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SEQUENCES WITH TRANSLATES CONTAINING MANY PRIMES

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ABSTRACT. Garrison [3], Forman [2], and Abel and Siebert [1] showed that for all positive integers k and N, there exists a positive integer λ such that $n^k + \lambda$ is prime for at least N positive integers n. In other words, there exists λ such that $n^k + \lambda$ represents at least N primes.

We give a quantitative version of this result. We show that there exists $\lambda \leq x^k$ such that $n^k + \lambda$, $1 \leq n \leq x$, represents at least $\left(\frac{1}{k} + o(1)\right)\pi(x)$ primes, as $x \longrightarrow \infty$. We also give some related results.

1. **Introduction.** In [1], Abel and Siebert make the wonderful observation that if $A = \{a_n\}$ is a sequence of natural numbers and $A(x) = \sum_{a_n \leq x} 1$, then

$$\sum_{\lambda \leq 2x} \sum_{a_n \leq x} \sum_{p = a_n + \lambda} 1 \geq [\pi(2x) - \pi(x)]A(x),$$

where p denotes a prime and $\pi(x)$ denotes the number of primes $p \leq x$. They used this inequality, together with Chebyshev's inequalities, to show that if $\limsup_{x \to \infty} \frac{A(x)}{\log x} = \infty$, then for all N there exists λ such that $a_n + \lambda$ represents at least N primes. Forman [2] obtained the same result with methods different from those of Abel and Siebert.

Earlier, Sierpenski [4] showed that $n^2 + \lambda$ represents arbitrarily many primes. Then Garrison [3] extended this to $n^k + \lambda$. Forman [2] and Abel and Siebert [1] showed that $g(n) + \lambda$ represents arbitrarily many primes, where $g(x)$ is any polynomial with integer coefficients and positive leading coefficient.

In this note we consider sums of the form

$$S(x) = \sum_{\lambda \leq 2x} \sum_{a_n \leq x} \sum_{b_m = a_n + \lambda} f(b_m) \quad \text{and} \quad T(x) = \sum_{\lambda \leq x} \sum_{a_n \leq x} \sum_{b_m = a_n + \lambda} f(b_m),$$

where $A = \{a_n\}$ and $B = \{b_m\}$ are given sequences of natural numbers and f is a given nonnegative function defined on the natural numbers. In particular, if B is the sequence of primes and $f \equiv 1$, then $T(x) = \big(1 + o(1)\big)A(x)\pi(x)$. This implies that if $A = \{n^k : n \geq 1\}$, then $T(x) = \big(1 + o(1)\big)x^{\frac{1}{k}}\pi(x)$. It follows that there exists a positive integer $\lambda \leq x^k$ such that $n^k + \lambda$, $n \leq x$, represents at least $\left(\frac{1}{k} + o(1)\right)\pi(x)$ primes.

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2. Results.

THEOREM 1. *Let $A = \{a_n\}$, $B = \{b_m\}$ be sequences of natural numbers, and let f be a nonnegative function defined on the natural numbers. Let $A(x) = \sum_{a_n \leq x} 1$, $B(x) = \sum_{b_m \leq x} f(b_m)$.*

Assume that $B(x) = \big(1 + o(1)\big)x^\alpha \varphi(x)$, where φ is monotonic and $\lim_{x\to\infty} \frac{\varphi(2x)}{\varphi(x)} = 1$.

Let $S(x)$ denote the sum

$$S(x) = \sum_{a_n \leq x} \sum_{\lambda \leq 2x} \sum_{b_m = a_n + \lambda} f(b_m)$$

Then

$$\big(2^\alpha - 1 + o(1)\big)A(x)B(x) \leq S(x) \leq \big(3^\alpha + o(1)\big)A(x)B(x).$$

PROOF. For the lower bound, we start with Abel and Siebert's inequality

$$S(x) \geq [B(2x) - B(x)]A(x).$$

Next,

$$\frac{B(2x) - B(x)}{B(x)} = \frac{\big(1 + o(1)\big)(2x)^\alpha \varphi(2x)}{\big(1 + o(1)\big)x^\alpha \varphi(x)} - 1 \longrightarrow 2^\alpha - 1,$$

hence $B(2x) - B(x) = \big(2^\alpha - 1 + o(1)\big)B(x)$.

For the upper bound, we write

$$\begin{aligned} S(x) &= \sum_{a_n \leq x} \sum_{a_n + 1 \leq b_m \leq a_n + 2x} f(b_m) \\ &= \sum_{a_n \leq x} [B(a_n + 2x) - B(a_n)] \leq \sum_{a_n \leq x} B(a_n + 2x). \end{aligned}$$

We now estimate $B(a_n + 2x)$ from above.

Let a be an integer, $1 \leq a \leq x$. Since φ is monotonic, $x \leq a + x \leq 2x$, and $\frac{\varphi(x)}{\varphi(x)} = 1$, $\frac{\varphi(2x)}{\varphi(x)} \longrightarrow 1$, it follows that for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\frac{\varphi(a + x)}{\varphi(x)} < 1 + \varepsilon, \quad x > N,\ 1 \leq a \leq x.$$

From this it follows that $\frac{\varphi(3x)}{\varphi(x)} = \frac{\varphi(3x)}{\varphi(2x)} \cdot \frac{\varphi(2x)}{\varphi(x)} \longrightarrow 1$.

Now since $2x \leq a + 2x \leq 3x$, φ is monotonic, and $\frac{\varphi(2x)}{\varphi(x)} \longrightarrow 1$, $\frac{\varphi(3x)}{\varphi(x)} \longrightarrow 1$, it follows that for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\frac{\varphi(a + 2x)}{\varphi(x)} < 1 + \varepsilon, \quad x > N, \quad 1 \leq a \leq x.$$

It now follows that for any $a = a(x)$, $1 \leq a \leq x$, and any $\varepsilon > 0$,

$$\frac{B(a + 2x)}{B(x)} = \frac{\big(1 + o(1)\big)(a + 2x)^\alpha \varphi(a + 2x)}{\big(1 + o(1)\big)x^\alpha \varphi(x)} < 3^\alpha + \varepsilon$$

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for sufficiently large x. Hence, independent of the choice of a, $1 \le a \le x$,

$$B(a+2x) \le \big(3^{\alpha} + o(1)\big)B(x),$$

and

$$S(x) \le \sum_{a_n \le x} B(a_n + 2x) \le \big(3^{a} + o(1)\big)A(x)B(x).$$

This completes the proof of Theorem 1.

Now we let B be the sequence of primes.

THEOREM 2. *Let $A = \{a_n\}$ be a sequence of natural numbers. Then*

$$S(x) = \sum_{\lambda \le 2x} \sum_{a_n \le x} \sum_{p = a_n + \lambda} 1 \ge \big(1 + o(1)\big)A(x)\pi(x),$$

where p denotes a prime. Hence there exists λ, $1 \le \lambda \le 2x$, such that $a_n + \lambda$, $1 \le a_n \le x$ represents at least $\left(\frac{1}{2} + o(1)\right)\frac{A(x)}{x}\pi(x)$ primes.

PROOF. This proof is a direct application of the method of Abel and Siebert. We have

$$S(x) = \sum_{\lambda \le 2x} \sum_{a_n \le x} \sum_{p = a_n + \lambda} 1 \ge \big(\pi(2x) - \pi(x)\big)A(x) = \big(1 + o(1)\big)A(x)\pi(x),$$

or

$$\frac{1}{2x}\sum_{\lambda=1}^{2x}\Big(\sum_{a_n \le x} \sum_{p = a_n + \lambda} 1\Big) \ge \Big(\frac{1}{2} + o(1)\Big)\frac{A(x)}{x}\pi(x),$$

so at least one λ, $1 \le \lambda \le 2x$, has the required property.

We now improve this result by using part of the method of proof of Theorem 1.

THEOREM 3. *Let $A = \{a_n\}$ be a sequence of natural numbers. Then*

$$T(x) = \sum_{\lambda \le x} \sum_{a_n \le x} \sum_{p = a_n + \lambda} 1 = \big(1 + o(1)\big)A(x)\pi(x),$$

where p denotes a prime. Hence there exists λ, $1 \le \lambda \le x$, such that $a_n + \lambda$, $1 \le a_n \le x$, represents at least $\big(1 + o(1)\big)\frac{A(x)}{x}\pi(x)$ primes.

PROOF. As in the proof of Theorem 1, we write

$$T(x) = \sum_{a_n \le x} \sum_{a_n + 1 \le p \le a_n + x} 1 = \sum_{a_n \le x} [\pi(a_n + x) - \pi(a_n)].$$

It is not hard to show that for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$1 - \varepsilon < \frac{\pi(a+x) - \pi(a)}{\pi(x)} < 1 + \varepsilon, \quad x > N,\ 1 \le a \le x.$$

(For fixed ε, divide $[1, x]$ into subintervals of length εx, and use the Prime Number Theorem to estimate $\frac{\pi(a+x)-\pi(x)}{\pi(x)}$ when $a \in [(i-1)\varepsilon x, i\varepsilon x]$.)

Summing this over all a_k, $a_k \le x$, gives

$$(1 - \varepsilon)A(x)\pi(x) < T(x) < (1 + \varepsilon)A(x)\pi(x), \quad x > N,$$

or $T(x) = \big(1 + o(1)\big)A(x)\pi(x)$. The rest follows as in the proof of Theorem 2.

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COROLLARY. *Let $k \geq 1$ be given. Then there exists a positive integer $\lambda \leq x^k$ such that $n^k + \lambda$, $n \leq x$, represents at least $\left(\frac{1}{k} + o(1)\right)\pi(x)$ primes.*

PROOF. Setting $a_n = n^k$ in Theorem 3, and replacing x by x^k in the conclusion of Theorem 3 shows that there exists λ, $1 \leq \lambda \leq x^k$, such that $n^k + \lambda$, $1 \leq n^k \leq x^k$, represents at least

$$\left(1 + o(1)\right)\frac{(x^k)^{\frac{1}{k}}}{x^k}\pi(x^k) = \left(1 + o(1)\right)\frac{x}{x^k}\frac{x^k}{\log x^k} = \left(1 + o(1)\right)\frac{x}{k\log x} = \left(\frac{1}{k} + o(1)\right)\pi(x)$$

primes.

We now apply our methods to the case when B is the sequence of square-free numbers.

THEOREM 4. *Let $A = \{a_n\}$ be a given sequence of natural numbers. Let $A(x) = \sum_{a_n \leq x} 1$, and let α be any fixed real number with $\frac{1}{2} < \alpha < 1$. Let $\varepsilon > 0$ be given. Then for all sufficiently large x, there exists λ, $1 \leq \lambda \leq x^\alpha$, such that more than $(\frac{6}{\pi^2} - \varepsilon)A(x)$ of the numbers $a_n + \lambda$, $a_n \leq x$, are square-free.*

PROOF. Let $B = \{b_m\}$ be the sequence of square-free numbers, and let $B(x) = \sum_{b_m \leq x} 1$. It is known (see [4]) that

$$B(x) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Let α be fixed, $1/2 < \alpha < 1$, and let L denote the number $L = [x^\alpha]$.

Let $\varepsilon > 0$ be given. Then

$$\begin{aligned}
\sum_{\lambda=1}^{L}\sum_{a_n \leq x}\sum_{b_m = a_n+\lambda} 1 &= \sum_{a_n \leq x}\sum_{\lambda \leq L}\sum_{b_m = a_n+\lambda} 1 \\
&= \sum_{a_n \leq x}\sum_{a_n+1 \leq b_m \leq a_n+L} 1 \\
&= \sum_{a_n \leq x}\left(B(a_n + L) - B(a_n)\right) \\
&= \sum_{a_n \leq x}\left(\frac{6L}{\pi^2} + O\left(\sqrt{x+L}\right)\right) \\
&= \sum_{a_n \leq x}\frac{6L}{\pi^2}\left(1 + o(1)\right) \\
&> \left(\frac{6}{\pi^2} - \varepsilon\right)L\sum_{a_n \leq x} 1 \\
&= \left(\frac{6}{\pi^2} - \varepsilon\right)LA(x)
\end{aligned}$$

holds for sufficiently large x. Hence there exists at least one λ, $1 \leq \lambda \leq L = [x^\alpha]$, for which

$$\sum_{a_n \leq x}\sum_{b_m = a_n+\lambda} 1 > \left(\frac{6}{\pi^2} - \varepsilon\right)A(x),$$

which was to be proved.

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