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ON THE FINITE TWO-DIMENSIONAL LINEAR GROUPS II.[1]

BY
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A group G is called a T_3-group if it contains subgroups K and H, $H \lhd K$, with the property that if g and g^b are members of $G-K$ there is exactly one $h \in H$ which satisfies the equation $g^h = g^b$. In these circumstances (G, K, H) is called a T_3-triple.

T_3-groups were studied by the author ([1], see also errata) and used there to give characterizations of the finite two-dimensional linear groups and in this paper we continue the study. In particular we will prove the following.

THEOREM. *If (G, K, H) is a T_3-triple, G is finite, a is an involution of $G-N(K)$ and $H \cap H^a \neq 1$, then either*

(1) *$K \cap K^a \lhd G$ and $G/K \cap K^a$ is isomorphic to a group of all similarity transformations over a finite field, or*

(2) *K has a conjugate K^b which is different from K and K^a, $K \cap K^a \cap K^b = Z(G)$ and $G/Z(G)$ is isomorphic to a group of all bilinear transformations over a finite field of characteristic 2.*

Throughout the paper we assume that (G, K, H) is a T_3-triple which satisfies the hypotheses of this theorem. In §2 we discuss the situation which gives rise to the first conclusion and in §3 we discuss the second.

Notations. We shall use the standard notations of [1]. In addition we denote the order of $H \cap K^a$ by α, the order of $H \cap H^a$ by β and the index of K in G by n.

By hypothesis we have $\beta > 1$ throughout the paper.

1. **Preliminaries.** For reference purposes we will list here, without proof, the results from [1] which we will be using in this paper. After each result the reference is to the place in [1] where this result may be found.

LEMMA 1.1. *If $g \in G-K$, then $C(g) \cap H = 1$.*
Lemma 3.1.

LEMMA 1.2. $N(K) = K$.
Theorem 4.1.

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LEMMA 1.3. *$H \cap H^a$ is abelian of odd order and if $x \in H \cap H^a$, then $x^a = x^{-1}$.*
Lemmas 6.4 and 6.5.

LEMMA 1.4. *$(H \cap K^a)(H^a \cap K)/H \cap H^a$ is abelian.*
Lemma 6.11.

LEMMA 1.5. *If H^x, H^y and H^z are three different conjugates of H, then $H^x \cap H^y \cap H^z = 1$.*
Lemma 6.6.

LEMMA 1.6. *G is doubly transitive on the right cosets of K in G.*
Lemma 6.2.

LEMMA 1.7. *H is transitive on the right cosets of K which are different from K.*
Lemma 6.3.

LEMMA 1.8. *$H - K^a$ intersects $n-2$ classes of K.*
Lemma 6.9.

Lemmas 1.6 and 1.7 have the following consequences. G is doubly transitive on the conjugates of K under conjugacy, and H is transitive on the conjugates of K different from K under conjugacy.

LEMMA 1.9. *H has order $\alpha(n-1)$.*
Lemma 6.8.

2. **A special case.** The proof of the theorem divides naturally into two parts, the first of which we investigate here. In this section we will assume that G has the additional property that if K^x, K^y and K^z are three different conjugates of K, then $H^x \cap H^y \subset K^z$. We will denote the intersection of all the subgroups conjugate to K by K^*. K^* is a normal subgroup of G and our aim will be to show that $(G/K^*, K/K^*, HK^*/K^*)$ is a T_3-triple to which we can apply some of our previous results.

An interpretation of the extra condition is

LEMMA 2.1. *If $x, y \in G$ and $K^x \neq K^y$, then $H^x \cap H^y \subset K^*$.*

However, we can prove more.

PROPOSITION 2.2. *If $x, y \in G$ and $K^x \neq K^y$, then $H^x \cap K^y \subset K^*$.*

Proof. By the double transitivity property of G (Lemma 1.6), it is sufficient to show that $H \cap K^a \subset K^*$.

Suppose $h \in (H \cap K^a) - K^*$. Since h is not contained in K^*, $h \in G - K^b$ for some conjugate K^b of K. Clearly $K^b \neq K$ and $K^b \neq K^a$. Consider the subgroup $H \cap H^b$. By assumption $H \cap H^b \subset K^* \subset K^a$ so that $H \cap H^b \subset H \cap K^a$. Also $h \in H \cap K^a$ so that the conjugates of h by members of $H \cap H^b$ are members of $H \cap K^a$. $H \cap H^b$ has order β and $h \in G - K^b$ so h has exactly β conjugates by members of $H \cap H^b$. Moreover, $H \cap H^a$ is a normal subgroup of $H \cap K^a$ having order β and $H \cap K^a / H \cap H^a$ is abelian (Lemma 1.4). Thus the conjugates of h by members of $H \cap H^b$ form the coset $h(H \cap H^a)$.

Now consider $H^a \cap H^b$ in the same fashion. We have $H^a \cap H^b \subset H^a \cap K$ and by Lemma 1.4, $H \cap H^a$ is a normal subgroup of the group $(H \cap K^a)(H^a \cap K)$ and the factor group is abelian. As in the last paragraph, the conjugates of h by members of $H^a \cap H^b$ also form the coset $h(H \cap H^a)$.

By hypothesis, $H \cap H^a$ has order $\beta > 1$ so we can find $s \in H \cap H^b$, $s \neq 1$ and $t \in H^a \cap H^b$, $t \neq 1$ such that $h^s = h^t$. Since $H \cap H^a \cap H^b = 1$ (by Lemma 1.5) we have $s \neq t$. h is not a member of K^b and hence the T_3-property of G is contradicted. Thus no such h exists. This proves the result.

PROPOSITION 2.3. *$(G/K^*, K/K^*, HK^*/K^*)$ is a T_3-triple.*

Proof. If $g, g^b \in G - K$, then by the property T_3, H contains a member h with the property $g^h = g^b$. Hence $(hK^*)^{-1}(gK^*)(hK^*) = (bK^*)^{-1}(gK^*)(bK^*)$. It remains to show that hK^* in this equation is unique. If also $h_1 \in H$ and h_1K^* satisfies this equation in place of hK^*, we have $g^{h_2}K^* = gK^*$ where $h_2 = hh_1^{-1}$. Then $gh_2g^{-1} \in h_2K^* \subset K$ so that $h_2 \in K^g$. But $g \in G - K$ so that $K \neq K^g$. Hence $h_2 \in H \cap K^g \subset K^*$ by Proposition 2.2. Thus $h_1K^* = hK^*$ which proves the proposition.

We can now combine this result with the results of [1] to prove the main theorem in the case considered in this section.

By Proposition 2.3 $(G/K^*, K/K^*, HK^*/K^*)$ is a T_3-triple. If $g \in G - K$, then $HK^* \cap K^g = (H \cap K^g)K^* \subset K^*$ by Proposition 2.2. i.e. $HK^*/K^* \cap (K/K^*)^g = 1$. Clearly aK^* is an involution of $G/K^* - K/K^*$. Applying Theorem 2 and Lemma 5.3 of [1] we deduce that $Z(G/K^*) = (K \cap K^a)/K^*$ and $(G/K^*)/Z(G/K^*)$ is isomorphic to a group of similarities. Hence $G/K \cap K^a$ is isomorphic to such a group.

3. **The main case.** We now treat the situation excluded in §2. We may assume that, for some $b \in G$, $(H \cap H^a) - K^b$ is not empty. In the rest of the section we shall assume that K^b has this property.

PROPOSITION 3.1. *If K^p, K^q and K^r are three different conjugates of K then either $H^p \cap H^q \subset K^r$ or $H^p \cap H^q \cap K^r = 1$.*

If $H^p \cap H^q \cap K^r = 1$, then H^r contains exactly one member s with the property $t^s = t^{-1}$ for all $t \in H^p \cap H^q$. Moreover $s^2 = 1$. In particular $H \cap H^a \cap K^b = 1$.

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Proof. Suppose u is a member of $H^p \cap H^q - K^r$. By Lemma 1.3 u^{-1} is a conjugate of u and, as $H^p \cap H^q$ has odd order, $u^{-1} \neq u$. Hence, by the property T_3, H^r contains exactly one member s with the property $u^s = u^{-1}$. From this it follows that $u \in C(s^2)$ and since $u \in G - K^r$, $s^2 \in H^r$ it follows from Lemma 1.1 that $s^2 = 1$. Since $u^s = u^{-1}$ we have $u \in (H^p \cap H^q) \cap (H^p \cap H^q)^s$. We have that the intersection of any three conjugates of H is 1 (Lemma 1.5) so either $H^{ps} = H^p$ and $H^{qs} = H^q$ or $H^{ps} = H^q$ and $H^{qs} = H^p$. In the first case we would have $s \in N(H^p) = K^p$ and $s \in N(H^q) = K^q$. Thus $s \in K^p \cap K^q$. By the double transitivity property of G it follows that a has a conjugate, say c, which is not in $K^p \cap K^q$ but has the property that $v^c = v^{-1}$ for all $v \in H^p \cap H^q$. Then $sc^{-1} \in C(u)$ and $sc^{-1} \in G - K^p$, $u \in H^p$ and $u \neq 1$. This contradicts Lemma 1.1. Hence we must have $H^{ps} = H^q$ and $H^{qs} = H^p$. This implies that $s \in N(H^p \cap H^q)$ and considering c as above, we have $s \in G - (K^p \cup K^q)$. It now easily follows that $(H^p \cap H^q) + s(H^p \cap H^q)$ is a generalized dihedral group and in particular $v^s = v^{-1}$ for all $v \in H^p \cap H^q$.

Suppose now that $w \in H^p \cap H^q \cap K^r$. Then $w \in K^r = N(H^r)$ and so $s^{-1}s^w \in H^r$. But $s^{-1}s^w = (w^{-1})^s w = w^2 \in H^p \cap H^q$. Thus $w^2 \in H^p \cap H^q \cap H^r = 1$. $H^p \cap H^q$ has odd order so we have $w = 1$. Hence $H^p \cap H^q \cap K^r = 1$.

By assumption $(H \cap H^a) - K^b$ is not empty which implies from the above, that $H \cap H^a \cap K^b = 1$.

This proves Proposition 3.1.

PROPOSITION 3.2. $\alpha \leq n-2$.

Proof. By Proposition 3.1, H^b contains exactly one involution, say y, with the property that $h^y = h^{-1}$ for all $h \in H \cap H^a$. Now y has at least α conjugates in $(K \cap K^a)y$, namely the conjugates y^k, $k \in H \cap K^a$. If z is such a conjugate of y, then $z^2 = 1$, $z \in N(H \cap H^a)$ and $C(z) \cap (H \cap H^a) = 1$. From this it follows that $h^z = h^{-1}$ for all $h \in H \cap H^a$. Also, $y \in G - (K \cup K^a)$ so that $z \in G - (K \cup K^a)$ and hence z is contained in a conjugate of H, not H or H^a, say $z \in H^c$. If H^c contains more than one of these conjugates of y, then so does H^b which is not possible. Hence the α conjugates of y are contained at most one each in $n-2$ subgroups. Hence $\alpha \leq n-2$.

PROPOSITION 3.3. *H contains α subgroups $H \cap H^c$ with the property $H \cap H^c \cap K^b = 1$.*

Proof. Denote by B the union of the subgroups of G which are conjugate to $H \cap H^a$. The intersection of any two of these subgroups is 1 by Lemma 1.5. Suppose that $(B \cap H) - K^b$ has order γ. $B \cap H$ has order $(n-1)(\beta-1)+1$ so that $B \cap H \cap K^b$ has order $(n-1)(\beta-1)+1-\gamma$. But $H \cap K^b$ has order α so we have $(n-1)(\beta-1)+1-\gamma \leq \alpha$.

We now consider the number γ in more detail. First we note that if $h \in (B \cap H)-K^b$, so is h^k for each $k \in H^b \cap K$ and hence γ is a multiple of α. By Lemma 3.1, $H \cap H^a$ contains $\beta-1$ members in $(B \cap H)-K^b$. Suppose $h \in (B \cap H)-K^b$. By Lemma 1.7, H is transitive on the right cosets of K in G which are not equal to K. Thus $h^s \in H \cap H^a$ for some $s \in H$. By Proposition 3.1, we then have h, $h^s \in G-K^b$ so that $h^s=h^t$ for some $t \in H^b$. Then $ts^{-1} \in C(h) \subset K$ so we obtain $t \in K$ and thus $t \in H^b \cap K$. Thus every member of $(B \cap H)-K^b$ is conjugate to a member of $H \cap H^a$ by a member of $H^b \cap K$. The number of such conjugates is at most $\alpha(\beta-1)$ so we obtain $\gamma \leq \alpha(\beta-1)$.

We have shown above that γ is a multiple of α, say $\gamma=\delta\alpha$. We now have

$$(n-1)(\beta-1)+1 \leq \alpha(\delta+1) \leq (n-2)(\delta+1)$$

by Proposition 3.2. This leads to

$$\delta+1 \geq \frac{(n-1)(\beta-1)}{n-2}+\frac{1}{n-2}$$

$$> \beta-1+\frac{1}{n-2}$$

Hence $\delta+1 \geq \beta$ as δ is an integer and so $\delta \geq \beta-1$, i.e. $\gamma \geq \alpha(\beta-1)$.

Hence $\gamma=\alpha(\beta-1)$.

Combining Proposition 3.1 with this result we obtain the proposition.

PROPOSITION 3.4. $C(H \cap H^a)=K \cap K^a$.

Proof. By Lemma 1.1 we have $C(H \cap H^a) \subset K \cap K^a$. We now prove the other inclusion.

Let h be a fixed member of $H \cap H^a$, $h \neq 1$ and let $H \cap H^c$ be a subgroup with the property $H \cap H^c \cap K^b=1$. H is transitive on the right cosets of K which are not equal to K so that $H \cap H^c=(H \cap H^a)^x$ for some $x \in H$. Then $h^x \in H \cap H^c$ so that $h \in G-K^b$ and $h^x \in G-K^b$. Hence by the property T_3 $h^x=h^t$ for some $t \in H^b$. Then $tx^{-1} \in C(h) \subset K$ so we have $t \in K$ i.e. $t \in H^b \cap K$.

We have thus shown that h has a conjugate in each of the subgroups $H \cap H^c$ with the property $H \cap H^c \cap K^b=1$ by a member of $H^b \cap K$. But there are α such subgroups and $H^b \cap K$ has order α. Thus h has exactly one such conjugate in each such subgroup and in particular, if $h^t \in H \cap H^a$ and $t \in H^b \cap K$, then $t=1$.

Suppose $k \in K \cap K^a$. We have $H \cap H^a \triangleleft K \cap K^a$ so that $h^k \in H \cap H^a$. As $H \cap H^a \cap K^b=1$ there exists $t \in H^b$ with the property $h^t=h^k$. Then $tk^{-1} \in C(h) \subset K$ so that $t \in K$. i.e. $t \in H^b \cap K$ and $h^t \in H \cap H^a$. By the above $t=1$ so we have $h^k=h$ i.e. $k \in C(h)$. This proves the proposition.

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PROPOSITION 3.5. *If $h \in H \cap H^a$, $h \neq 1$, then h has $n(n-1)$ conjugates in G and the only conjugates of h in $H \cap H^a$ are h and h^{-1}.*

Proof. By Proposition 3.4, $K \cap K^a \subset C(h)$ and by Lemma 1.1 $C(h) \subset K \cap K^a$. $K \cap K^a$ has index $n(n-1)$ so that this is the number of conjugates of h in G.

Now $H \cap H^a$ has $\frac{1}{2}n(n-1)$ conjugate subgroups and by Proposition 1.5 each pair of these intersects in 1. Hence $H \cap H^a$ contains two conjugates of h. $h^a = h^{-1}$ so that h and h^{-1} are two conjugates of h in $H \cap H^a$. $H \cap H^a$ has odd order so that $h \neq h^{-1}$. This proves the result.

PROPOSITION 3.6. *$\alpha = n-2$ and for each c with $K^c \neq K$, $K^c \neq K^a$, $H^c \cap K \cap K^a = 1$.*

Proof. Suppose $h \in H \cap H^a$, $h \neq 1$. $h \in G-K^b$ so that h has at least one conjugate in $G-K$. By the property T_3, h then has exactly $\alpha(n-1)$ conjugates in $G-K$. Since each conjugate of $H \cap H^a$ contains two of these conjugates of h, and by Proposition 3.1, it follows that there are exactly $\frac{1}{2}\alpha(n-1)$ conjugates of $H \cap H^a$ which intersect K in 1 and the rest are contained in K. Hence the number of conjugates contained in K is $\frac{1}{2}(n-\alpha)(n-1)$. K has n conjugate subgroups so it follows that each conjugate of $H \cap H^a$ is contained in exactly $n-\alpha$ of the conjugates of K. Hence for α subgroups K^c we have $H \cap H^a \cap K^c = 1$, and so $C(H \cap H^a) \cap H^c = 1$ by Lemma 1.1. Thus by Proposition 3.4 $K \cap K^a \cap H^c = 1$. Now considering these α subgroups which intersect $K \cap K^a$ in 1, it is clear that these subgroups contain $\alpha(n-1) - \frac{1}{2}\alpha(\alpha-1)$ conjugates of $H \cap H^a$.

If $K \cap K^a$ contains γ conjugates of $H \cap H^a$, then using the fact that each of these is contained in $n-\alpha$ conjugates of K we obtain $\gamma = \frac{1}{2}(n-\alpha)(n-\alpha-1)$.

Now $\alpha(n-1) - \frac{1}{2}\alpha(\alpha-1) + \frac{1}{2}(n-\alpha)(n-\alpha-1) = \frac{1}{2}n(n-1)$ which is the total number of subgroups conjugate to $H \cap H^a$. Hence we have proved that if $H^p \cap H^q$ is a conjugate of $H \cap H^a$, then either $H^p \cap H^q \subset K \cap K^a$ or both $H^p \cap K \cap K^a = 1$ and $H^q \cap K \cap K^a = 1$.

Suppose now that $\alpha < n-2$. Then for some H^p with $H^p \neq H$, $H^p \neq H^a$ we have $H^p \cap K \cap K^a \neq 1$. From the above it follows that $K \cap K^a$ then contains all the subgroups $H^p \cap H^q$, $H^p \neq H^q$, p fixed. These subgroups contain $(n-1)(\beta-1)+1$ members of G so that $H^p \cap K \cap K^a$ contains at least this number of members. $H^p \cap K \cap K^a \subset H^p \cap K$ which has order α. Thus

$$(n-1)(\beta-1)+1 \leq \alpha$$

But $\alpha \leq n-2$ by Proposition 3.2 so we have

$$(n-1)(\beta-1) \leq n-3$$

which is a contradiction as $\beta > 1$.

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This implies that $\alpha=n-2$ so by Proposition 3.3 we have $H \cap H^c \cap K^b=1$ for each H^c with $H\neq H^c$, $H^c\neq K^b$ which leads to the result

PROPOSITION 3.7. *If K^p, K^q and K^r are different conjugates of K then $H^p \cap K^q \cap K^r=1$.*

Proof. G is doubly transitive on the right cosets of K in G and by Proposition 3.6 $H^c \cap K \cap K^b=1$ for all H^c with K^c, K and K^b different. This is sufficient to prove the result.

PROPOSITION 3.8. *H is a Frobenius group with Frobenius kernel of order $n-1$ and $H \cap K^a$ is a Frobenius complement.*

Proof. Consider $H \cap K^a$. If $h \in H-K^a$, then $K^{ah}\neq K^a$ so that $(H \cap K^a) \cap (H \cap K^a)^h=H \cap K^a \cap K^{ah}=1$ by Proposition 3.7. Hence $H \cap K^a$ is a Frobenius complement in H. $H \cap K^a$ has order $n-2$ and H has order $(n-2)(n-1)$ so the Frobenius kernel has order $n-1$.

PROPOSITION 3.9. *$H \cap K^a$ is abelian and H is isomorphic to a group of similarities over a finite field. Moreover $C(H \cap K^a)=K \cap K^a$.*

Proof. By Proposition 3.8, H is a Frobenius group with kernel of order $n-1$ and complement of order $n-2$. Hence the kernel contains 1 and one class of H. By Lemma 1.9, $H-K^a$ contains $n-2$ classes of K. Hence, from the properties of Frobenius groups, $H \cap K^a$ contains 1 and intersects $n-3$ other classes of K. Thus $H \cap K^a$ is abelian and $C(H \cap K^a)=K \cap K^a$.

PROPOSITION 3.10. *$K \cap K^a \cap K^b=Z(G)$ and $G/Z(G)$ has a triply transitive representation of degree n in which only the identity fixes 3 members. Also $H \cap H^a=H \cap K^a$.*

Proof. Put $L=K \cap K^a \cap K^b$. By Proposition 3.9 we have both $L \subset C(H \cap K^a)$ and $L \subset C(H \cap K^b)$. H is a Frobenius group and $H \cap K^a$, $H \cap K^b$ are complements of orders greater than one and so these two subgroups generate H. Hence $L \subset C(H)$, and similarly $L \subset C(H^a)$.

Let G_1 be the subgroup of G generated by H, H^a and L. Clearly $L \subset Z(G_1)$, by the above. We have $H^{ah} \subset G_1$ for each $h \in H$ so by Proposition 1.7, G_1 contains every conjugate of H. Let γ be the index of L in G_1. We have $H \cap L \subset H \cap K^a \cap K^b=1$, $H^a \cap L=1$ similarly and $H \cap H^a$ has order β, so that $\gamma\geq 1/\beta((n-1)(n-2))^2$. Now $K \cap K^a \cap K^b$ has index $\leq n(n-1)(n-2)$ in G so that the index of G_1 in G is less than the quotient of these two numbers, i.e. $n\beta/(n-1)(n-2)$ which is $\leq n/n-1$ as $\beta\leq n-2$. Hence this index is ≤ 1 and so $G=G_1$.

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We thus have $K \cap K^a \cap K^b \subset Z(G)$ and the other inclusion is clear from Proposition 1.1.

From the above, we also have the inequality

$$n(n-1)(n-2) \geq \frac{1}{\beta}((n-1)(n-2))^2$$

and so $\beta \geq (n-1)(n-2)/n$. Now $n>2$ as $H \cap K^a$ has order $n-2$ and so $\beta > \frac{1}{2}(n-2)$. β is the order of $H \cap H^a$ which is a subgroup of $H \cap K^a$ which has order $n-2$. Hence, by Lagrange's theorem $\beta = n-2$ and we deduce that $H \cap H^a = H \cap K^a$.

Applying this result to the above inequalities we have $\gamma \geq (n-1)^2(n-2)$. But G is doubly transitive on the right cosets of K in G and $N(K)=K$ so that γ is a divisor of $n(n-1)(n-2)$. Thus $\gamma = n(n-1)(n-2)$ as $n>2$.

The representation of $G/Z(G)$ on the right cosets of K in G has the desired properties.

Our theorem now follows from the results of Zassenhaus [2]. From this paper and the two previous propositions, we deduce that $G/Z(G)$ is isomorphic to a group of all bilinear transformations over a finite field of order $n-1$. By Proposition 3.10 and Lemma 1.3, $n-2=\beta$ has odd order so that $n-1$ is even and thus a power of 2. Thus the finite field referred to has characteristic 2 which proves the main theorem.

References

1. Peter Lorimer, *On the finite two dimensional linear groups*, J. Algebra **10** (1968), 419–435.

2. H. Zassenhaus, *Kennzeichnung endlicher linearer Gruppen als Permutationsgruppen*, Abh. Math. Sem. Univ. Hamburg **2** (1936), 17–40.

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