ON ALMOST REGULAR HOMEOMORPHISMS

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1. Let (X, d) be a metric space with metric d, and h be a homeomorphism of X onto itself. Any point y in X is called a *regular* point **(2)** under h if for any given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(h^n(x), h^n(y)) < \epsilon$ for all integers n, where h^n is the composition of h or h^{-1} with itself $|n|$ times, depending upon whether n is positive or negative, and h^0 is the identity on X. If y is not regular under h, then y is called *irregular.* We shall denote the set of regular points by $R(h)$ and the set of irregular points by $I(h)$. The homeomorphism h is called *almost regular* if $I(h)$ is zero dimensional and compact. Note that $I(h)$ is therefore non-empty. We use the terms Lim sup and Lim inf as defined in **(5)**.

One of the aims of this paper is to prove the following:

MAIN THEOREM. *Let X be a locally compact, locally connected and connected space, and h be an almost regular homeomorphism of X onto itself. If $R(h)$ is connected, then $I(h)$ consists of at most two points, both of which are fixed under h.*

This result is related to Theorem 1 of **(4)**, where $I(h)$ was assumed to be finite and X connected and compact. The case $I(h) = \emptyset$ is considered in **(4)**.

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LEMMA 1. *Let (X, d) be a locally compact metric space and h be an almost regular homeomorphism of X onto itself. If $p \in I(h)$ and X is locally connected at p, then there exists a $y \in R(h)$ such that*

$$p \in \operatorname*{Lim\,sup}_{n\to\pm\infty} h^n(y).$$

Proof. Since p is an irregular point under h, there exists an $\epsilon > 0$ such that for any open set V containing p, diam $h^n(V) > \epsilon$ for infinitely many integers n.

Let

$$Q = \operatorname*{Lim\,sup}_{n\to\pm\infty} h^n(p).$$

It is easy to see that $h^n[I(h)] = I(h)$ for each integer n. This together with the fact that $I(h)$ is compact implies that Q is a non-empty compact subset of $I(h)$. Since $I(h)$ is zero dimensional and X is locally compact, there exists a finite open covering $\{U_i\}$ $(i = 1, \ldots, m)$ of Q such that for each i, diam $U_i < \epsilon$, $\bar{U}_i$ is compact, and boundary of $U_i \cap I(h)$ is $\emptyset$. Since

$$Q = \operatorname*{Lim\,sup}_{n\to\pm\infty} h^n(p),$$

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there exists a natural number N such that for all $|n| \geqslant N$,

$$h^n(p) \in \bigcup_{i=1}^{m} U_i.$$

Since X is locally connected at p, there exists a sequence $\{V_i\}$ of connected open sets containing p, such that $V_{i+1} \subset V_i$ for $i = 1, 2, \ldots$, and $\operatorname{diam}[V_i] \to 0$ as $i \to \infty$. Let $\{n_{ij}: j = 1, 2, \ldots\}$ be the set of all the integers such that $\operatorname{diam} h^{n_{ij}}[V_i] > \epsilon$. Since $V_{i+1} \subset V_i$, we have $\{n_{i+1,j}\} \subset \{n_{ij}\}$ for each i. It is easy to see then that there exists a member U of $U_1, \ldots, U_m$ and a subsequence $\{i_k: k = 1, 2, \ldots\}$ of the natural numbers i such that for each member i_k of this infinite set, $U \cap h^{n_{i_kj}}[V_{i_k}] \neq 0$ for infinitely many values of j. For each k let m_k denote an element of $\{n_{i_kj}: j = 1, 2, \ldots\}$ such that $h^{m_k}[V_{i_k}] \cap U \neq \emptyset$, and $|m_{k+1}| > |m_k|$ $(k = 1, 2, \ldots)$. Since $\operatorname{diam} h^{m_k}[V_{i_k}] > \epsilon$, $h^{m_k}[V_{i_k}]$ is connected, and $\operatorname{diam} U < \epsilon$, $h^{m_k}[V_{i_k}] \cap \operatorname{bdry} U \neq \emptyset$. Let

$$y_k \in h^{m_k}[V_{i_k}] \cap \operatorname{bdry} U \quad (k = 1, 2, \ldots).$$

The sequence of points $\{y_k\} \subset \operatorname{bdry} U$ has a limit point y in it. Let us assume for convenience that $\{y_i\}$ converges to y (or we work with a subsequence converging to y). We shall show that

$$p \in \operatorname*{Lim\,sup}_{n \to \pm\infty} h^n(y).$$

Let $\eta > 0$ be arbitrary. Since $y \in \operatorname{bdry} U \subset R(h)$, there exists a $\gamma > 0$, such that, for $d(x, y) < \gamma$, $d(h^n(x), h^n(y)) < \eta/2$ for all integers n. Let K be large enough so that $d(y, y_k) < \gamma$ for $k \geqslant K$. Hence (i) $d(h^{-m_k}(y), h^{-m_k}(y_k)) < \eta/2$ for $k \geqslant K$. Let N be large enough so that $\operatorname{diam} V_{i_k} < \eta/2$ for $k \geqslant N$. Hence (ii) $d(h^{-m_k}(y_k), p) < \eta/2$ for $k \geqslant N$. Thus

$$d(h^{-m_k}(y), p) \leqslant d(h^{-m_k}(y), h^{-m_k}(y_k)) + d(h^{-m_k}(y_k), p) < \tfrac{1}{2}\eta + \tfrac{1}{2}\eta = \eta$$

for $k \geqslant \max(K, N)$ from (i) and (ii). Hence

$$p = \operatorname*{Lim}_{k \to \infty} h^{-m_k}(y),$$

which completes the proof.

THEOREM 1. *Let X be a locally connected, connected and locally compact metric space, and h be an almost regular homeomorphism of X onto itself. If $R(h)$ is connected, then for any $y \in R(h)$,*

$$\operatorname*{Lim\,sup}_{n \to \pm\infty} h^n(y) \cap R(h) = \emptyset.$$

Proof. The proof follows immediately from Theorems 1 and 2 of **(4)** and Lemma 1 above.

Henceforth we assume in this paper that (X, d) is a locally connected, connected and locally compact space, h is an almost regular homeomorphism of X onto itself and $R(h)$ is connected. Note that under these conditions the above results are true.

2. For the purposes of this article let U be an open set in X such that $\bar{U}$ is compact and bdry $U \cap I(h) = \emptyset$. Let $\{m_i\}$ be any sequence of integers, and

$$F = \liminf_{i\to\infty} h^{m_i}[U] \qquad (m_i, i = 1, 2, \ldots, \text{distinct}).$$

LEMMA 2. *Let $y \in F \cap R(h)$. Then $y \in h^{m_i}[U]$ for all except finitely many i.*

Proof. Suppose there is a subsequence $\{n_i\} \subset \{m_i\}$ such that $y \notin h^{n_i}[U]$. Let $\{\epsilon_k\}$ be a sequence of positive numbers converging to zero. Since $y \in R(h)$, for each integer k there exists a $\delta_k > 0$ such that for $d(x, y) < \delta_k$, $d(h^n(x), h^n(y)) < \epsilon_k$ for all integers n. Again since $y \in F$ and $\{n_i\}$ is a subsequence of $\{m_i\}$ there exists for each k an integer n_k in $\{n_i\}$ such that for $|n_i| \geqslant n_k$, $h^{n_i}[U] \cap U_k \neq \emptyset$ where U_k is the δ_k-neighbourhood of y. Let $y_k \in h^{n_k}(U) \cap U_k$ for $k = 1, 2, \ldots$. Since $d(y, y_k) < \delta_k$, $d(h^{-n_k}(y_k), h^{-n_k}(y)) < \epsilon_k$, $h^{-n_k}(y_k) \in U$, and $h^{-n_k}(y) \notin U$ $(k = 1, 2, \ldots)$. But, since $\epsilon_k \to 0$ as $k \to \infty$,

$$\operatorname*{Lim\,sup}_{k\to\infty} h^{-n_k}(y_k) = \operatorname*{Lim\,sup}_{k\to\infty} h^{-n_k}(y).$$

Clearly,

$$\operatorname*{Lim\,sup}_{k\to\infty} h^{-n_k}(y_k) \neq \emptyset$$

for $h^{-n_k}(y_k) \in \bar{U}$ and $\bar{U}$ is compact; hence

$$\operatorname*{Lim\,sup}_{k\to\infty} h^{-n_k}(y) \neq \emptyset$$

and is contained in the boundary of U. This contradicts Theorem 1 above and completes the proof.

LEMMA 3. *If $y \in h^{m_i}[U] \cap R(h)$ for all values of i, then there exists an open set V in X containing y such that $V \subset h^{m_i}[U]$ for all but a finite number of values of i.*

Proof. Suppose the lemma is false; that is for any open set V containing y there exist infinitely many values of i for which $V - h^{m_i}[U] \neq \emptyset$. Let $\{\epsilon_k\}$ be a sequence of real positive numbers converging to zero. Since $y \in R(h)$, for every integer k there exists a $\delta_k > 0$ such that for $d(x, y) < \delta_k$, $d(h^n(y), h^n(x)) < \epsilon_k$ for all integers n. But for each integer k $(k = 1, 2, \ldots)$ there exists an n_k in $\{m_i\}$ such that $U_k - h^{n_k}[U] \neq \emptyset$ and $|n_k| < |n_{k+1}|$, where U_k is the δ_k-neighbourhood of y in X. Let y_k be any point in $U_k - h^{n_k}[U]$. Then $d(y, y_k) < \delta_k$ implies that $d(h^{-n_k}(y), h^{-n_k}(y_k)) < \epsilon_k$ where $h^{-n_k}(y) \in U$ and $h^{-n_k}(y_k) \notin U$. Since $\epsilon_k \to 0$ as $k \to \infty$.

$$\operatorname*{Lim\,sup}_{k\to\infty} h^{-n_k}(y_k) = \operatorname*{Lim\,sup}_{k\to\infty} h^{-n_k}(y) \subset \text{bdry } U \subset R(h).$$

This, as in Lemma 2, leads to a contradiction and completes the proof.

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From Lemmas 2 and 3, we have immediately

THEOREM 2. *If $y \in F \cap R(h)$, then there exists an open set V in X containing y such that $V \subset h^{m_i}[U]$ for all but a finite number of values of i,* i.e.

$$V \subset F = \operatorname*{Lim\,inf}_{i\to\infty} h^{m_i}[U].$$

THEOREM 3. *If $F \cap R(h) \neq \emptyset$, then $R(h) \subset F$.*

Proof. From Theorem 2 it follows that $F \cap R(h)$ is open in $R(h)$. Since F is closed in X, $F \cap R(h)$ is closed in $R(h)$. Since $R(h)$ is connected and $F \cap R(h) \neq \emptyset$, the result follows.

3. THEOREM 4. *If $p \in I(h)$, then for any open set U containing p there exists a sequence of integers $\{m_i\}$ such that*

$$\operatorname*{Lim\,inf}_{i\to\infty} h^{m_i}[U] = X.$$

Proof. Since Lemma 1 is true, there exists a point $y \in R(h)$ and a sequence of integers $\{m_i\}$ such that

$$\operatorname*{Lim}_{i\to\infty} h^{-m_i}(y) = p.$$

Let V be an open set such that $p \in V \subset U$, $\bar{V}$ is compact, and

$$\text{bdry } V \cap I(h) = \emptyset.$$

Now $h^{-m_i}(y) \in V$ for all but a finite number of values of i; hence $y \in h^{m_i}[V]$ for all but a finite number of values of i, that is,

$$y \in \operatorname*{Lim\,inf}_{i\to\infty} h^{m_i}[V].$$

Since

$$R(h) \cap \operatorname*{Lim\,inf}_{i\to\infty} h^{m_i}[V] \neq \emptyset,$$

it follows from Theorem 3 that

$$R(h) \subset \operatorname*{Lim\,inf}_{i\to\infty} h^{m_i}[V].$$

But since

$$\operatorname*{Lim\,inf}_{i\to\infty} h^{m_i}[V]$$

is closed in X, it contains $\overline{R(h)}$. Finally, since $R(h)$ is dense in X and $V \subset U$, the theorem follows.

LEMMA 4. *Let $p \in I(h)$ and V be any open set containing p. Then there exists a sequence of integers $\{m_i\}$ such that given any $y \in R(h)$, $h^{m_i}(y) \in V$ for $i \geqslant j$ for some positive integer j.*

Proof. Let U be an open set containing p such that $\bar{U}$ is compact,

$$\text{bdry } U \cap I(h) = \emptyset,$$

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and $U \subset V$. From Theorem 4 there exists a sequence of integers $\{m_i\}$ such that

$$\operatorname*{Lim\,inf}_{i\to\infty} h^{-m_i}[U] = X.$$

If $y \in R(h)$, then

$$y \in R(h) \cap \operatorname*{Lim\,inf}_{i\to\infty} h^{-m_i}[U].$$

Hence, from Lemma 2, $y \in h^{-m_i}[U]$ for all $i \geqslant j$ for some integer j. Thus $h^{m_i}(y) \in U \subset V$ for $i \geqslant j$. This completes the proof.

THEOREM 5. *If $p \in I(h)$, then $h(p) = p$.*

Proof. Suppose $h(p) \neq p$. Then there exists an open set U containing p such that $U \cap h[U] = \emptyset$. From Lemma 4 there exists a sequence of integers $\{m_i\}$ such that for any $y \in R(h)$, $h^{m_i}(y) \in U$ for $i \geqslant j$ for some integer j.

Consider y and $y_1 = h^{-1}(y)$ in $R(h)$. Then there exists an integer j such that for $i \geqslant j$, $h^{m_i}(y)$ and $h^{m_i}(y_1)$ are both in U. But $h^{m_i}(y_1) = h^{-1}(h^{m_i}(y))$ gives $h^{m_i}(y) \in h[U]$ for $i \geqslant j$. Hence $U \cap h[U] \neq \emptyset$. This contradiction completes the proof.

Proof of the Main Theorem. Suppose $I(h)$ consists of more than two points. We shall establish a contradiction.

It is not difficult to see that every point of $I(h)$ is a non-cut point of X since $R(h)$ is connected and dense in X. Hence for any $p \in I(h)$ there exists an arbitrarily small open set in X containing p such that its complement is connected (**5**, (4.15), p. 50). Let V be an open set containing some point $p \in I(h)$ such that $X - V$ is connected and contains at least two points, say p_1 and p_2, of $I(h)$. Since every point of $I(h)$ is fixed under h (Theorem 5) and $h[I(h)] = I(h)$, p_1 and p_2 do not belong to $h^n[V]$ for any integer n. Note also that for any integer n, $X - h^n[V]$ is a connected set, and also for any two integers m, n, $(X - h^n[V]) \cap (X - h^m[V]) \neq \emptyset$.

Let U be an open set containing p such that $p \in U \subset V$, $\bar{U}$ is compact, and bdry $U \cap I(h) = \emptyset$. Let n_i denote the sequence of integers such that

$$\operatorname*{Lim\,inf}_{i\to\infty} h^{n_i}[U] = X$$

(see Theorem 4).

Consider

$$B_j = X - \bigcap_{i=j}^{\infty} h^{n_i}[V] = \bigcup_{i=j}^{\infty} h^{n_i}[X - V].$$

Then B_j is a connected set, and $B_j \supset B_{j+1}$ $(j = 1, 2, \ldots)$. Set

$$B = \bigcap_{j=1}^{\infty} B_j = X - \bigcup_{j=1}^{\infty} \bigcap_{i=j} h^{n_i}[V].$$

Then B contains at least two points, p_1 and p_2.

Let W be an open set containing p_1 but not p_2, $\bar{W}$ be compact, and bdry $W \cap I(h) = \emptyset$. Since $B_j \supset B$, and B_j is connected, $B_j \cap$ bdry $W \neq \emptyset$.

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Let $y_i \in B_j \cap \text{bdry } W$ $(j = 1, 2, \ldots)$. Since $y_j \in B_j$,

$$y_j \notin \bigcap_{i=j}^{\infty} h^{n_i}[V], \qquad \text{that is } y_j \notin \bigcap_{i=j}^{\infty} h^{n_i}[U]$$

since $U \subset V$. Hence there exists, for each j, an $i_j \geqslant j$ such that $y_j \notin h^{n_{i_j}}[U]$. The sequence of points $\{y_j\}$ contained in the boundary of W must have at least one limit point y in it. Let us suppose for convenience that it converges to y.

Let Z be an open set containing y. Since

$$\lim_{j\to\infty} y_j = y,$$

there exists an integer k such that, for $j \geqslant k$, $y_j \in Z$. But, since $y_j \notin h^{n_{i_j}}[U]$, there exist infinitely many values of i for which $h^{n_i}[U]$ does not contain Z. Since Z is arbitrary, this contradicts Theorem 2. Hence for any arbitrarily small open set containing p its complement contains at most one point of $I(h)$. Thus $I(h)$ consists of at most two points. That these points are fixed is a consequence of Theorem 5 above. This completes the proof.

An immediate consequence of the Main Theorem and Theorems 6 and 7 of **(3)** is the following:

THEOREM 6. *If X is a closed connected topological n-manifold and there exists an almost regular homeomorphism h of X onto itself such that $R(h)$ is connected, then X is an n-sphere.*

Remark. If, in Theorem 6, $n \geqslant 2$, then the condition that $R(h)$ be connected is redundant.

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