} + \sum_{j=0}^{L-l-1} c_j\big(u_{l+j-1}^n - 2u_{l+j}^n + u_{l+j+1}^n\big)\right],$$

where $c_j = (j+1)^{2-\beta} - j^{2-\beta}$.

Substituting the above expressions into (3.2) and (3.3), we obtain a finite-difference approximation for Equation (3.1) as

$$\begin{aligned}\frac{u_l^{n+1} - u_l^n}{\tau} = -u_l^n - \frac{h^{-\beta}}{2\cos(\beta\pi)\Gamma(3-\beta)}\Bigg[& u_0^n(1-\beta)(2-\beta)l^{-\beta} \\ &+ \sum_{j=0}^{l-1} c_j(u_{l-j+1}^n - 2u_{l-j}^n + u_{l-j-1}^n) - (2-\beta)(L-l)^{1-\beta}(u_L^n - u_{L-1}^n) \\ &+ \sum_{j=0}^{L-l-1} c_j(u_{l+j-1}^n - 2u_{l+j}^n + u_{l+j+1}^n)\Bigg].\end{aligned} \tag{3.4}$$

The above equation together with the boundary conditions can be written as the following EFDA:

$$\begin{aligned}u_l^{n+1} = u_l^n - \tau u_l^n + d_1(l)u_0^n + k\sum_{j=0}^{l-1} c_j\big(u_{l-j+1}^n - 2u_{l-j}^n + u_{l-j-1}^n\big)& \\ + k\sum_{j=0}^{L-l-1} c_j\big(u_{l+j-1}^n - 2u_{l+j}^n + u_{l+j+1}^n\big) + d_2(l)u_{L-1}^n,&\end{aligned} \tag{3.5}$$

for $l = 1, \ldots, L-1$, where k, $d_1(l)$ and $d_2(l)$ are given by the expressions

$$\begin{cases} k = -\dfrac{\tau h^{-\beta}}{2\cos(\beta\pi/2)\Gamma(3-\beta)} > 0, \\ d_1(l) = k(1-\beta)(2-\beta)l^{-\beta} < 0, \\ d_2(l) = k(2-\beta)(L-l)^{1-\beta} > 0. \end{cases} \tag{3.6}$$

The EFDA (3.5) can be written in the matrix form $U^{n+1} = BU^n$, where $U^n = (u_1^n, u_2^n, \ldots, u_{L-1}^n)^{\mathrm{T}}$ and $B = (b_{ij})_{(L-1)\times(L-1)}$ is a matrix of coefficients.

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4. Analysis of stability

Let U represent the exact solution of the partial differential equation (3.1), and let u be the exact solution of the EFDA, then the error $e = U - u$. To prove the stability and the convergence, we need the following lemmas.

LEMMA 4.1. *Let $A \in C^{m\times n}$ and let $\rho(A)$ be the spectral radius of the matrix A, then $\rho(A) \le \|A\|$ for any matrix norm.*

PROOF. See [24]. □

LEMMA 4.2. *Let $c_l = (l+1)^{2-\beta} - l^{2-\beta}$ $(l \ge 0)$, and let k, $d_1(l)$ and $d_2(l)$ be as defined in (3.6). Then:*

(1) $c_{l-1} > c_l > 0$, $c_{l-1} - 2c_l + c_{l+1} > 0$, $l \ge 1$;
(2) $d_2(l) + k(-2c_{L-l-1} + c_{L-l-2}) > 0$, $0 \le l \le L-1$;
(3) $d_2(l) - c_{L-l-1}k < 0$, $0 \le l \le L-1$;
(4) $d_1(l) + k(c_{l-2} - c_{l-1}) > 0$, $l \ge 2$.

PROOF. (1) Let $f(l) = c_l = (l+1)^{2-\beta} - l^{2-\beta}$. Then for any $l \ge 0$,

$$f'(l) = (2-\beta)[(l+1)^{1-\beta} - l^{1-\beta}] = (2-\beta)(1-\beta)\xi^{-\beta} < 0, \tag{4.1}$$
$$f''(l) = (2-\beta)(1-\beta)[(l+1)^{-\beta} - l^{-\beta}] = (2-\beta)(1-\beta)(-\beta)\eta^{-\beta-1} > 0, \tag{4.2}$$

where $\xi, \eta \in (l, l+1)$. It follows from (4.1) and (4.2) that

$$c_{l-1} > c_l > 0, \quad c_{l-1} - 2c_l + c_{l+1} > 0, \quad l \ge 1.$$

(2) Owing to $c_l = (l+1)^{2-\beta} - l^{2-\beta} = (2-\beta)\xi^{1-\beta}$, $\xi \in (l, l+1)$, it follows that

$$(2-\beta)(l+1)^{1-\beta} < c_l < (2-\beta)l^{1-\beta}.$$

Hence,

$$\begin{aligned} &d_2(l) + k(-2c_{L-l-1} + c_{L-l-2}) \\ &\quad = k(2-\beta)(L-l)^{1-\beta} + k(-2c_{L-l-1} + c_{L-l-2}) \\ &\quad = k[(2-\beta)(L-l)^{1-\beta} - c_{L-l}] + k(c_{L-l} - 2c_{L-l-1} + c_{L-l-2}) > 0. \end{aligned} \tag{4.3}$$

(3) From (4.3), $d_2(l) - kc_{L-l-1} = k[(2-\beta)(L-l)^{1-\beta} - c_{L-l-1}] < 0$.
(4) Owing to

$$\begin{aligned} c_{l-2} - c_{l-1} &= f(l-2) - f(l-1) \\ &= -(2-\beta)[(\xi+1)^{1-\beta} - \xi^{1-\beta}] \\ &= -(2-\beta)(1-\beta)\eta^{-\beta} > -(2-\beta)(1-\beta)l^{-\beta}, \end{aligned}$$

where $\xi \in (l-2, l-1)$ and $\eta \in (\xi, \xi+1)$, we obtain $(1-\beta)(2-\beta)l^{-\beta} + (c_{l-2} - c_{l-1}) > 0$. Noting that $k > 0$ and $d_1(l) = k(1-\beta)(2-\beta)l^{-\beta}$, we have $d_1(l) + k(c_{l-2} - c_{l-1}) > 0$. □

THEOREM 4.3. *Suppose that*

$$\tau + \frac{\tau h^{-\beta}}{\cos(\beta\pi/2)\Gamma(3-\beta)}(-3 + 2^{2-\beta}) < 1,$$

then the explicit finite-difference method (3.5) *for Equation* (3.1) *is stable.*

PROOF. Noting that $k > 0$, $d_1(l) < 0$, $d_2(l) > 0$ and

$$\sum_{j=1}^{L-1} |b_{1j}| = |1 - \tau + d_1(1) - 2k + kc_1| + |k + k(c_0 - 2c_1 + c_2)| + \sum_{j=1}^{L-4} |k(c_j - 2c_{j+1} + c_{j+2})| + |d_2(1) + k(-2c_{L-2} + c_{L-3})|, \tag{4.4}$$

applying Lemma 4.2, we conclude that, if $1 - \tau + d_1(1) - 2k + kc_1 > 0$, then $b_{11} \geq 0$. Thus,

$$\sum_{j=1}^{L-1} |b_{1j}| = 1 - \tau + d_1(1) + d_2(1) - kc_{L-2} < 1. \tag{4.5}$$

Similarly, if $1 - \tau + 2k(-2c_0 + c_1) > 0$, then when $2 \leq i \leq L-2$,

$$\sum_{j=1}^{L-1} |b_{ij}| = 1 - \tau + d_1(i) + d_2(i) - kc_{L-i-1} < 1, \tag{4.6}$$

and if $1 - \tau + d_2(L-1) + k(-4c_0 + c_1) > 0$,

$$\sum_{j=1}^{L-1} |b_{L-1\,j}| = 1 - \tau + d_1(L-1) + d_2(L-1) + k(c_1 - 2) < 1. \tag{4.7}$$

Combining (4.5)–(4.7), we easily conclude that, if $1 - \tau + 2k(-2c_0 + c_1) > 0$, that is,

$$\tau + \frac{\tau h^{-\beta}}{\cos(\beta\pi/2)\Gamma(3-\beta)}(-3 + 2^{2-\beta}) < 1,$$

then $\|B\|_\infty = \max_{1 \leq i \leq L-1} \sum_{j=1}^{L-1} |b_{ij}| < 1$. Furthermore, using Lemma 4.1 and according to the Lax–Richtmer definition of stability [24], we obtain that the EFDA (3.5) is conditionally stable. □

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5. Analysis of convergence

In order to prove convergence, we introduce the following propositions.

PROPOSITION 5.1. *Let* $u_j = u(jh, t)$,

$$I_+^{-\beta} u(x,t) = \frac{u(0,t)x^{-\beta}}{\Gamma(1-\beta)} + \frac{1}{\Gamma(2-\beta)} \int_0^x (x-\xi)^{1-\beta} \frac{\partial^2 u(\xi,t)}{\partial \xi^2}\, d\xi,$$

$$\tilde{I}_+^{-\beta} u(x,t) = \frac{h^{-\beta}}{\Gamma(3-\beta)} \left[u_1 \frac{(1-\beta)(2-\beta)}{l^\beta} + \sum_{j=0}^{l-1} c_j (u_{l-j+1} - 2u_{l-j} + u_{l-j-1}) \right]$$

and assume that $u(x,t)$ *is a smooth function. Then* $I_+^{-\beta} u(x,t) - \tilde{I}_+^{-\beta} u(x,t) = O(h^{2-\beta})$, *where* $x = lh$, $l = 1, \ldots, L-1$.

PROOF. Considering the standard central difference formula, we have

$$\begin{aligned} &\frac{h^{-\beta}}{\Gamma(3-\beta)} \sum_{j=0}^{l-1} c_j (u_{l-j+1} - 2u_{l-j} + u_{l-j-1}) \\ &\quad = \frac{h^{2-\beta}}{\Gamma(3-\beta)} \sum_{j=0}^{l-1} c_j \frac{\partial^2 u(x-jh,t)}{\partial z^2} + \frac{Cx^{2-\beta}}{\Gamma(3-\beta)} h^2. \end{aligned} \tag{5.1}$$

The mean value theorem of differential calculus then yields

$$\begin{aligned} &\frac{1}{\Gamma(2-\beta)} \int_0^x (x-\xi)^{1-\beta} \frac{\partial^2 u(\xi,t)}{\partial \xi^2}\, d\xi \\ &\quad = \frac{1}{\Gamma(2-\beta)} \sum_{j=0}^{l-1} \int_{jh}^{(j+1)h} z^{1-\beta} \frac{\partial^2 u(x-z,t)}{\partial z^2}\, dz \\ &\quad = \frac{h^{2-\beta}}{\Gamma(3-\beta)} \sum_{j=0}^{l-1} c_j \frac{\partial^2 u(x-\theta_1 h,t)}{\partial z^2}, \end{aligned} \tag{5.2}$$

where $\theta_1 \in [j, j+1]$.

From (5.1) and (5.2),

$$\begin{aligned} &\left| \frac{1}{\Gamma(2-\beta)} \int_0^x (x-\xi)^{1-\beta} \frac{\partial^2 u(\xi,t)}{\partial \xi^2}\, d\xi - \frac{h^{-\beta}}{\Gamma(3-\beta)} \sum_{j=0}^{l-1} c_j \big(u_{l-j+1} - 2u_{l-j} + u_{l-j-1}\big) \right| \\ &\quad = \left| \frac{h^{2-\beta}}{\Gamma(3-\beta)} \sum_{j=0}^{l-1} c_j \left[\frac{\partial^2 u(x-\theta_1 h,t)}{\partial z^2} - \frac{\partial^2 u(x-jh,t)}{\partial z^2} \right] - \frac{Cx^{2-\beta}}{\Gamma(3-\beta)} h^2 \right| \\ &\quad = \left| \frac{h^{2-\beta}}{\Gamma(3-\beta)} \sum_{j=0}^{l-1} c_j \frac{\partial^3 u(x-\theta_2 h,t)}{\partial z^3} (j-\theta_1) h - \frac{Cx^{2-\beta}}{\Gamma(3-\beta)} h^2 \right| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{Mh^{3-\beta}}{\Gamma(3-\beta)} \sum_{j=0}^{l-1} c_j + \frac{|C|b^{2-\beta}}{\Gamma(3-\beta)} h^2 \\
&= \frac{Mx^{2-\beta}}{\Gamma(3-\beta)} h + \frac{|C|b^{2-\beta}}{\Gamma(3-\beta)} h^2 \le \frac{Mb^{2-\beta}}{\Gamma(3-\beta)} h + \frac{|C|b^{2-\beta}}{\Gamma(3-\beta)} h^2 = O(h)
\end{aligned} \tag{5.3}
$$

where $\theta_2 \in [j, j+1]$.

Hence,

$$
\begin{aligned}
&\left| I_+^{-\beta} u(x,t) - \tilde{I}_+^{-\beta} u(x,t) \right| \\
&\quad = \Bigg| \frac{(u_1 + C' h^2) l^{-\beta} h^{-\beta}}{\Gamma(1-\beta)} + \frac{1}{\Gamma(2-\beta)} \int_0^x (x-\xi)^{1-\beta} \frac{\partial^2 u(\xi,t)}{\partial \xi^2}\, d\xi \\
&\qquad - \frac{h^{-\beta}}{\Gamma(3-\beta)} \left[u_1 \frac{(1-\beta)(2-\beta)}{l^\beta} + \sum_{j=0}^{l-1} c_j \big(u_{l-j+1} - 2u_{l-j} + u_{l-j-1} \big) \right] \Bigg| \\
&\quad \le \left| \frac{C' x^{-\beta}}{\Gamma(1-\beta)} h^2 \right| + Ch \le \left| \frac{C'}{\Gamma(1-\beta)} \right| h^{2-\beta} + Ch = O(h^{2-\beta}).
\end{aligned} \tag{5.4}
$$

□

PROPOSITION 5.2. *Let*

$$
I_-^{-\beta} u(x,t) = -\frac{u'(b,t)(b-x)^{1-\beta}}{\Gamma(2-\beta)} + \frac{1}{\Gamma(2-\beta)} \int_x^b (\xi - x)^{1-\beta} \frac{\partial^2 u(\xi,t)}{\partial \xi^2}\, d\xi,
$$

$$
\tilde{I}_-^{-\beta} u(x,t) = \frac{h^{-\beta}}{\Gamma(3-\beta)} \left[\sum_{j=0}^{L-l-1} c_j \big(u_{l+j-1} - 2u_{l+j} + u_{l+j+1} \big) - \frac{(2-\beta)(u_L - u_{L-1})}{(L-l)^{\beta-1}} \right].
$$

Then $I_+^{-\beta} u(x,t) - \tilde{I}_-^{-\beta} u(x,t) = O(h^{2-\beta})$, *where* $x = lh$, $l = 1, \ldots, L-1$.

PROOF. The proof is similar to that of Proposition 5.1. □

From Propositions 5.1 and 5.2, we obtain the following result.

PROPOSITION 5.3. *Let*

$$
\begin{aligned}
{}_x\tilde{D}_0^\beta u(x,t) &= -\frac{1}{2\cos(\beta\pi/2)} [\tilde{I}_+^{-\beta} u(x,t) + \tilde{I}_-^{-\beta} u(x,t)] \quad \textit{and} \\
{}_x D_0^\beta u(x,t) &= -\frac{1}{2\cos(\beta\pi/2)} [I_+^{-\beta} u(x,t) + I_-^{-\beta} u(x,t)].
\end{aligned}
$$

Then ${}_x\tilde{D}_0^\beta u(x,t) = {}_x D_0^\beta u(x,t) + O(h^{2-\beta})$, *where* $x = lh$, $l = 1, \ldots, L-1$.

REMARK 1. The explicit finite-difference scheme (3.5) has a local truncation error of $e = O(\tau) + O(h^{2-\beta})$.

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THEOREM 5.4. *If*

$$\tau+\frac{\tau h^{-\beta}}{\cos(\beta\pi/2)\Gamma(3-\beta)}(-3+2^{2-\beta})<1,$$

then the explicit finite-difference method (3.5) *for the RSFRDE* (3.1) *is convergent.*

PROOF. At the mesh points (x_l, t_n), $u_l^n=U_l^n-e_l^n$. Substituting into (3.4) and using the Taylor theorem and Proposition 5.3, we obtain

$$\begin{aligned}\frac{e_l^{n+1}-e_l^n}{\tau}&=-e_l^n-\frac{h^{-\beta}}{2\cos(\beta\pi/2)\Gamma(3-\beta)}\Big[e_1^n(1-\beta)(2-\beta)l^{-\beta}\\&\quad+\sum_{j=0}^{l-1}c_j\big(e_{l-j+1}^n-2e_{l-j}^n+e_{l-j-1}^n\big)-(2-\beta)(L-l)^{1-\beta}(e_L^n-e_{L-1}^n)\\&\quad+\sum_{j=0}^{L-l-1}c_j\big(e_{l+j-1}^n-2e_{l+j}^n+e_{l+j+1}^n\big)\Big]+O(h^{2-\beta})+O(\tau),\end{aligned}\tag{5.5}$$

and the initial and boundary conditions are

$$e_l^0=0,\ \ (l=0,\ldots,L),\qquad e_0^n=e_1^n+O(h^2)\quad\text{and}\quad e_L^n=0,\ n\in N.$$

Equation (5.5) can be rewritten in matrix form as

$$E_{n+1}=BE_n+R,\qquad E_0=O_{(L-1)\times 1},$$

where $E_n=(e_1^n,e_2^n,\ldots,e_{L-1}^n)^{\mathrm T}$, $R=\tau\big(O(h^{2-\beta})+O(\tau)\big)(1,\ldots,1)^{\mathrm T}$. Thus, we have

$$\begin{aligned}&E_{n+1}=BE_n+R=\cdots=(B^n+B^{n-1}+\cdots+B^2+B+I)R\quad\text{and}\\&\|E_{n+1}\|_\infty\le(\|B^n\|_\infty+\|B^{n-1}\|_\infty+\cdots+\|B\|_\infty+\|I\|_\infty)\|R\|_\infty.\end{aligned}$$

According to Theorem 4.3, if $\tau+\tau h^{-\beta}(-3+2^{2-\beta})/\cos(\beta\pi/2)\Gamma(3-\beta)<1$, then $\|B\|_\infty\le 1$. We thus obtain

$$\|E_{n+1}\|_\infty\le(n+1)\tau|O(h^{2-\beta})+O(\tau)|\le C(h^{2-\beta}+\tau).$$

This inequality completes the proof. □

6. Numerical results

In order to demonstrate the efficiency of the RSFRDE, the method of lines (MoL) for RSFRDE is now presented. This method was introduced by Liu *et al.* [11, 12, 14] to solve fractional partial differential equations successfully. The MoL for the RSFRDE can be written in the following form: for $1<\beta<2$, $l=1,\ldots,L-1$,

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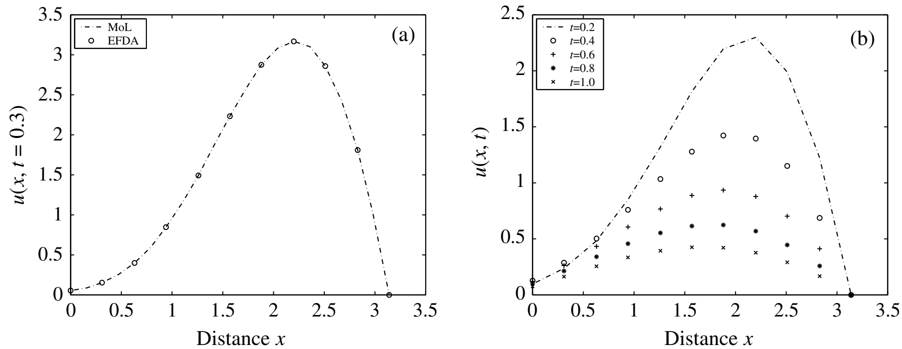

FIGURE 1. (a) The numerical solutions obtained using the MoL and the EFDA for $\beta = 1.7$ and $T = 0.3$; (b) the evolution result using the EFDA with $\beta = 1.7$ $(0 \le t \le 1, 0 \le x \le \pi)$.

$$\begin{aligned}\frac{du_l}{dt} = -u_l^n - \frac{h^{-\beta}}{2\cos(\beta\pi/2)\Gamma(3-\beta)}\Bigg[&u_0(1-\beta)(2-\beta)l^{-\beta}\\ &+\sum_{j=0}^{l-1} c_j\big(u_{l-j+1} - 2u_{l-j} + u_{l-j-1}\big) - (2-\beta)(L-l)^{1-\beta}(u_L - u_{L-1})\\ &+\sum_{j=0}^{L-l-1} c_j\big(u_{l+j-1} - 2u_{l+j} + u_{l+j+1}\big)\Bigg],\end{aligned}$$

with $u_0 = u_1$, $u_L = 0$ and $u_l = u(x_l, t)$.

To test the numerical scheme, it is preferable to use a simple analytical model. In this section we present an example in a bounded domain to demonstrate that the RSFRDE can be applied to simulate the behaviour of a fractional reaction–diffusion equation. Such a numerical technique overcomes the problem of not being able to evaluate the analytical solution for $1 < \beta \le 2$ owing to the nature of the Mittag–Leffler function. We consider the system

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t} = -u(x,t) +_x D_0^{\beta} u(x,t), & 0 \le x \le \pi,\ 0 \le t \le T,\\ u(x,0) = g(x) = x^2 \sin x, & 0 \le x \le \pi,\\ \dfrac{\partial u(0,t)}{\partial x} = 0, \quad u(\pi,t) = 0, & 0 \le t \le T.\end{cases}$$

REMARK 2. We take $L = 100$, that is, $h = \pi/100$, $\tau = 0.0001$ and $\beta = 1.7$. Then $\tau + \tau h^{-\beta}(-3 + 2^{2-\beta})/\big(\cos(\beta\pi/2)\Gamma(3-\beta)\big) = 7.936 \times 10^{-2} < 1$. Thus, τ, h and β satisfy the convergence condition.

Figure 1(a) shows the numerical solutions using the MoL and the RSFRDE with $h = \pi/100$ and $\tau = 0.0001$ for $\beta = 1.7$ and $T = 0.3$. It is seen that the EFDA is in good agreement with the MoL.

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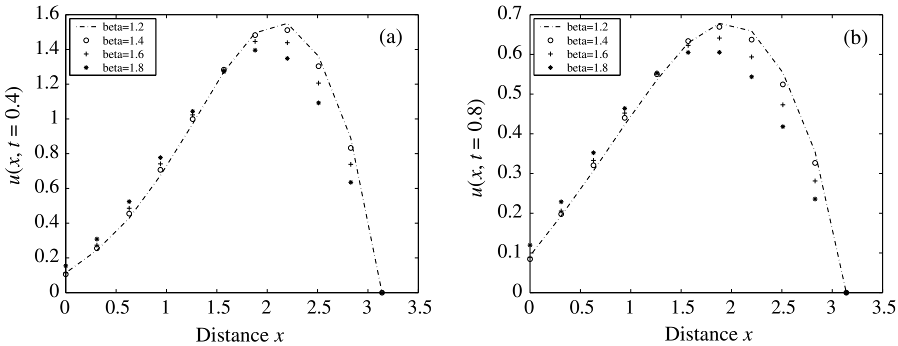

FIGURE 2. Comparison of the response of the RSFRDE for real orders $\beta = 1.2$, 1.4, 1.6 and 1.8 at: (a) $T = 0.4$; (b) $T = 0.8$.

Figure 1(b) shows the evolution result using the EFDA with $h = \pi/100$, $\tau = 0.0001$ and $\beta = 1.7$, $0 \leq t \leq 1$, $0 \leq x \leq \pi$. It is apparent that the order $\beta = 1.7$ exhibits diffusive behaviour for different times.

Figures 2(a) and (b) compare the response of the RSFRDE equation for different orders $1.2 \leq \beta \leq 1.8$ at $T = 0.4$ and $T = 0.8$, respectively.

7. Conclusions

In this paper we have given the fundamental solution for a RSFRDE and have provided an EFDA in a bounded domain. The difference approximation has been proved to be conditionally stable and convergent.

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References

[1] D. A. Benson, “The fractional advection–dispersion equation”, Ph. D. thesis, University of Nevada, Reno, NV, 1998.

[2] D. A. Benson, S. W. Wheatcraft and M. M. Meerschaert, “Application of a fractional advection–despersion equation”, *Water Resources Res.* **36** (2000a) 1403–1412.

[3] D. A. Benson, S. W. Wheatcraft and M. M. Meerschaert, “The fractional-order governing equation of Lévy motion”, *Water Resources Res.* **36** (2000b) 1413–1423.

[4] W. Feller, “On a generalization of Marcel Riesz potentials and the semi-groups generated by them” Meddelanden Lunds Universities Matematiska Seminarium (Comn. Sem. Mathem. Universite de Lund), Lund, 1952, 73–81.

[5] R. Gorenflo, Yu. Luchko and F. Mainardi, “Wright function as scale-invariant solutions of the diffusion-wave equation”, *J. Comput. Appl. Math.* **118** (2000) 175–191.

[6] R. Gorenflo and F. Mainardi, “Random walk models for space-fractional diffusion processes”, *Fract. Calc. Appl. Anal.* **1** (1998) 167–191.

[7] R. Gorenflo and F. Mainardi, “Approximation of Lévy–Feller diffusion by random walk”, *Z. Anal. Anwendungen* **18** (1999) 231–246.

[8] B. I. Henry and S. L. Wearne, “Existence of Turing instabilities in a two-species fractional reaction–diffusion system”, *SIAM J. Appl. Math.* **62** (2002) 870–887.

[9] R. Hilfer, *Application of fractional calculus in physics* (World Scientific, Singapore, 2000).

[10] R. Lin and F. Liu, “Fractional high order methods for the nonlinear fractional ordinary differential equation”, *Nonlinear Anal.* **66** (2007) 856–869.

[11] F. Liu, V. Anh and I. Turner, “Numerical solution of the fractional-order advection–dispersion equation”, *Proc. Int. Conf. on Boundary and Interior Layers – Computational and Asymptotic Methods*, Perth, Australia, 2002, 159–164.

[12] F. Liu, V. Anh and I. Turner, “Numerical solution of space-fractional Fokker–Planck equation”, *J. Comput. Appl. Math.* **166** (2004) 209–219.

[13] F. Liu, V. Anh, I. Turner and P. Zhuang, “Time-fractional advection–dispersion equation”, *J. Appl. Math. Comput.* **13** (2003) 233–246.

[14] F. Liu, V. Anh, I. Turner and P. Zhuang, “Numerical simulation for solute transport in fractal porous media”, *ANZIAM J.* **45(E)** (2004) 461–473.

[15] V. E. Lynch, B. A. Carreras, D. del Castillo-Negrete, K. M. Ferreira-Mejias and H. R. Hicks, “Numerical methods for the solution of partial differential equations of fractional order”, *J. Comput. Phys.* **192** (2003) 406–421.

[16] F. Mainardi, “The fundamental solutions for the fractional diffusion-wave equation”, *Appl. Math.* **9** (1996) 23–28.

[17] F. Mainardi, Yu. Luchko and G. Pagnini, “The fundamental solution of the space-time-fractional diffusion equation”, *Fract. Calc. Appl. Anal.* **4** (2001) 153–192.

[18] M. M. Meerschaert and C. Tadjeran, “Finite difference approximations for two-sided space-fractional partial differential equations”, *Appl. Numer. Math.* **56** (2006) 80–90.

[19] I. Podlubny, *Fractional differential equations* (Academic Press, New York, 1999).

[20] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications* (Gordon and Breach, Newark, NJ, 1993).

[21] E. Scalas, R. Gorenflo and F. Mainardi, “Fractional calculus and continuous-time finance”, *Physica A* **284** (2000) 376–384.

[22] R. Schumer and D. A. Benson, “Eulerian derivative of the fractional advection–dispersion equation”, *J. Contaminant* **48** (2001) 69–88.

[23] S. Shen and F. Liu, “Error analysis of an explicit finite difference approximation for the space-fractional diffusion equation with insulated ends”, *ANZIAM J.* **46** (2005) 871–887.

[24] G. D. Smith, *Numerical solution of partial differential equations: Finite difference methods* (Clarendon Press, Oxford, 1985).

[25] W. Wyss, “The fractional Black–Scholes equation”, *Fract. Calc. Appl. Anal.* **3** (2000) 51–61.