Glasgow Math. J. **51** (2009) 651–657. © 2009 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089509990061. Printed in the United Kingdom

THE CENTRE OF THE MAXIMAL p-SUBGROUP OF $\mathcal{U}(\mathbb{F}_{p^k} D_{2p})$

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(Received 2 May 2008; accepted 29 May 2009)

Abstract. The centre of the maximal p-subgroup of $\mathcal{U}(\mathbb{F}_{p^k} D_{2p})$ is described as an elementary abelian p-group, where p is a prime.

2000 *Mathematics Subject Classification.* 20C05 16S34 20H30 15A33.

1. Introduction. The set of all the invertible elements of a ring S form a group called the unit group of S, denoted by $\mathcal{U}(S)$. Let $\mathbb{F}_{p^k} D_{2p^m}$ be the group algebra of D_{2p^m} over $\mathbb{F}_{p^k}$, where $\mathbb{F}_{p^k}$ is the Galois field of p^k-elements, D_{2p^m} is the dihedral group of order $2p^m$, p is a prime and $m \in \mathbb{N}$. For further details on group algebras see [**8**].

In [**5**], the order of $\mathcal{U}(\mathbb{F}_{p^k} D_{2p^m})$ is established to be $p^{2k(p^m-1)}(p^k-1)^2$, where p is an odd prime and $m \in \mathbb{N}_0$. The structure of $\mathcal{U}(\mathbb{F}_{3^k} D_6)$ is described as split extensions of cyclic groups in [**3**]. Additionally in [**7**], it is shown that V_1 and $V_1/Z(V_1)$ are elementary abelian 3-groups, where $V_1 = 1 + J(\mathbb{F}_{3^k} D_6)$, $J(\mathbb{F}_{3^k} D_6)$ is the Jacobson radical of $\mathbb{F}_{3^k} D_6$ and $Z(V_1)$ is the centre of V_1.

Let C_n be the cyclic group of order n. Let $M_n(R)$ be the ring of $n \times n$ matrices over a ring R. Using an established isomorphism between RG and a subring of $M_n(R)$ [**6**] and other techniques, we establish the centre of the Maximal p-subgroup of $\mathcal{U}(\mathbb{F}_{p^k} D_{2p})$ to be ${C_p}^{k(\frac{p+1}{2})}$.

The techniques described in this paper can also be used to study the structure of the σ-unitary groups $V_\sigma(KG)$, where σ are involutions of the group ring KG. Some of these non-classical involutions were introduced and studied in [**1**] and [**2**].

2. Background.

Definition 2.1. A circulant matrix over a ring R is a square $n \times n$ matrix, which takes the form

$$\operatorname{circ}(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_n & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \ldots & a_1 \end{pmatrix},$$

where $a_i \in R$.

For further details on circulant matrices see Davis [**4**].

Let $\{g_1, g_2, \ldots, g_n\}$ be a fixed listing of the elements of a group G. Then the matrix

$$\begin{pmatrix} g_1^{-1}g_1 & g_1^{-1}g_2 & g_1^{-1}g_3 & \ldots & g_1^{-1}g_n \\ g_2^{-1}g_1 & g_2^{-1}g_2 & g_2^{-1}g_3 & \ldots & g_2^{-1}g_n \\ g_3^{-1}g_1 & g_3^{-1}g_2 & g_3^{-1}g_3 & \ldots & g_3^{-1}g_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n^{-1}g_1 & g_n^{-1}g_2 & g_n^{-1}g_3 & \ldots & g_n^{-1}g_n \end{pmatrix}$$

is called the matrix of G (relative to this listing) and is denoted by $M(G)$. Let $w = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$, where R is a ring. Then the matrix

$$\begin{pmatrix} \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \ldots & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \alpha_{g_2^{-1}g_3} & \ldots & \alpha_{g_2^{-1}g_n} \\ \alpha_{g_3^{-1}g_1} & \alpha_{g_3^{-1}g_2} & \alpha_{g_3^{-1}g_3} & \ldots & \alpha_{g_3^{-1}g_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{g_n^{-1}g_1} & \alpha_{g_n^{-1}g_2} & \alpha_{g_n^{-1}g_3} & \ldots & \alpha_{g_n^{-1}g_n} \end{pmatrix}$$

is called the RG-matrix of w and is denoted by $M(RG, w)$.

The following theorem can be found in [**6**].

THEOREM 2.2. *Given a listing of the elements of a group G of order n there is a bijective ring homomorphism between RG and the $n \times n$ G-matrices over R. This bijective ring homomorphism is given by $\sigma : w \mapsto M(RG, w)$.*

EXAMPLE 2.3. Let $D_{2n} = \langle x, y \mid x^n = 1, y^2 = 1, yx = x^{-1}y \rangle$ and $\kappa = \sum_{i=0}^{n-1} a_i x^i + \sum_{j=0}^{n-1} b_j x^j y \in \mathbb{F}_{p^k} D_{2n}$, where $a_i, b_j \in \mathbb{F}_{p^k}$ and p is a prime; then

$$\sigma(\kappa) = \begin{pmatrix} A & B \\ B^T & A^T \end{pmatrix},$$

where $A = \text{circ}(a_0, a_1, \ldots, a_{n-1})$ and $B = \text{circ}(b_0, b_1, \ldots, b_{n-1})$.

The next two results appear in [**5**]

PROPOSITION 2.4. *Let $A = circ\,(a_1, a_2, \ldots, a_{p^m})$, where $a_i \in \mathbb{F}_{p^k}$, p is a prime and $m \in \mathbb{N}_0$. Then*

$$A^{p^m} = \sum_{i=1}^{p^m} a_{i\,p^m}^{p^m}.$$

THEOREM 2.5. $|\mathcal{U}(\mathbb{F}_{p^k} D_{2p^m})| = p^{2k(p^m-1)}(p^k - 1)^2$, *where p is an odd prime and $m \in \mathbb{N}_0$.*

3. The maximal p-subgroup of $\mathcal{U}(\mathbb{F}_{p^k} D_{2p})$. Define the ring homomorphism $\theta : \mathbb{F}_{p^k} D_{2p} \longrightarrow \mathbb{F}_{p^k} C_2$ by $\sum_{i=0}^{p-1} a_i x^i + \sum_{j=0}^{p-1} b_j x^j y \longmapsto \sum_{i=0}^{p-1} a_i + \sum_{j=0}^{p-1} b_j \cdot \overline{y}$. Now define the group epimorphism $\theta' : \mathcal{U}(\mathbb{F}_{p^k} D_{2p}) \longrightarrow \mathcal{U}(\mathbb{F}_{p^k} C_2)$, where θ' is θ restricted to $\mathcal{U}(\mathbb{F}_{p^k} D_{2p})$. Let $\psi : \mathcal{U}(\mathbb{F}_{p^k} C_2) \longrightarrow \mathcal{U}(\mathbb{F}_{p^k} D_{2p})$ be the group homomorphism defined by $a + b \cdot \overline{y} \mapsto a + b \cdot y$. Then $\theta' \circ \psi(a + b \cdot \overline{y}) = \theta\,(a + b \cdot y) = a + b \cdot \overline{y}$. Therefore $\mathcal{U}(\mathbb{F}_{p^k} D_{2p})$ is a split extension of $\mathcal{U}(\mathbb{F}_{p^k} C_2)$ by $\ker(\theta')$. Thus $\mathcal{U}(\mathbb{F}_{p^k} D_{2p}) \cong H \rtimes \mathcal{U}(\mathbb{F}_{p^k} C_2)$, where $H = \ker(\theta')$; $|H| = p^{2k(p-1)}$ by Theorem 2.5. Clearly H is the maximal p-subgroup of $\mathcal{U}(\mathbb{F}_{p^k} D_{2p})$.

PROPOSITION 3.1. $|C_H(x)| = p^{kp}$.

Proof. $C_H(x) = \{h \in H \mid hx = xh\}$. Let $h = \sum_{i=0}^{p-1} a_i x^i + \sum_{j=0}^{p-1} b_j x^j y \in H$, where $\sum_{i=0}^{p-1} a_i = 1$, $\sum_{j=0}^{p-1} b_j = 0$ and $a_i, b_j \in \mathbb{F}_{p^k}$. Then

$$\begin{aligned} hx - xh &= \left(\sum_{i=0}^{p-1} a_i x^i + \sum_{j=0}^{p-1} b_j x^j y\right) x - x\left(\sum_{i=0}^{p-1} a_i x^i + \sum_{j=0}^{p-1} b_j x^j y\right), \\ &= \sum_{i=0}^{p-1} a_i x^{i+1} + \sum_{j=0}^{p-1} b_j x^{j-1} y - \sum_{i=0}^{p-1} a_i x^{i+1} - \sum_{j=0}^{p-1} b_j x^{j+1} y, \\ &= \sum_{j=0}^{p-1} b_j x^{j-1} y - \sum_{j=0}^{p-1} b_j x^{j+1} y, \\ &= 0 \iff \sum_{j=0}^{p-1} b_j x^{j-1} y = \sum_{j=0}^{p-1} b_j x^{j+1} y; \end{aligned}$$

$\sum_{j=0}^{p-1} b_j x^{j-1} y = \sum_{j=0}^{p-1} b_j x^{j+1} y \iff b_i = b_{i-2}\, \forall\, i \iff b_0 = b_2 = b_4 = \cdots = b_{p-1} = b_1 = b_3 = \cdots = b_{p-2} = b_0$. If $\kappa = \sum_{l=0}^{p-1} c_l x^l + d \sum_{m=0}^{p-1} x^m y$, where $\sum_{l=0}^{p-1} c_l = 1$, then $\kappa x = x\kappa$. Thus every element of $C_H(x)$ is of the form $\sum_{i=0}^{p-1} a_i x^i + b \sum_{j=0}^{p-1} x^j y$, where $\sum_{i=0}^{p-1} a_i = 1$. Hence $|C_H(x)| = (p^k)^{p-1}(p^k) = p^{kp}$. □

PROPOSITION 3.2. *Let A be a $p \times p$ circulant matrix with row sums* 1. *Suppose that*

$$D(A - A^T) = 0$$

for every $p \times p$ circulant D with row sum 0. *Then $A = A^T$.*

Proof. Let $A - A^T = \operatorname{circ}(0, a_2, a_3, a_4, \ldots, a_{p-1}, a_p)$. If $D = \operatorname{circ}(1, -1, 0, \ldots, 0)$, then $D(A - A^T) = \operatorname{circ}(-a_p, a_2, a_3 - a_2, a_4 - a_3, \ldots, a_p - a_{p-1})$. If $D(A - A^T) = 0$, then $a_2 = a_3 = a_4 = \ldots = a_{p-1} = a_p = 0$. Therefore $A - A^T = 0$ and $A = A^T$. □

PROPOSITION 3.3. $|Z(H)| = p^{\frac{k(p+1)}{2}}$.

Proof. $Z(H)$ is contained in $C_H(x)$. Let $\alpha = \sum_{i=0}^{p-1} a_i x^i + b \sum_{j=0}^{p-1} x^j y \in C_H(x)$ and $\beta = \sum_{l=0}^{p-1} c_l x^l + \sum_{m=0}^{p-1} d_m x^m y \in H$, where $a_i, b, c_l, d_m \in \mathbb{F}_{p^k}$. Therefore $Z(H) = \{\alpha \in C_H(x) \mid \alpha h = h\alpha\}$. Then

$$\begin{aligned} \sigma(\alpha)\sigma(\beta) - \sigma(\beta)\sigma(\alpha) &= \begin{pmatrix} A & B \\ B & A^T \end{pmatrix}\begin{pmatrix} C & D \\ D^T & C^T \end{pmatrix} - \begin{pmatrix} C & D \\ D^T & C^T \end{pmatrix}\begin{pmatrix} A & B \\ B & A^T \end{pmatrix} \\ &= \begin{pmatrix} AC+BD^T & AD+BC^T \\ BC+A^TD^T & BD+A^TC^T \end{pmatrix} - \begin{pmatrix} AC+BD & BC+DA^T \\ AD^T+BC^T & BD^T+A^TC^T \end{pmatrix}, \end{aligned}$$

where $A = \operatorname{circ}(a_0, a_1, \ldots, a_{p-1})$, $B = \operatorname{circ}(b, b, \ldots, b)$, $C = \operatorname{circ}(c_0, c_1, \ldots, c_{p-1})$, $D = \operatorname{circ}(d_0, d_1, \ldots, d_{p-1})$. Now

- $BC = \operatorname{circ}(b, b, \ldots, b)\operatorname{circ}(c_0, c_1, \ldots, c_{p-1}) = B = BC^T$ since $\sum_{l=0}^{p-1} c_l = 1$;
- $BD = \operatorname{circ}(b, b, \ldots, b)\operatorname{circ}(d_0, d_1, \ldots, d_{p-1}) = 0_p = BD^T$ since $\sum_{m=0}^{p-1} d_m = 0$.

Therefore $\sigma(\alpha)\sigma(\beta) - \sigma(\beta)\sigma(\alpha) = \left(\begin{smallmatrix} 0 & D(A - A^T) \\ D^T(A^T - A) & 0 \end{smallmatrix}\right)$. Thus $\sigma(\alpha)\sigma(\beta) - \sigma(\beta)\sigma(\alpha) = 0 \iff A = A^T \iff a_i = a_{-i} \bmod p \; \forall\, i$ by Proposition 3.2.

Thus every element of $Z(H)$ is of the form $\gamma_0 + \gamma_1 x^1 + \gamma_2 x^2 + \ldots + \gamma_{\frac{p-1}{2}} x^{\frac{p-1}{2}} + \gamma_{\frac{p-1}{2}} x^{\frac{p+1}{2}} + \ldots + \gamma_2 x^{p-2} + \gamma_1 x^{p-1} + \delta \sum_{j=0}^{p-1} x^j y$ where $\gamma_i, \delta \in \mathbb{F}_{p^k}$. Thus $|Z(H)| = (p^k)^{(\frac{p-1}{2})}.(p^k)^1 = p^{\frac{k(p+1)}{2}}$. □

PROPOSITION 3.4. *Let* $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ *be a* 2×2 *matrix over* $\mathbb{R}$. *Then*

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^n = \begin{pmatrix} \sum_{x=1}^{\frac{n+1}{2}} \binom{n}{2x-2} a^{n-(2x-2)} b^{2x-2} & \sum_{x=1}^{\frac{n+1}{2}} \binom{n}{2x-1} a^{n-(2x-1)} b^{2x-1} \\ \sum_{x=1}^{\frac{n+1}{2}} \binom{n}{2x-1} a^{n-(2x-1)} b^{2x-1} & \sum_{x=1}^{\frac{n+1}{2}} \binom{n}{2x-2} a^{n-(2x-2)} b^{2x-2} \end{pmatrix}$$

when n is odd.

Proof. Clearly $\begin{pmatrix} a & b \\ b & a \end{pmatrix}^1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

Assume for $n = k$ (k-odd) that

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^k = \begin{pmatrix} \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-2} a^{k-(2x-2)} b^{2x-2} & \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-1} a^{k-(2x-1)} b^{2x-1} \\ \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-1} a^{k-(2x-1)} b^{2x-1} & \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-2} a^{k-(2x-2)} b^{2x-2} \end{pmatrix}.$$

We must show that

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^{k+2} = \begin{pmatrix} \sum_{x=1}^{\frac{k+3}{2}} \binom{k+2}{2x-2} a^{(k+2)-(2x-2)} b^{2x-2} & \sum_{x=1}^{\frac{k+3}{2}} \binom{k+2}{2x-1} a^{(k+2)-(2x-1)} b^{2x-1} \\ \sum_{x=1}^{\frac{k+3}{2}} \binom{k+2}{2x-1} a^{(k+2)-(2x-1)} b^{2x-1} & \sum_{x=1}^{\frac{k+3}{2}} \binom{k+2}{2x-2} a^{(k+2)-(2x-2)} b^{2x-2} \end{pmatrix}.$$

Now

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^{k+2} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}^{k} \begin{pmatrix} a & b \\ b & a \end{pmatrix}^{2} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}^{k} \begin{pmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ \delta & \gamma \end{pmatrix}.$$

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First let's deal with γ and then with δ.

$$(1)\ \gamma = \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-2} a^{k-(2x-2)} b^{2x-2}(a^2+b^2) + \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-1} a^{k-(2x-1)} b^{2x-1}(2ab),$$

$$= \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-2} a^{(k+2)-(2x-2)} b^{2x-2} + \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-2} a^{k-(2x-2)} b^{2x}$$

$$+ 2\sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-1} a^{(k+1)-(2x-1)} b^{2x},$$

$$= \binom{k}{0} a^{k+2} b^0 + \binom{k}{2} a^k b^2 + \binom{k}{4} a^{k-2} b^4 + \cdots + \binom{k}{k-1} a^3 b^{k-1}$$

$$+ \binom{k}{0} a^k b^2 + \binom{k}{2} a^{k-2} b^4 + \cdots + \binom{k}{k-3} a^3 b^{k-1} + \binom{k}{k-1} a^1 b^{k+1}$$

$$+ 2\left[\binom{k}{1} a^k b^2 + \binom{k}{3} a^{k-2} b^4 + \cdots + \binom{k}{k-2} a^3 b^{k-1} + \binom{k}{k} a^1 b^{k+1}\right],$$

$$= a^{k+2} + \sum_{x=1}^{\frac{k-1}{2}} \left[\binom{k}{2x} + \binom{k}{2x-2} + 2\binom{k}{2x-1}\right] a^{k-(2x-2)} b^{2x} + (k+2)ab^{k+1},$$

$$= a^{k+2} + \sum_{x=1}^{\frac{k-1}{2}} \binom{k+2}{2x} a^{k-(2x-2)} b^{2x} + (k+2)b^{k+2},$$

$$= \sum_{x=1}^{\frac{k+3}{2}} \binom{k+2}{2x-2} a^{(k+2)-(2x-2)} b^{2x-2}.$$

$$(2)\ \delta = \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-2} a^{k-(2x-2)} b^{2x-2}(2ab) + \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-1} a^{k-(2x-1)} b^{2x-1}(a^2+b^2),$$

$$= 2\sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-2} a^{(k+1)-(2x-1)} b^{2x-1} + \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-1} a^{(k+2)-(2x-1)} b^{2x-1}$$

$$+ \sum_{x=1}^{\frac{k+1}{2}} \binom{k}{2x-1} a^{k-(2x-1)} b^{2x+1},$$

$$= 2\left[\binom{k}{0} a^{k+1} b^1 + \binom{k}{2} a^{k-1} b^3 + \cdots + \binom{k}{k-3} a^4 b^{k-2} + \binom{k}{k-1} a^2 b^k\right],$$

$$+ \binom{k}{1} a^{k+1} b^1 + \binom{k}{3} a^{k-1} b^3 + \cdots + \binom{k}{k-2} a^4 b^{k-2} + \binom{k}{k} a^2 b^k$$

$$+ \binom{k}{1} a^{k-1} b^3 + \cdots + \binom{k}{k-4} a^4 b^{k-2} + \binom{k}{k-2} a^2 b^k + \binom{k}{k} a^0 b^{k+2},$$

$$= (k+2)a^{k+1} b^1 + \sum_{x=1}^{\frac{k-1}{2}} \left[2\binom{k}{2x} + \binom{k}{2x+1} + \binom{k}{2x-1}\right] a^{k-(2x-1)} b^{2x+1} + a^0 b^{k+2},$$

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$$= (k+2)a^{k+1}b^1 + \sum_{x=1}^{\frac{k-1}{2}} \binom{k+2}{2x+1} a^{k-(2x-1)} b^{2x+1} + b^{k+2},$$

$$= \sum_{x=1}^{\frac{k+3}{2}} \binom{k+2}{2x-1} a^{(k+2)-(2x-1)} b^{2x-1}.$$

□

PROPOSITION 3.5. *Let A and B be $p \times p$ circulant matrices over $\mathbb{F}_{p^k}$ where p is a prime. Then*

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}^p = \begin{pmatrix} A^p & B^p \\ B^p & A^p \end{pmatrix}.$$

Proof. We can apply the previous lemma, since circulant matrices commute. Therefore

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}^p = \begin{pmatrix} \sum_{x=1}^{\frac{p+1}{2}} \binom{p}{2x-2} A^{p-(2x-2)} A^{2x-2} & \sum_{x=1}^{\frac{p+1}{2}} \binom{p}{2x-1} A^{p-(2x-1)} B^{2x-1} \\ \sum_{x=1}^{\frac{p+1}{2}} \binom{p}{2x-1} A^{p-(2x-1)} B^{2x-1} & \sum_{x=1}^{\frac{p+1}{2}} \binom{p}{2x-2} A^{p-(2x-2)} B^{2x-2} \end{pmatrix};$$

$$\sum_{x=1}^{\frac{p+1}{2}} \binom{p}{2x-2} A^{p-(2x-2)} B^{2x-2} = \binom{p}{0} A^p B^0 + \binom{p}{2} A^{p-2} B^2 + \cdots + \binom{p}{p-1} A^1 B^{p-1}$$
$$= A^p \bmod p;$$

$$\sum_{x=1}^{\frac{p+1}{2}} \binom{p}{2x-1} A^{p-(2x-2)} B^{2x-2} = \binom{p}{1} A^{p-1} B^1 + \binom{p}{3} A^{p-2} B^3 + \cdots + \binom{p}{p} A^0 B^p$$
$$= B^p \bmod p.$$

□

THEOREM 3.6. *$Z(H)$ has exponent p.*

Proof. Let $\alpha = a_0 + a_1 x + a_2 x^2 + \ldots + a_{\frac{p-1}{2}} x^{\frac{p-1}{2}} + a_{\frac{p-1}{2}} x^{\frac{p+1}{2}} + \ldots + a_2 x^{p-2} + a_1 x^{p-1} + b\sum_{j=0}^{p-1} x^j y \in Z(H)$, where $a_i, b \in \mathbb{F}_{p^k}$. Then $(\sigma(\alpha))^p = \begin{pmatrix} A & B \\ B & A \end{pmatrix}^p = \begin{pmatrix} A^p & B^p \\ B^p & A^p \end{pmatrix}$, where $A = \mathrm{circ}(a_0, a_1, a_2, \ldots, a_{\frac{p-1}{2}}, a_{\frac{p-1}{2}}, \ldots, a_2, a_1)$ and $B = \mathrm{circ}(b, b, \ldots, b)$. From proposition 2.4, $A^p = (a_0 + 2\sum_{i=1}^{\frac{p-1}{2}} a_i{}^p)^p = I$ and $B^p = (pb^p)^p = 0$ since $\alpha \in H$. □

Clearly $Z(H) \cong C_p{}^{k(\frac{p+1}{2})}$ by Proposition 3.3 and Theorem 3.6.

https://doi.org/10.1017/S0017089509990061 Published online by Cambridge University Press

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