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HIGH LEVEL OCCUPATION TIMES FOR GAUSSIAN STOCHASTIC PROCESSES WITH SAMPLE PATHS IN ORLICZ SPACES

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Let X be a complete separable metric space, and $\{P_\epsilon\}$ a family of probability measures on the Borel subsets of X. We say that $\{P_\epsilon\}$ obeys the large deviation principle (LDP) with a rate function $I(\cdot)$ if there exists a function $I(\cdot)$ from X into $[0, \infty]$ satisfying:

(i) $0 \leqq I(x) \leqq \infty$ for all $x \in X$.

(ii) $I(\cdot)$ is lower semicontinuous.

(iii) For each $l < \infty$ the set $\{x:I(x) \leqq l\}$ is a compact set in X.

(iv) For each closed set $C \subset X$

$$\limsup_{\epsilon \to 0} \epsilon \log P_\epsilon(C) \leqq -\inf_{x \in C} I(x).$$

(v) For each open set $G \subset X$

$$\liminf_{\epsilon \to 0} \epsilon \log P_\epsilon(G) \geqq -\inf_{x \in G} I(x).$$

It is easy to see that if A is a Borel set such that

$$\inf_{x \in A^0} I(x) = \inf_{x \in A} I(x) = \inf_{x \in \bar{A}} I(x)$$

then

$$\lim_{\epsilon \to 0} \epsilon \log P_\epsilon(A) = -\inf_{x \in A} I(x)$$

where A^0 and $\bar{A}$ are respectively the interior and the closure of the Borel set A.

1. Proposition [**12**]. *Let P_ϵ satisfy the large deviation principle with a rate function $I(\cdot)$. Let F be a continuous map from $X \to L$ where L is another complete separable metric space. Then if we define Q_ϵ on L by $Q_\epsilon = P_\epsilon \circ F^{-1}$, then Q_ϵ satisfies the large deviation principle with a rate function $\mathscr{I}(\cdot)$ defined by*

$$\mathscr{I}(y) = \inf_{x:F(x)=y} I(x).$$

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2. PROPOSITION [**3**, **11**]. *Let* $(B, \mathscr{B}(B), \mu)$ *be a real separable Banach space with a mean-zero Gaussian measure* μ *defined on the Borel* σ*-algebra* $\mathscr{B}(B)$. *Let* H_μ *be the closure in* $L^2(\mu)$ *of the set* $\{x^*(\cdot):x^* \in B^*\}$; *and, for* $h \in H_\mu$ *let us define*

$$S_\mu(h) = \int xh(x)\mu(dx).$$

Let $\{X_i\}_{i=1}^\infty$ *be a sequence of independent B-valued random elements, each with distribution* μ. *Set*

$$S_n = \sum_1^n X_m$$

and let μ_n *be the distribution of* S_n/n, *then* $\{\mu_n:n \geqq 1\}$ *satisfies the large deviation principle with the rate function* $I_\mu(\cdot)$ *defined as follows*

$$I_\mu(x) = \begin{cases} \frac{1}{2}\|S_\mu^{-1}x\|_{H_\mu}^2 & \text{if } x \in S_\mu(H_\mu) \\ \infty & \text{if } x \in B\setminus S_\mu(H_\mu), \end{cases}$$

and for any closed set F:

$$\limsup_{\epsilon\to 0} \epsilon \log \mu(\epsilon^{-1/2}F) \leqq -\inf_{x\in F} I_\mu(x),$$

for any open set G:

$$\liminf_{\epsilon\to 0} \epsilon \log \mu(\epsilon^{-1/2}G) \geqq -\inf_{x\in G} I_\mu(x).$$

In this paper we are going to show that Proposition 2 is true for Orlicz spaces L_ϕ such that $\phi(\sqrt{t})$ is equivalent to $\tilde{\phi}(t)$ concave. It is easy to see that this class of Orlicz spaces includes some non-locally convex vector spaces. By applying the L.D.P. for Orlicz spaces we extend Kallianpur's and Oodaira's (1978), Marlow's (1973) results concerning some asymptotic estimates for the probabilities of high level occupation times for continuous Gaussian stochastic processes to the class of Gaussian stochastic processes with sample paths in Orlicz spaces.

Let $(T, \mathscr{F}, m)$ be an arbitrary σ-finite measure space with σ-algebra $\mathscr{F}$ and a separable measure m. Let S be the space of equivalence classes in measure m of all real valued $\mathscr{F}$ measurable functions. By ϕ let us denote a continuous, non-negative, non-decreasing function defined for $u \geqq 0$ such that $\phi(u) = 0$ if and only if $u = 0$. We assume additionally that the function $\phi(u)$ satisfies the so-called Δ_2 condition, i.e., there is a positive constant k such that for any u

$$\phi(2u) \leqq k\phi(u).$$

For $x \in S$ let us define

$$R_\phi(x) = \int_T \phi(|x(t)|)m(dt)$$

and let L_ϕ be the set of all $x \in S$ such that $R_\phi(ax) < \infty$ for some positive scalar a. The set L_ϕ is a linear space under the usual addition and scalar multiplication. Moreover it becomes a complete, separable metric space under the (usually non-homogeneous) seminorm $\|\cdot\|_\phi$:

$$\|x\|_\phi = \inf\{c:c > 0, R_\phi(c^{-1}x) < c\}.$$

The space $(L_\phi, \|\cdot\|_\phi)$ is called an Orlicz space. It is easy to see that convergence in the L_ϕ seminorm implies convergence in measure. In the case that ϕ is a convex function L_ϕ is a Banach space [**10**]. We say that $\phi(\sqrt{u})$ is equivalent to a concave function $\tilde{\phi}(u)$ if for all $u \geqq 0$

$$A\phi(c_1\sqrt{u}) \leqq \tilde{\phi}(u) \leqq B\phi(c_2\sqrt{u})$$

for some c_1, c_2, A, B positive constants. In this case Theorem 7.2.5 [**5**] implies that $\phi(u)$ satisfies Δ_2-condition. The best known examples of the Orlicz spaces are $L_p(T, \mathcal{F}, m)$ spaces for $0 \leqq p < \infty$ [**10**].

For convenience let us recall some necessary facts concerning probability measures on $(L_\phi, \mathcal{B}(L_\phi))$ spaces.

A. For each probability measure μ on $(L_\phi, \mathcal{B}(L_\phi))$ can be constructed a measurable stochastic process $\xi = \{\xi(t):t \in T\}$ on

$$(\Omega, \Sigma, P) = L_\phi, \mathcal{B}(L_\phi), \mu)$$

with sample paths in L_ϕ such that $\tilde{\xi}(x) = x$ μ a.e.; induced measure μ_ξ is equal to μ, and for every pair (s, u) of real numbers

$$\xi(t; sx \pm uy) = s\xi(t, x) \pm u\xi(t, y) \quad m \times \mu \times \mu \text{ a.e.}$$

Conversely, each jointly measurable stochastic process $\xi(t, \omega)$, defined on T, with almost all its sample paths in L_ϕ induces an $L_\phi(T, \mathcal{F}, m)$ valued random element [**1**].

B. An L_ϕ-valued r.e. $\tilde{\xi}$ (or p.m. μ on $(L_\phi, \mathcal{B}(L_\phi))$ is Gaussian if for any pair of independent copies of $\tilde{\xi}$, X_1 and X_2, the random elements $X_1 + X_2$ and $X_1 - X_2$ are independent; this is equivalent to: the process $\tilde{\xi}$ with sample paths in L_ϕ is Gaussian if and only if there exists a measurable subset T_0, $m(T_0) = 0$ such that for all finite sets $\{t_1, \dots, t_k\} \subset T\setminus T_0$ the random vector $\langle \xi(t_1), \dots, \xi(t_k) \rangle$ is Gaussian [**1**].

C. Let $\xi = \{\xi(t):t \in T\}$ be a measurable Gaussian stochastic process and let

$$\theta(t) = E\xi(t), \quad K(s, t) = E(\xi(s) - \theta(s))(\xi(t) - \theta(t)).$$

Then for almost every ω, $\xi(\cdot, \omega) \in L_\phi$ if and only if $\theta(t) \in L_\phi$ and

$K^{1/2}(t, t) \in L_\phi$. If almost all sample paths of the process ξ belong to the space L_ϕ then the measure μ_ξ induced by ξ on $(L_\phi, \mathscr{B}(L_\phi))$ is Gaussian [**1**].

D. Let μ be a mean-zero non-degenerate Gaussian measure on $(L_\phi, \mathscr{B}(L_\phi))$ and let $\xi = \{\xi(t):t \in T\}$ be a measurable stochastic process, such as in A, inducing the measure μ. By A there exists a measurable subset T_0, $m(T_0) = 0$ such that for any $t \in T\setminus T_0$

$$\xi(t, x \pm y) = \xi(t, x) \pm \xi(t, y) \quad \mu \times \mu \text{ a.e.}$$

Let

$$H_\mu = \overline{\text{lin}\{\xi(t):t \in T\setminus T_0\}}^{L_2(\mu)}.$$

From [7] it follows that the space H_μ does not depend on the version of the stochastic process inducing the measure μ and consists of quasi-additive measurable functionals (q.m.f.) F [7], i.e.,

$$H_\mu = \{F:F:L_\phi \to R, \text{ measurable}, \quad F(x \pm y) = F(x) \pm F(y) \quad \mu \times \mu \text{ a.e.}\}.$$

For each $F \in H_\mu$ let

$$(\Lambda F)(\cdot) = \left[\int \xi(\cdot, x)F(x)\mu(dx)\right] = [\Lambda_\xi F(\cdot)]$$

where $[\cdot]$ denotes the class of functions equivalent m a.e. In [7] it was shown that Λ is a one-to-one map which embeds continuously the space H_μ into L_ϕ, and this embedding does not depend on the version of the stochastic process inducing the measure μ. When L_ϕ is a Banach space from [7] it follows that S_μ defined in Proposition 2 equals Λ and the rate function $I_\mu(\cdot)$ is expressed as follows:

$$I_\mu(x) = \begin{cases} \frac{1}{2}\|\Lambda^{-1}x\|^2_{H_\mu} & \text{if } x \in \Lambda(H_\mu) \\ \infty & \text{if } x \notin \Lambda(H_\mu). \end{cases}$$

Let $\{E_j\}$ be a C.O.N.S. in H_μ and $\psi_j(t) = \langle \xi(t), E_j \rangle$, then by [**1**, **2**]

$$\xi(t, x) = \sum_{j=1}^{\infty} \psi_j(t)E_j(x)$$

μ a.e. in the seminorm of L_ϕ.

3. PROPOSITION [**8**]. *Any mean-zero, non-degenerate Gaussian measure μ defined on $(L_\phi, \mathscr{B}(L_\phi))$ such that $\phi(\sqrt{t})$ is equivalent to $\tilde{\phi}(t)$ concave is the image under a continuous linear map of a centered Gaussian measure on a separable real Hilbert space.*

Sketch of the proof. Let $b > 0$ be an arbitrary constant such that

$$R_\phi(bK^{1/2}(t, t)) < \infty,$$

then

$$v_\phi(A) = \int_A \widetilde{\phi}(b^2 c_2^{-2} K(t, t))m(dt), \quad A \in \mathcal{F}$$

defines a non-negative, finite measure on $\mathcal{F}$. Let

$$L_{2,\phi} = L_2(T, \mathcal{F}, v_\phi)$$

be a real, separable Hilbert space and u be a map defined on $L_{2,\phi}$ as follows:

$$L_{2,\phi} \ni f(t) \mapsto (uf)(t) = f(t)K^{1/2}(t, t)$$

then u is a linear, continuous map with values in L_ϕ. Let

$$f_j(t) = \begin{cases} \psi_j(t)K^{-1/2}(t, t) & \text{if } K(t, t) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

then

$$S = \sum_{j=1}^{\infty} f_j(t)E_j$$

is a mean-zero Gaussian random element with values in $L_{2,\phi}$ such that $uS = \xi$ a.e.

4. THEOREM. *Let μ be a mean-zero, non-degenerate Gaussian measure defined on $(L_\phi, \mathcal{B}(L_\phi))$ such that*:

(i) *$\phi(t)$ is a convex function,*

or

(ii) *$\phi(\sqrt{t})$ is equivalent to $\widetilde{\phi}(t)$ concave function. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent L_ϕ-valued random elements, each with distribution μ. Set*

$$S_n = \sum_{i=1}^{n} X_i$$

and let μ_n be the distribution of S_n/n, then $\{\mu_n : n \geqq 1\}$ satisfies the large deviation principle with the rate function $I_\mu(x)$, defined as follows:

$$I_\mu(x) = \begin{cases} \frac{1}{2} \int F^2(x)\mu(dx) & \text{if } x = \Lambda F \\ \infty & \text{if } x \notin \Lambda H_\mu. \end{cases}$$

Proof. In the case that $\phi(t)$ is a convex function L_ϕ is a Banach space and the theorem follows from Remark D. We have to prove only the case where $\phi(\sqrt{t})$ is equivalent to $\widetilde{\phi}(t)$ concave function.

In the proof we use the same notation as in the sketch of the proof of Proposition 3. Let

$$L_{\phi,0} = \{f(t):f(t) \in L_{\phi}, f(t) = 0 \text{ on the set } \{t:K(t, t) = 0\}\}.$$

It is easy to see that $L_{\phi,0}$ is a closed linear subspace of L_{ϕ}, such that $u(L_{2,\phi}) \subseteq L_{\phi,0}$. Since the measures v_{ϕ} and m are absolutely continuous with respect to each other on the set $\{t:K(t, t) \neq 0\}$, then u is a one-to-one map with u^{-1} defined as follows:

$$(u^{-1}f)(t) = \begin{cases} K^{-1/2}(t, t)f(t) & \text{if } K(t, t) \neq 0 \\ 0 & \text{if } K(t, t) = 0 \end{cases}$$

for any $f \in L_{\phi,0}$.

Proposition 3 implies that the measure μ is concentrated on the subspace $L_{\phi,0}$ and $\mu(A) = \mu_S(u^{-1}A)$ for any measurable subset A, where μ_S denotes the distribution of the random element S.

Since u is a continuous linear map [**8**], $\mu = \mu_S \circ u^{-1}$, μ_S is a mean-zero Gaussian measure defined on the Hilbert space $L_{2,\phi}$, then by Propositions 1 and 2 $\{\mu_n:n \geqq 1\}$ satisfies the L.D.P. with the rate function

$$I_{\mu}(x) = \inf_{y:u(y)=x} I_S(y)$$

where I_S is a rate function for the measures $\mu_{S,n}$. We will prove that

$$I_{\mu}(x) = \begin{cases} \frac{1}{2}||\Lambda^{-1}x||^2 & \text{if } x \in \Lambda H_{\mu} \\ \infty & \text{if } x \notin \Lambda H_{\mu} \end{cases}$$

where $||\cdot||$ denotes the norm in the Hilbert space H_{μ}.

Let us denote by $\xi = \{\xi(t):t \in T\}$, $\eta = \{\eta(t):t \in T\}$ a measurable stochastic processes such as in A, inducing the measures μ and μ_S respectively. Since

$$\eta(t, x) \in x \text{ for } \mu_S \text{ a.e. } x;$$

$$\xi(t, x) \in x \text{ for } \mu \text{ a.e. } x$$

then

$$u\eta(\cdot, x) = \xi(\cdot, ux) \qquad m \text{ a.e. for } \mu_S \text{ a.e. } x$$
$$u^{-1}\xi(\cdot, x) = \eta(\cdot, u^{-1}x)\, v_{\phi} \text{ a.e. for } \mu \text{ a.e. } x.$$

Let H_{μ} be the space of quasi-additive measurable functionals defined on $(L_{\phi}, \mathscr{B}(L_{\phi}), \mu)$ and H_S the space of quasi-additive measurable functionals defined on $(L_{2,\phi}, \mathscr{B}(L_{2,\phi}), \mu_S)$. Since

$$|\Lambda_\xi F(t)| \leqq \left(\int \xi^2(t)d\mu\right)^{1/2}\left(\int F^2 d\mu\right)^{1/2} = K^{1/2}(t, t)\,\|F\|$$

then

$$\Lambda H_\mu \subseteq L_{\phi,0}.$$

If F is a q.m.f. on the space $(L_\phi, \mathscr{B}(L_\phi), \mu)$ then $F \circ u$ is a q.m.f. on the space $(L_{2,\phi}, \mathscr{B}(L_{2,\phi}), \mu_S)$ and if G is a q.m.f. on the space $(L_{2,\phi}, \mathscr{B}(L_{2,\phi}), \mu_S)$ then $G \circ u^{-1}$ is a q.m.f. on the space $(L_\phi, \mathscr{B}(L_\phi), \mu)$. These follow from

$$\begin{aligned}0 &= \mu \times \mu(\{(x, y):F(x \pm y) \neq F(x) \pm F(y)\})\\ &= \mu_S \times \mu_S(\{(u^{-1}x, u^{-1}y):F(x \pm y) \neq F(x) \pm F(y)\})\\ &= \mu_S \times \mu_S(\{(z, s):F(u(z \pm s)) \neq F(uz) \pm F(us)\}).\end{aligned}$$

$$\begin{aligned}&\mu \times \mu(\{(x, y):G(u^{-1}(x \pm y)) \neq G(u^{-1}x) \pm G(u^{-1}y)\})\\ &= \mu_S \times \mu_S(\{(u^{-1}x, u^{-1}y):G(u^{-1}(x \pm y)) \neq G(u^{-1}x)\\ &\qquad\qquad \pm G(u^{-1}y)\})\\ &= \mu_S \times \mu_S(\{(z, s):G(z \pm s) \neq G(z) \pm G(s)\}) = 0.\end{aligned}$$

Let G be μ_S q.m.f., then

$$\begin{aligned}u(\Lambda_\eta G)(t) &= u\left(\int \eta(t, x)G(x)\mu_S(dx)\right)\\ &= \int K^{1/2}(t, t)\eta(t, x)G(x)\mu_S(dx)\\ &= \int \xi(t, ux)G(x)\mu_S(dx) = \int \xi(t, y)G(u^{-1}y)\mu(dy)\ m \text{ a.e.}\end{aligned}$$

Since $G \circ u^{-1}$ is μ q.m.f., then $u(\Lambda_\eta G) \in \Lambda H_\mu$, which implies $u(\Lambda_\eta H_S) \subseteq \Lambda H_\mu$. We use the same notation for a function and the corresponding equivalence class in measure.

Let F be μ q.m.f., then

$$\begin{aligned}u^{-1}(\Lambda_\xi F)(t) &= u^{-1}\left(\int \xi(t, x)F(x)\mu(dx)\right)\\ &= \int K^{-1/2}(t, t)\xi(t, x)F(x)\mu(dx)\\ &= \int \eta(t, u^{-1}x)F(x)\mu(dx)\\ &= \int \eta(t, y)F(uy)\mu_S(dy) \quad \nu_\phi \text{ a.e.}\end{aligned}$$

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Since $F \circ u$ is μ_S q.m.f., then

$$u^{-1}(\Lambda_\xi F) \in \Lambda_\eta H_S, \quad \text{and} \quad u^{-1}(\Lambda_\xi H_\mu) \subseteq \Lambda_\eta H_S,$$

this implies that

$$u(\Lambda_\eta H_S) = \Lambda H_\mu \subseteq L_{\phi,0}.$$

From Propositions 1, 2 and Remark D it follows that

$$I_\mu(x) = \inf_{y:u(y)=x} I_S(y)$$

where $I_S(\cdot)$ is the rate function for the sequence $\{\mu_{S,n}:n \geqq 1\}$.

If $x \in \Lambda H_\mu$ then there exists a q.m.f. F such that $x = \Lambda F$. Since u is a one-to-one map,

$$I_\mu(x) = I_S(u^{-1}(\Lambda F))$$
$$= \frac{1}{2}\int [F(u(z))]^2 \mu_S(dz) = \frac{1}{2}\int F^2(y)\mu(dy).$$

If $x \notin \Lambda H_\mu$ then there is no $y \in \Lambda_\eta H_S$ such that $u(y) = x$ and this implies that $I_\mu(x) = \infty$. This finishes the proof of the theorem that

$$I_\mu(x) = \begin{cases} \frac{1}{2}\|\Lambda^{-1}x\|^2 & \text{if } x \in \Lambda H_\mu \\ \infty & \text{if } x \notin \Lambda H_\mu. \end{cases}$$

5. COROLLARY. *Let μ be a mean-zero, non-degenerate Gaussian measure defined on $(L_\phi, \mathscr{B}(L_\phi))$, such that $\phi(\sqrt{t})$ is equivalent to $\tilde{\phi}(t)$ concave, then for any closed subset E*

$$\limsup_{\epsilon \to 0} \epsilon \log \mu(\epsilon^{-1/2}E) \leqq -\inf_{x \in E} I_\mu(x)$$

and for any open subset D

$$\liminf_{\epsilon \to 0} \epsilon \log \mu(\epsilon^{-1/2}D) \geqq -\inf_{x \in D} I_\mu(x).$$

Proof. The proof of this corollary is an immediate consequence of Theorem 4 and Theorem 3.48 in [**11**], namely for any closed subset E and an open subset D

$$\limsup_{\epsilon \to 0} \epsilon \log \mu(\epsilon^{-1/2}E) = \limsup_{\epsilon \to 0} \epsilon \log \mu_S(\epsilon^{-1/2}u^{-1}E)$$
$$\leqq -\inf_{x \in u^{-1}E} I_S(x) = -\inf_{y \in E} I_\mu(y),$$

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$$\liminf_{\epsilon \to 0} \epsilon \log \mu(\epsilon^{-1/2} D) = \liminf_{\epsilon \to 0} \epsilon \log \mu_S(\epsilon^{-1/2} u^{-1} D)$$
$$\geqq -\inf_{x \in u^{-1} D} I_S(x) = -\inf_{y \in D} I_\mu(y),$$

because for any subset B

$$\inf_{x \in u^{-1} B} I_S(x) = \begin{cases} \inf_{x \in u^{-1}(B \cap \Lambda H_\mu)} I_S(x) & \text{if } B \cap \Lambda H_\mu \neq \phi \\ \infty & \text{if } B \cap \Lambda H_\mu = \phi \end{cases}$$
$$= \begin{cases} \inf_{y \in B \cap \Lambda H_\mu} I_S(u^{-1} y) & \text{if } B \cap \Lambda H_\mu \neq \phi \\ \infty & \text{if } B \cap \Lambda H_\mu = \phi \end{cases}$$
$$= \begin{cases} \inf_{y \in B \cap \Lambda H_\mu} I_\mu(y) & \text{if } B \cap \Lambda H_\mu \neq \phi \\ \infty & \text{if } B \cap \Lambda H_\mu = \phi \end{cases}$$
$$= \inf_{y \in B} I_\mu(y).$$

6. PROPOSITION. *Let $(L_\Phi, \mathscr{B}(L_\Phi), \mu)$ be an Orlicz space with mean-zero, non-degenerate Gaussian measure μ and a rate function $I_\mu(\cdot)$:*

$$I_\mu(x) = \begin{cases} \frac{1}{2}\|\Lambda^{-1} x\|^2 & \text{if } x \in \Lambda H_\mu \\ \infty & \text{if } x \notin \Lambda H_\mu \end{cases}$$

then

(i) *the set $K_r = \{\Lambda F : I_\mu(\Lambda F) \leqq r^2\}$, $0 < r < \infty$ is compact in L_Φ.*

(ii) *$I_\mu(y)$ is lower-semicontinuous on ΛH_μ with respect to $\|\cdot\|_\Phi$-norm convergence, i.e., if $\|\Lambda F_n - \Lambda F\|_\Phi \to 0$ as $n \to \infty$, F_n, $F \in H_\mu$, then*

$$I_\mu(\Lambda F) \leqq \liminf_{n \to \infty} I_\mu(\Lambda F_n).$$

Proof. First we show that K_r, for any $0 < r < \infty$ is a compact subset of L_Φ. Let $\{\Lambda F_n\} \subset K_r$ be an arbitrary sequence. By the Banach-Alaoglu Theorem $\{F_n\}$ contains a subsequence $\{F_{n'}\}$ which is weakly convergent to F from $\Lambda^{-1} K_r$. Remark D implies that there exists a measurable subset T_0, $m(T_0) = 0$, such that for any $t \in T \setminus T_0$, $\xi(t) \in H_\mu$ and

$$\Lambda_\xi F_{n'}(t) = \int \xi(t) F_{n'} d\mu \mapsto \int \xi(t) F d\mu = \Lambda_\xi F(t).$$

Since

$$|\Lambda_\xi F_{n'}(t)| \leqq K^{1/2}(t, t) \|F_{n'}\| \leqq \sqrt{2} r K^{1/2}(t, t)$$

for m a.e. t, and $K^{1/2}(t, t) \in L_\Phi$ then by the Lebesgue Dominated Convergence Theorem, $\Lambda F_{n'} \mapsto \Lambda F$ in L_Φ, which proves that K_r is a compact subset of L_Φ.

Proof of part (ii). By $\{F_{n'}\}$ let us denote a subsequence such that

$$\liminf_{n} I_\mu(\Lambda F_n) = \lim_{n'} I_\mu(\Lambda F_{n'}).$$

Since

$$\|\Lambda F_{n'} - \Lambda F\|_\phi \mapsto 0 \quad \text{as } n' \to \infty,$$

then there exists a subsequence $\{n''\} \subset \{n'\}$ and a measurable subset T_0, $m(T_0) = 0$ such that for any $t \in T \setminus T_0$, $\xi(t)$ is a q.m.f. and

$$\langle \xi(t), F_{n''} \rangle = \Lambda_\xi F_{n''}(t) \mapsto \Lambda_\xi F(t) = \langle \xi(t), F \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in H_μ. Let

$$G = \text{lin}\{\xi(t):t \in T \setminus T_0\}$$

then G is a dense subset of H_μ [7] and for any $g \in G$

$$\langle g, F_{n''} \rangle \mapsto \langle g, F \rangle \quad \text{as } n'' \to \infty.$$

Since

$$\|F_{n''}\| = \sup\{ \langle g, F_{n''} \rangle :g \in G, \|g\| = 1\},$$

then for any $g \in G$, $\|g\| = 1$

$$\lim_{n''} \|F_{n''}\| \geqq \lim_{n''} \langle g, F_{n''} \rangle = \langle g, F \rangle.$$

This implies that

$$\lim_{n''} \|F_{n''}\| \geqq \sup\{ \langle g, F \rangle :g \in G, \|g\| = 1\} = \|F\|$$

which proves part (ii), because

$$\liminf_{n} \|F_n\| = \lim_{n''} \|F_{n''}\| \geqq \|F\|.$$

7. *Remark*. In the case that $\phi(t)$ satisfies additionally

$$\liminf_{t \to \infty} \inf\{c > 0:2\phi(ct) > \phi(t) \} > 0$$

the space L_ϕ is locally bounded (i.e., contains a bounded neighbourhood of zero) and for certain p, $0 < p \leqq 1$, there exists a p-homogeneous F-norm $\|\cdot\|_1$ equivalent to the original $\|\cdot\|_\phi$ **[10]**.

8. PROPOSITION. *Let μ be a mean-zero, non-degenerate Gaussian measure defined on $(L_\phi, \mathscr{B}(L_\phi))$ such that*

(i) *$\phi(t)$ is a convex function,*

or

(ii) *$\phi(\sqrt{t})$ is equivalent to a $\tilde{\phi}(t)$ concave function and*

$$\liminf_{t \to \infty} \inf\{c > 0:2\phi(ct) \geqq \phi(t) \} > 0.$$

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Let $a = \inf\{I_\mu(x):\|x\|_1 \geqq 1\}$ *where* $\|\cdot\|_1$ *is p-homogeneous*, $0 < p \leqq 1$, *F-norm equivalent to* $\|\cdot\|_\Phi$, *then* $0 < a < \infty$, *and*

$$\lim_{R\to\infty} R^{-2}\log \mu(\{x:\|R^{-1}x\|_1 > 1\}) = -a.$$

Proof. Let $\epsilon = R^{-2}$ and $B = \{x:\|x\|_1 < 1\}$, then by Corollary 5

$$\overline{\lim_{R\to\infty}} R^{-2} \log \mu(\{x:\|R^{-1}x\|_1 \geqq 1\})$$

$$= \overline{\lim_{\epsilon\to 0}} \epsilon \log \mu(\{x:\|\epsilon^{1/2}x\|_1 \geqq 1\})$$

$$= \overline{\lim_{\epsilon\to 0}} \epsilon \log \mu(\epsilon^{-1/2}B^c)$$

$$\leqq -\inf_{x\in B^c} I_\mu(x) = -\inf_{\|x\|_1\geqq 1} I_\mu(x).$$

$$\underline{\lim_{R\to\infty}} R^{-2} \log \mu(\{x:\|R^{-1}x\|_1 > 1\})$$

$$= \underline{\lim_{\epsilon\to 0}} \epsilon \log \mu(\{x:\|\epsilon^{1/2}x\|_1 > 1\})$$

$$= \underline{\lim_{\epsilon\to 0}} \epsilon \log \mu(\epsilon^{-1/2}\bar{B}^c)$$

$$\geqq -\inf_{x\in \bar{B}^c} I_\mu(x) = -\inf_{\|x\|_1 > 1} I_\mu(x).$$

Therefore

$$\begin{aligned} -\inf_{\|x\|_1>1} I_\mu(x) &\leqq \underline{\lim_{R\to\infty}} R^{-2} \log \mu(\{x:\|R^{-1}x\|_1 > 1\}) \\ &\leqq \overline{\lim_{R\to\infty}} R^{-2} \log \mu(\{x:\|R^{-1}x\|_1 > 1\}) \\ &\leqq -\inf_{\|x\|_1\geqq 1} I_\mu(x). \end{aligned}$$

Since

$$\begin{aligned} a &= \inf\{I_\mu(x):\|x\|_1 > 1, x \in \Lambda H_\mu\} \\ &= \inf\{c^2 I_\mu(x):\|x\|_1 = 1, c > 1, x \in \Lambda H_\mu\} \\ &= \inf\{c^2 I_\mu(x):\|x\|_1 = 1, c \geqq 1, x \in \Lambda H_\mu\} \\ &= \inf\{I_\mu(x):\|x\|_1 \geqq 1, x \in \Lambda H_\mu\} \end{aligned}$$

then

$$\lim_{R\to\infty} R^{-2} \log \mu(\{x:\|R^{-1}x\|_1 > 1\}) = -a.$$

If $a = 0$, then there exists a sequence $\{F_n\}$ of q.m.f.'s such that $\|\Lambda_\xi F_n\|_1 > 1$ and $\|F_n\| \leqq 1/n$. This implies that

$$|\Lambda_\xi F_n(t)| \leqq K^{1/2}(t, t)\,\|F_n\| \leqq \frac{1}{n} K^{1/2}(t, t).$$

Since $K^{1/2}(t, t) \in L_\phi$, then by the Lebesgue Dominated Convergence Theorem $\|\Lambda_\xi F_n\|_\phi \to 0$ as $n \to \infty$, which implies that $\|\Lambda_\xi F_n\|_1 \to 0$ as $n \to \infty$ contradicting the assumption that $\|\Lambda_\xi F_n\|_1 > 1$, therefore $0 < a < \infty$.

LEMMA 1. *Let $(T, \mathcal{F}, m)$ be a measurable space with a σ-finite measure m, and $L_\phi(T, \mathcal{F}, m)$ an Orlicz space, then for any $\beta > 0$*

$$D_\beta = \{f(t):f(t) \in L_\phi, m(\{t:f(t) > 1\}) > \beta\}$$

is an open set in L_ϕ.

Proof. It is enough to prove that for some $\beta > 0$

$$D^c_\beta = \{f(t):f(t) \in L_\phi, m(\{t:f(t) > 1\}) \leqq \beta\}$$

is a closed set in L_ϕ.

Let $\{f_n\} \subset D^c_\beta$ and $f_n \mapsto f$ in L_ϕ as $n \to \infty$, then there exists a subsequence $\{n_k\}$ such that $f_{n_k}(t) \to f(t)$ m a.e. By Egoroff's theorem [**4**], there exists an increasing sequence of measurable subsets $\{E_i\}$ such that the sequence $\{f_{n_k}\}$ converges uniformly on each E_i $i = 1, 2, \ldots$, and

$$m\left(T \setminus \bigcup_{i=1}^{\infty} E_i\right) = 0.$$

Let

$$T_n = \left\{t:f(t) > 1 + \frac{1}{n}\right\},$$

then T_n is an increasing sequence of subsets and

$$S = \{t:f(t) > 1\} = \bigcup_{n=1}^{\infty} T_n.$$

Therefore

$$m(S) = \lim_n m(T_n)$$

and we finish the proof by showing that $m(T_n) \leqq \beta$ for each n.

Let n be an arbitrary but fixed, then

$$\forall i \;\exists\, n_i \;\forall\, n_k > n_i \;\forall\, t \in E_i \quad f(t) - \frac{1}{n} < f_{n_k}(t).$$

This implies that

$$\left\{t:f(t) > 1 + \frac{1}{n}\right\} \cap E_i \subset \bigcap_{n_k > n_i} \{t:f_{n_k}(t) > 1\} \cap E_i = A_i.$$

Since $\{E_i\}$ is an increasing sequence of sets then $\{n_i\}$ is a non-decreasing sequence implying that $\{A_i\}$ is an increasing sequence of sets. Since

$$m\left(T \setminus \bigcup_{i=1}^{\infty} E_i\right) = 0,$$

then

$$m\left(\left\{t:f(t) > 1 + \frac{1}{n}\right\}\right) = m\left(\bigcup_{i=1}^{\infty} \left\{t:f(t) > 1 + \frac{1}{n}\right\} \cap E_i\right)$$
$$\leqq m\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} m(A_i).$$

Since

$$m(A_i) = m(\bigcap_{n_k > n_i} \{t:f_{n_k}(t) > 1\} \cap E_i) \leqq \beta \quad \text{for each } i,$$

then

$$m\left(\left\{t:f(t) > 1 + \frac{1}{n}\right\}\right) \leqq \beta \quad \text{for each } n,$$

which proves the lemma.

LEMMA 2. *Let $(T, \mathscr{F}, m)$ be a measurable space with a σ-finite measure m, and $L_\Phi(T, \mathscr{F}, m)$ an Orlicz space, then for any $\beta > 0$ the L_Φ-closure of D_β, $\bar{D}_\beta$ is contained in D^*_β where*

$$D^*_\beta = \left\{f(t):f(t) \in L_\Phi,\right.$$
$$\left.\forall k = 1, 2, \dots, m\left(\left\{t:f(t) > 1 - \frac{1}{k}\right\}\right) \geqq \beta\right\}.$$

Proof. Let $\{f_n\} \subset D$ and $f_n \to f$ in L_Φ as $n \to \infty$, then $f_n \to f$ in measure m as $n \to \infty$, i.e.,

$$\forall k \forall \epsilon \exists n_{k,\epsilon} \forall n > n_{k,\epsilon}\ m\left(\left\{t:|f_n(t) - f(t)| \geqq \frac{1}{k}\right\}\right) < \epsilon$$

which implies that

$$m\left(\left\{t:f_n(t) > 1, |f_n(t) - f(t)| < \frac{1}{k}\right\}\right) > \beta - \epsilon.$$

Therefore

$$m\left(\left\{t:f(t) > 1 - \frac{1}{k}\right\}\right) > \beta - \epsilon \quad \text{for each } k \text{ and } \epsilon$$

and

$$m\left(\left\{t:f(t) > 1 - \frac{1}{k}\right\}\right) \geqq \beta \quad \text{for each } k,$$

which proves that $f \in D^*_\beta$.

LEMMA. 3. *Let* $(L_\Phi, \mathscr{B}(L_\Phi), \mu)$ *be an Orlicz space with a mean-zero, non-degenerate Gaussian measure* μ *and a rate function* $I_\mu(\cdot)$:

$$I_\mu(x) = \begin{matrix} \frac{1}{2}\|\Lambda^{-1}x\|^2 & \text{if } x \in \Lambda H_\mu \\ \infty & \text{if } x \notin \Lambda H_\mu. \end{matrix}$$

Let

$$a_\beta = \inf\{I_\mu(x):x \in D_\beta\},$$

$$\bar{a}_\beta = \inf\{I_\mu(x):x \in \bar{D}_\beta\},$$

$$a^*_\beta = \inf\{I_\mu(x):x \in D^*_\beta\},$$

then $0 < a^*_\beta \leqq \bar{a}_\beta \leqq a_\beta$. *If the covariance function* $K(s, t)$ *of a measurable stochastic process* $\xi = \{\xi(t):t \in T\}$ *inducing the measure* μ *is such that*

(*) $\forall\beta > 0 \quad m(\{s:m(\{t:K(s, t) > 0\}) > \beta\}) > 0$

then $a_\beta < \infty$ *for every* $\beta > 0$.

Proof. If $a^*_\beta = 0$ then there exists a sequence $\{\Lambda F_n\} \subset D^*_\beta$ such that $\|F_n\| < 1/n$ and for almost every t

$$|\Lambda F_n(t)| \leqq K^{1/2}(t, t)\,\|F_n\| < \frac{1}{n}K^{1/2}(t, t).$$

This implies that for each k

$$\left\{t:\Lambda F_n(t) > 1 - \frac{1}{k}\right\} \subset \left\{t:\frac{1}{n}K^{1/2}(t, t) > 1 - \frac{1}{k}\right\},$$

and for each k and n

$$m\left(\left\{t:K^{1/2}(t, t) > \left(1 - \frac{1}{k}\right)n\right\}\right) \geqq \beta.$$

Let k be an arbitrary but fixed and

$$A_n = \left\{t:K^{1/2}(t, t) > \left(1 - \frac{1}{k}\right)n\right\}$$

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for every n, since $K^{1/2}(t, t) \in L_\phi$ then there exists $a > 0$ such that

$$m(A_n)\phi\left(a\left(1 - \frac{1}{k}\right)n\right) \leqq \int \phi(aK^{1/2}(t, t))m(dt) < \infty$$

which implies that $m(A_n) < \infty$ for every n.

Since $\{A_n\}$ is a decreasing sequence then

$$\lim_n m(A_n) = m\left(\bigcap_{n=1}^{\infty} A_n\right) \quad \text{and}$$

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) \geqq \beta$$

implying that

$$m(\{t:K^{1/2}(t, t) = \infty\}) \geqq \beta$$

which is impossible. Therefore $0 < a^* \leqq \bar{a}_\beta \leqq a_\beta$.

Let $\xi = \{\xi(t):t \in T\}$ be a measurable stochastic process such as in A, inducing the measure μ with the covariance function $K(s, t)$ satisfying (*). There exists a measurable subset T_0, $m(T_0) = 0$ such that for every $s \in T\backslash T_0$, $\xi(s)$ is a q.m.f. Let $\beta > 0$ be an arbitrary but fixed, then there exists a q.m.f. $\xi(s)$ such that

$$m(\{t:\Lambda\xi(s)(t) > 0\}) > \beta.$$

Let

$$A_n = \left\{t:\Lambda\xi(s)(t) > \frac{1}{n}\right\},$$

$$\{t:\Lambda\xi(s)(t) > 0\} = \bigcup_{n=1}^{\infty} A_n.$$

Since $\{A_n\}$ is an increasing sequence, then there exists n such that $m(A_n) > \beta$ implying that for a q.m.f.

$$F = n\xi(s), \quad m(\{t:\Lambda F(t) > 1\}) > \beta \quad \text{and} \quad a_\beta \leqq \frac{1}{2}\|F\|^2.$$

9. THEOREM. *Let* $\xi = \{\xi(t):t \in T\}$ *be a mean-zero Gaussian stochastic process with almost all sample paths in an Orlicz space* L_ϕ *such that*

(i) $\phi(t)$ *is a convex function,*

or

(ii) $\phi(\sqrt{t})$ *is equivalent to* $\tilde{\phi}(t)$ *concave function. Let for any* $\beta > 0$

$$D_\beta = \{f(t):f(t) \in L_\phi, m(\{t:f(t) > 1\}) > \beta\},$$

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$$a_\beta = \inf\{I_\mu(x):x \in D_\beta\},\ \bar{a}_\beta = \inf\{I_\mu(x):x \in \bar{D}_\beta\},$$

then

$$-a_\beta \leqq \varliminf_{\alpha\to\infty} \alpha^{-2} \log P(\{\omega:m(\{t:\xi(t, \omega) > \alpha\}) > \beta\})$$
$$\leqq \varlimsup_{\alpha\to\infty} \alpha^{-2} \log P(\{\omega:m(\{t:\xi(t, \omega) > \alpha\}) > \beta\}) \leqq \bar{a}_\beta.$$

If T is a metric space with the measure m such that for any open set U, $m(U) > 0$, the covariance function $K(s, t)$ of the process $\xi = \{\xi(t):t \in T\}$ is continuous and for each $\beta > 0$

$$m(\{s:m(\{t:K(s, t) > 0\}) > \beta\}) > 0$$

then $0 < a_\beta < \infty$ *and*

$$\lim_{\alpha\to\infty} \alpha^{-2} \log P(\{\omega:m(\{t:\xi(t, \omega) > \alpha\}) > \beta\}) = -a_\beta.$$

Proof. Let μ denote Gaussian measure generated by the stochastic process $\xi = \{\xi(t):t \in T\}$. By Lemma 1 D_β is an open set. Since

$$\mu(\alpha D) = P(\{\omega:m(\{t:\xi(t, \omega) > \alpha\}) > \beta\})$$

and $\mu(\alpha\bar{D}) \geqq \mu(\alpha D)$ then by Corollary 5

$$-a_\beta \leqq \varliminf_{\alpha\to\infty} \alpha^{-2} \log \mu(\alpha D) \leqq \varlimsup_{\alpha\to\infty} \alpha^{-2} \log \mu(\alpha D) \leqq -\bar{a}_\beta.$$

Under the additional assumptions by Lemma 3, $0 < \bar{a}_\beta \leqq a_\beta < \infty$. Since the covariance function $K(s, t)$ is continuous the space ΛH_μ consists of continuous functions.

To finish the proof of the theorem, by Lemma 3 it is sufficient to show that $a_\beta^* \geqq a_\beta$. Let F be an arbitrary q.m.f. such that for each k

$$m\left(\left\{t:\Lambda_\xi F(t) > 1 - \frac{1}{k}\right\}\right) \geqq \beta.$$

Let

$$G_k = \left(1 + \frac{1}{k-1}\right)F$$

then for any k

$$m(\{t:\Lambda_\xi G_k(t) > 1\}) \geqq \beta.$$

Since for any open set U, $m(U) > 0$, and for each k $\Lambda_\xi G_k(t)$ is a continuous function, then for any k and n

$$m\left(\left\{t:1 - \frac{1}{n} < \Lambda_\xi G_k(t) < 1\right\}\right) > 0.$$

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Let

$$H_{n,k} = \left(1 + \frac{1}{n-1}\right)G_k = \left(1 + \frac{1}{n-1}\right)\left(1 + \frac{1}{k-1}\right)F,$$

then

$$m(\{t:\Lambda_\xi H_{n,k}(t) > 1\}) > \beta.$$

Since for each n, k $H_{n,k} \in D_\beta$, $\|H_{n,k}\| \mapsto \|F\|$ when $n \to \infty$, $k \to \infty$ this implies that

$$\inf\{I_\mu(x):x \in D_\beta,\ x \in \Lambda H_\mu\} = \inf\{I_\mu(x):x \in D_\beta^*,\ x \in \Lambda H_\mu\}$$

and $a_\beta^* = a_\beta = \bar{a}_\beta$.

10. COROLLARY. *Let $(T, \mathcal{F}, m)$ be a real line with Borel σ-algebra $\mathcal{F}$ and Lebesgue measure m. Let $\xi = \{\xi(t):t \in T\}$ be a mean-zero Gaussian stochastic process, continuous in probability with almost all its sample paths in $L_p = L_p(T, \mathcal{F}, m)$, $0 < p < \infty$, such that for any $\beta > 0$*

$$m(\{s:m(\{t:K(s, t) > 0\}) > \beta\}) > 0$$

then for any $\beta > 0$ there exists $0 < a_\beta < \infty$ such that

$$\lim_{\alpha \to \infty} \alpha^{-2} \log P(\{\omega:m(\{t:\xi(t, \omega) > \alpha\}) > \beta\}) = -a_\beta.$$

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