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ON FUNCTIONAL CESÀRO AND HÖLDER METHODS OF SUMMABILITY

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1. Cesàro and Hölder-type methods of summability. Suppose that $f(x)$ is integrable L in every finite interval $[0, X]$, and that $\delta > 0$. Define

$$f_\delta(x) = \{\Gamma(\delta)\}^{-1}\int_0^x (x-t)^{\delta-1}f(t)dt, \quad \text{and} \quad g(x) = e^x f(x).$$

Definition. If $\Gamma(\delta+1)x^{-\delta}f_\delta(x) \to \sigma$ as $x \to \infty$, then we say that the (C, δ) limit of $f(x)$ is σ, and write $f(x) \to \sigma(C, \delta)$.

Definition. If $e^{-x}g_\delta(x) \to \sigma$ as $x \to \infty$, then we say that the $(\hat{C}, \delta)$ limit of $f(x)$ is σ, and write $f(x) \to \sigma(\hat{C}, \delta)$.

Note that (C, δ) is the standard Cesàro method of summability, that

$$\begin{aligned} e^{-x}g_\delta(x) &= \{\Gamma(\delta)\}^{-1}e^{-x}\int_0^x (x-t)^{\delta-1}e^t f(t)dt \\ &= \{\Gamma(\delta)\}^{-1}X^{-1}\int_1^X (\log X/T)^{\delta-1}f(\log T)dT, \end{aligned}$$

and that this final integral is the functional Hölder transform of $f(\log T)$.

It is well known [**4**] that $f(x) \to \sigma(C, \delta)$ *if and only if* $f(e^x) \to \sigma(\hat{C}, \delta)$. Our primary objective is to prove that *if* $f(x) \to \sigma(\hat{C}, \delta)$ *then* $f(x) \to \sigma(C, \delta)$, *and that there is a function whose* (C, δ) *limit exists but whose* $(\hat{C}, \delta)$ *limit does not exist.*

We need two lemmas. The first is due to M. Riesz [**3**].

LEMMA 1. *For* $x > t > 0$ *and* $0 < \delta < 1$,

$$\Gamma(1-\delta)\int_0^t (x-v)^{\delta-1}f(v)dv = \delta\int_0^t f_\delta(v)dv\int_t^x (x-w)^{\delta-1}(w-v)^{-\delta-1}dw.$$

LEMMA 2. *If* $0 < \delta \leqq 1$ *and* $e^{-x}g_\delta(x) \to \sigma$ *as* $x \to \infty$, *then*

$$\delta x^{-\delta}\int_0^x (x-t)^{\delta-1}e^{-t}g(t)dt \to \sigma \quad \text{as } x \to \infty.$$

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Proof. Suppose first that $0 < \delta < 1$. Using the result of Lemma 1, we have

$$\int_0^x (x-t)^{\delta-1}e^{-t}g(t)dt - \Gamma(\delta)e^{-x}g_\delta(x)$$

$$= \int_0^x e^{-t}dt \int_0^t (x-u)^{\delta-1}g(u)du$$

$$= \frac{\delta}{\Gamma(1-\delta)} \int_0^x e^{-t}dt \int_0^t g_\delta(v)dv \int_t^x (x-w)^{\delta-1}(w-v)^{-\delta-1}dw$$

$$= \frac{\delta}{\Gamma(1-\delta)} \int_0^x e^{-v}g_\delta(v)dv \int_v^x (x-w)^{\delta-1}(w-v)^{-\delta-1}dw \int_v^w e^{-(t-v)}dt$$

$$= \int_0^x J(x-v)e^{-v}g_\delta(v)dv, \text{ where}$$

$$J(y) = \frac{\delta}{\Gamma(1-\delta)} \int_0^y (y-u)^{\delta-1}u^{-\delta-1}(1-e^{-u})du.$$

It now suffices to show that

$$\delta x^{-\delta} \int_0^x J(x-v)\phi(v)dv \to \sigma$$

whenever $\phi(x)$ is integrable L in every finite interval $[0, X]$ and tends to σ as $x \to \infty$. This is true since (see, for example [**2**, Theorem 6])

$$\delta x^{-\delta} \int_0^x J(x-v)dv = \frac{\delta}{\Gamma(1-\delta)} x^{-\delta} \int_0^x (x-u)^{\delta}u^{-\delta-1}(1-e^{-u})du$$

$$\to \frac{\delta}{\Gamma(1-\delta)} \int_0^\infty u^{-\delta-1}(1-e^{-u})du = 1 \quad \text{as } x \to \infty,$$

and since, for each fixed $y > 0$,

$$\delta x^{-\delta} \int_0^y J(x-v)dv \to 0 \quad \text{as } x \to \infty.$$

When $\delta = 1$, we have

$$x^{-1} \int_0^x e^{-t}g(t)dt - x^{-1}e^{-x}g_1(x) = x^{-1} \int_0^x e^{-t}g_1(t)dt,$$

and the desired result now follows from the regularity of the $(C, 1)$ method.

We now prove two theorems which show the relation between the methods $(\hat{C}, \alpha)$ and (C, α).

THEOREM 1. *For $\alpha > 0$, if $f(x) \to \sigma(\hat{C}, \alpha)$ then $f(x) \to \sigma(C, \alpha)$.*

Proof. First suppose that $0 < \alpha \leqq 1$. Since

$$\Gamma(\alpha)f_\alpha(x) = \int_0^x (x-t)^{\alpha-1}e^{-t}g(t)dt,$$

the result follows from Lemma 2.

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Now suppose that $\alpha > 1$. Set $\alpha = k + \delta$ where $0 < \delta \leqq 1$ and $k = 1, 2, \ldots$. Integration by parts k times yields

$$\begin{aligned}\Gamma(\alpha)f_\alpha(x) &= (-1)^k \int_0^x g_k(t)\left(\frac{d}{dt}\right)^k [e^{-t}(x-t)^{\alpha-1}]dt \\ &= a_0 \int_0^x g_k(t)e^{-t}(x-t)^{\delta-1}dt + \sum_{r=1}^{k} a_r \int_0^x g_k(t)e^{-t}(x-t)^{\delta-1+r}dt \\ &= a_0 \int_0^x g_k(t)e^{-t}(x-t)^{\delta-1}dt + \sum_{r=0}^{k-1} b_r \int_0^x g_{k+1}(t)e^{-t}(x-t)^{\delta-1+r}dt \\ &\qquad + \int_0^x g_{k+1}(t)e^{-t}(x-t)^{\alpha-1}dt,\end{aligned}$$

where the a_r and b_r are constants.

By assumption, $e^{-x}g_\alpha(x) \to \sigma$ as $x \to \infty$, and so since $k + 1 \geqq \alpha$, it is easy to show that $e^{-x}g_{k+1}(x) \to \sigma$ as $x \to \infty$. Thus using Lemma 1 for the term involving a_0 and the regularity of the Cesàro methods for the other terms, it follows that $\Gamma(\alpha + 1)x^{-\alpha}f_\alpha(x) \to \sigma$ as $x \to \infty$. This completes the proof of Theorem 1.

THEOREM 2. *For $\alpha > 0$, $e^{ix} \to 0(C, \alpha)$, but the $(\hat{C}, \alpha)$ limit of e^{ix} does not exist.*

Proof. By the Riemann-Lebesgue theorem,

$$x^{-\alpha}\int_0^x (x-t)^{\alpha-1}e^{it}dt = \int_0^1 (1-u)^{\alpha-1}e^{iux}du \to 0 \quad \text{as } x \to \infty,$$

so that $e^{ix} \to 0(C, \alpha)$. On the other hand,

$$\begin{aligned} e^{-x}\int_0^x (x-t)^{\alpha-1}e^t e^{it}dt &= e^{ix}\int_0^x t^{\alpha-1}e^{-t(1+i)}dt \\ = e^{ix}\int_0^\infty t^{\alpha-1}e^{-t(1+i)}dt + o(1) &= \frac{e^{ix}\Gamma(\alpha)}{(1+i)^\alpha} + o(1),\end{aligned}$$

which does not tend to a limit as $x \to \infty$; that is, the $(\hat{C}, \alpha)$ limit of e^{ix} does not exist.

2. Application to the Borel-type methods of summability.

Suppose that $\lambda > 0$, that μ is real and that N is a non-negative integer greater than $-\mu/\lambda$. Let ρ, s_n $(n = 0, 1, \ldots)$ be complex numbers. Define

$$S_{\lambda,\mu}(x) = \lambda e^{-x} \sum_{n=N}^{\infty} \frac{s_n x^{\lambda n+\mu-1}}{\Gamma(\lambda n + \mu)}.$$

Definition [1]. *If $S_{\lambda,\mu}(x) \to \rho$* as $x \to \infty$, then we say that the (B, λ, μ) limit of the sequence $\{s_n\}$ is ρ, and writes $s_n \to \rho(B, \lambda, \mu)$.

The following two theorems are known.

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THEOREM 3 [**5**]. *The* $(\hat{C}, \alpha)$ (B, λ, μ) *transform of the sequence* $\{s_n\}$ *is equal to the* $(B, \lambda, \mu + \delta)$ *transform of the sequence* $\{s_n\}$; *that is*

$$e^{-x}\int_0^x (x-t)^{\delta-1}e^t S_{\lambda,\mu}(t)dt = S_{\lambda,\mu+\delta}(x).$$

From this it follows that $s_n \to \rho(\hat{C}, \delta)$ (B, λ, μ) if and only if $s_n \to \rho(B, \lambda, \mu + \delta)$.

THEOREM 4. *If* $s_n \to \rho(B, \lambda, \mu)$ *then* $s_n \to \rho(C, \delta)$ (B, λ, μ).

This is trivial since (C, δ) is a regular method. See also [**6**].

The following theorem, which follows immediately from Theorem 3 and the results of §1, extends Theorem 4.

THEOREM 5. (i) *If* $s_n \to \rho(B, \lambda, \mu + \delta)$ *then* $s_n \to \rho(C, \delta)$ (B, λ, μ);

(ii) *There is a sequence whose* (C, δ) (B, λ, μ) *limit exists but whose* $(B, \lambda, \mu + \delta)$ *limit does not exist.*

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