ON A GENERALIZATION OF THE FIRST CURVATURE OF A CURVE IN A HYPERSURFACE OF A RIEMANNIAN SPACE

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1. Introduction. The unit tangent vector at a point of a curve in a hypersurface of a Riemannian space has two derived vectors along the curve, one with respect to the Riemannian space in which the hypersurface is imbedded and one with respect to the hypersurface itself. When the former vector is decomposed along the directions normal and tangent to the hypersurface, its tangential component, which is called the first curvature vector of the curve at the point in the hypersurface, is exactly the latter vector. In this respect, the first curvature at a point of a curve in a hypersurface, that is, the magnitude of the first curvature vector, is related to the unit normal vector of the hypersurface at the point. Since the normal direction to the hypersurface used in the above decomposition can be replaced by a general direction orthogonal to the curve, it is obvious that the concept of the first curvature of a curve in a hypersurface can be generalized. This note deals with such generalization and its consequences.

The notation of Eisenhart **(2)** will be used for the most part except that Γ^i_{jk} will be employed for Christoffel Symbols of the second kind.

2. Definition. Let V_n be a Riemannian space with positive definite first fundamental form $g_{ij}\,dx^i\,dx^j$ $(i, j = 1, \ldots, n)$ imbedded in a Riemannian space V_{n+1} with positive definite first fundamental form $a_{\alpha\beta}\,dy^\alpha\,dy^\beta$ $(\alpha, \beta = 1,\ldots, n+1)$. Let $C\colon x^i = x^i(s)$ be a curve in V_n, where s is its arc length. If q^α and p^i represent the derived vectors of the unit tangent vector t of C with respect to V_{n+1} and V_n respectively, we have

$$q^\alpha = y^\alpha{}_{,i}\,p^i + \left(\Omega_{ij}\,\frac{dx^i}{ds}\,\frac{dx^j}{ds}\right)\xi^\alpha \tag{2.1}$$

where ξ^α are the contravariant components of the unit vector normal to V_n and where Ω_{ij} the second fundamental quadratic tensor for V_n**(2**, p. 151**)**.

Let λ be unit vectors in V_{n+1}, which are not in V_n except possibly in its asymptotic directions and whose contravariant components λ^α at a point P in V_n are analytic functions of x^i and dx^i at P. The totality of these vectors λ associated with V_n is called a λ-*congruence*, which is a congruence of unit vectors if λ^α are functions of x^i only, or a congruence of hypercones of unit vectors if λ^α are functions of both x^i and dx^i, in which case we *assume* that λ is in V_n if and

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only if λ and its corresponding dx^i at x^i are coincident with an asymptotic direction in V_n. Each point of V_n is associated with one unit vector or one hypercone of unit vectors of the λ-congruence. We always assume such association between the λ-congruence and V_n in the following discussion. Expressing λ as a linear combination of independent vectors in V_{n+1}, we may write

$$\lambda^\alpha = y^\alpha{}_{,i}\, w^i + w\, \xi^\alpha \tag{2.2}$$

where w^i are the components of a contravariant vector in V_n and where w is a scalar. The case $w = 0$ which corresponds to λ in an asymptotic direction of V_n will be discussed at an appropriate place. Unless explicitly indicated w is always assumed to be different from zero. Since $a_{\alpha\beta}\,\lambda^\alpha\,\lambda^\beta = 1$, we have

$$g_{ij}\, w^i\, w^j + (w)^2 = 1. \tag{2.3}$$

Let N^α be a unit vector at a point P of the curve C in V_n, which satisfies the conditions: (1) it is linearly dependent on λ and the unit tangent vector t at P of C, (2) it is orthogonal to t. Hence $a_{\alpha\beta}\, N^\alpha\, N^\beta = 1$ and $a_{\alpha\beta}\, N^\alpha\, t^\beta = 0$. With the help of (2.3) and the fact that the contravariant components of t in V_{n+1} are $y^\alpha{}_{,i}\, dx^i/ds$, we have

$$N^\alpha = \pm \frac{y^\alpha{}_{,i}\, \{-g_{hk} w^h (dx^k/ds)(dx^i/ds) + w^i\} + w\xi^\alpha}{\{1 - g_{ij}\, g_{hk}\, w^i\, w^h (dx^j/ds)(dx^k/ds)\}^{\frac{1}{2}}}. \tag{2.4}$$

The plus sign in (2.4) is to be taken when $w > 0$, the minus sign when $w < 0$. Thus (2.4) will reduce to $N^\alpha = \xi^\alpha$ as is expected, when λ is linearly dependent on t and ξ^α; that is, $w^i = k dx^i/ds$, k being any constant different from unity. Elimination of ξ^α from (2.1) and (2.4) gives

$$q^\alpha = y^\alpha{}_{,i}\left(p^i - K_n\, \rho^i + K_n\, g_{hk}\, \rho^h\, \frac{dx^k}{ds}\, \frac{dx^i}{ds}\right) + N^\alpha K_n \left(1 - g_{ij}\, g_{hk}\, w^i\, w^h\, \frac{dx^j}{ds}\, \frac{dx^k}{ds}\right)^{\frac{1}{2}} \Big/ |w| \tag{2.5}$$

where K_n is the normal curvature of C and where $\rho^i = w^i/w$. Thus, when q^α is decomposed along N^α and a direction in V_n, we have from (2.5) the tangential component, $\bar{K}_g \mu^i$, defined by

$$\bar{K}_g\, \mu^i = p^i - K_n\, \rho^i + K_n\, g_{hk}\, \rho^h\, \frac{dx^k}{ds}\, \frac{dx^i}{ds} \tag{2.6}$$

where μ^i is the unit first curvature vector of C in V_n at P. We call $\bar{K}_g \mu^i$ the first curvature vector of C in V_n at P relative to the λ-congruence and $\bar{K}_g$ the first curvature of C in V_n at P relative to the λ-congruence. For convenience, they may be called the *relative first curvature vector* and the *relative first curvature* of the curve C at P respectively.

When $w = 0$, we have $w^i = dx^i/ds$ coincident with an asymptotic direction of V_n. Equations (2.4) and consequently equations (2.6) are then undefined. In this case, equations (2.1) reduce to $q^\alpha = y^\alpha{}_{,i}\, p^i$, which yield immediately

$$\bar{K}_g\, \mu^i = p^i. \tag{2.7}$$

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Since equations (2.7) are derivable from (2.6) when t is asymptotic and $w \neq 0$, we see that any result obtained from (2.6) for an asymptotic t with $w \neq 0$ holds true for an asymptotic t with $w = 0$. With this understanding we may say that equations (2.6) do give us a complete knowledge of the relative first curvature vector and the relative first curvature of the curve C in a hypersurface.

A simple calculation shows that the relative first curvature of C at P is given by

$$\bar{K}_g = K_g + K_d \tag{2.8}$$

where K_g is the first curvature of C in V_n and where K_d is defined by

$$K_d = 0 \quad \text{or} \quad K_d = -K_n\,\rho_j\,\mu^j \qquad (\rho_j = g_{ij}\,\rho^i) \tag{2.9}$$

according as $w = 0$ or $w \neq 0$. When C is a geodesic in V_n, we have $K_g = 0$. Consequently, $\bar{K}_g = K_d$, which is then the relative first curvature of a geodesic in V_n. Hence we have

THEOREM 2.1. *The relative first curvature at a point P of a curve in V_n differs from its first curvature at P in V_n by the relative first curvature at P of the geodesic in V_n, which passes through P in the same direction as the curve.*

If t is asymptotic, we have $\bar{K}_g = K_g$ and $K_g = K$, where K is the first curvature of C in V_{n+1}. Conversely, if $K = K_g$, then t is asymptotic and $\bar{K}_g = K_g$. Hence we have

THEOREM 2.2. *The three curvatures at a point P of a curve in V_n in V_{n+1}—the relative first curvature, the first curvature in V_n, and the first curvature in V_{n+1}—are identical if and only if the direction of the curve at P is an asymptotic direction of V_n.*

From the definition of K_d in equations (2.9) it is seen that K_d vanishes if and only if: (1) t is asymptotic and $w = 0$, or (2) t is asymptotic and $w \neq 0$, or (3) t is not asymptotic and $w^i = kt$ where k is any constant different from unity. Hence we have

THEOREM 2.3. *The relative first curvature of a geodesic in V_n is zero if and only if the geodesic is an asymptotic curve, or at each point P of the geodesic, every vector of the λ-congruence associated with P is linearly dependent on the unit vector normal to V_n at P and the unit tangent vector of the geodesic at P.*

3. Pseudogeodesics. A curve in V_n is called a pseudogeodesic in V_n relative to a λ-congruence or simply a pseudogeodesic in V_n, if the relative first curvature at each point of the curve is zero. From (2.5) a curve in V_n in V_{n+1} is a pseudogeodesic in V_n if and only if the principal normal vector of the curve in V_{n+1} coincides with N^α, which according to its definition is determined by the direction of the curve and the vector of the λ-congruence. Since the osculating geodesic surface at a point of a curve in a space is the surface formed by the geodesics through the point in the pencil of directions determined by the tangent and the principal normal to the curve (**2**, p. 62), we have the following geometric property of a pseudogeodesic.

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THEOREM 3.1. *A necessary and sufficient condition that a curve in V_n in V_{n+1} be a pseudogeodesic in V_n is that the osculating geodesic surface at every point of the curve considered as a curve in V_{n+1} is tangent to the unit vector of the λ-congruence associated with the curve at the corresponding point.*

Since $p^i = d^2x^i/ds^2 + \Gamma^i_{jk}(dx^j/ds)(dx^k/ds)$, we have from (2.6) that pseudogeodesics in a hypersurface with respect to the λ-congruence are defined by

$$(3.1) \qquad \frac{d^2x^i}{ds^2} + \left(\Gamma^i_{jk} - \Omega_{jk}\,\rho^i + \Omega_{jk}\,\rho_m \frac{dx^m}{ds}\frac{dx^i}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

These equations are n differential equations of the second order. Their complete integral involves $2n$ arbitrary constants. These may be determined by initial values of x^i and dx^i/ds, that is, values for $s = 0$, or by the initial values of x^i denoted by x^i_0 and other values of x^i such that $|x^i - x^i_0|$ are less than some fixed quantity. Hence we have

'THEOREM 3.2. *Through each point and in any given direction in a hypersurface there passes a unique pseudogeodesic.*

THEOREM 3.3. *Through two sufficiently near points in a hypersurface there passes one and only one pseudogeodesic.*

By Theorem 2.2 we have

THEOREM 3.4. *If an asymptotic curve of V_n is a geodesic in V_{n+1}, it is both a geodesic in V_n and a pseudogeodesic in V_n. If a geodesic in V_n is a geodesic in V_{n+1}, it is a pseudogeodesic in V_n. An asymptotic geodesic in V_n is always a pseudogeodesic in V_n.*

By Theorem 3.1, geodesics, union curves, and hypergeodesics on a surface in ordinary space appear as special pseudogeodesics **(5)** in V_2 in S_3. Substitution into (2.8) and (3.1) for λ the unit normal vectors to the surface, or the unit vectors along the specified congruence in the definition of union curves, or the unit vectors along generators of the osc-cones associated with a family of hypergeodesics gives respectively the geodesic curvature of a curve and the differential equations of geodesics, or the union curvature of a curve and the differential equations of union curves **(4)**, or the hypergeodesic curvature of a curve and the differential equations of hypergeodesics **(1)**. *The definition of the relative first curvature of a curve has the characteristic of being independent of the differential equations of pseudogeodesics.*

4. Relative parallelism. Let C be an arbitrary curve in V_n with unit tangent vector dx^i/ds. Let v^i be contravariant components of a family of unit vectors v along C in V_n. We say that the vectors v are parallel along C relative to the λ-congruence if and only if

$$(4.1) \qquad \frac{dv^i}{ds} + \left(\Gamma^i_{jk} - \Omega_{jk}\,\rho^i + \Omega_{jk}\,\rho_m \frac{dx^m}{ds}\frac{dx^i}{ds}\right) v^j \frac{dx^k}{ds} = 0.$$

Such parallelism of v along C is called a *relative parallelism*. When the λ-con-

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gruence becomes a congruence of unit normal vectors to V_n, a relative parallelism of v along C reduces to parallelism of v along C in the sense of Levi-Civita.

Comparing equations (3.1) with equations (4.1), we see that the unit tangent vector to a pseudogeodesic suffers a relative parallel displacement along the pseudogeodesic. Hence we have

THEOREM 4.1. *Pseudogeodesics in V_n are relative auto-parallel curves in V_n.*

If v are not parallel along C relative to the λ-congruence, we put

$$\frac{dv^i}{ds} + \left(\Gamma^i{}_{jk} - \Omega_{jk}\,\rho^i + \Omega_{jk}\,\rho_m\,\frac{dx^m}{ds}\,\frac{dx^i}{ds}\right) v^j\,\frac{dx^k}{ds} = \nu^i. \tag{4.2}$$

Then ν^i are obviously contravariant components of a vector. We call ν^i the *relative associate curvature vector* or the *relative angular spread vector* of v along C and its magnitude $(g_{ij}\nu^i\nu^j)^{\frac{1}{2}}$ the *relative associate curvature* or the *relative angular spread* of v along C.

Let Q be a neighboring point of P on C. Let Δs be the arc length of C from P to Q. Let a unit vector v^i in V_n undergo a local displacement along C from P to Q into a unit vector $v_1|^i$ and undergo a relative parallel displacement along C from P to Q into a unit vector $v_2|^i$. It is obvious that along C dx^i/ds, its unit tangent vector, and any function of x^i are functions of s. We assume all functions involved in the following discussion to be analytic along C in a certain common interval of s. Let g_{ij} at Q be denoted by $\bar{g}_{ij}$. Let Φ^i denote certain terms in (4.1) as follows

$$\Phi^i = \left(-\Gamma^i{}_{jk} + \Omega_{jk}\,\rho^i - \Omega_{jk}\,\rho_m\,\frac{dx^m}{ds}\,\frac{dx^i}{ds}\right) v^j\,\frac{dx^k}{ds}.$$

Then we can express $v_1|^i$, $v_2|^i$ and $\bar{g}_{ij}$ as follows in Taylor's series:

$$\begin{aligned} v_1|^i &= v^i + \frac{dv^i}{ds}\,(\Delta s) + \tfrac{1}{2}\,\frac{d^2v^i}{ds^2}\,(\Delta s)^2 + \dots, \\ v_2|^i &= v^i + \Phi^i\,(\Delta s) + \tfrac{1}{2}\,\frac{d\Phi^i}{ds}\,(\Delta s)^2 + \dots, \\ \bar{g}_{ij} &= g_{ij} + \frac{dg_{ij}}{ds}\,(\Delta s) + \tfrac{1}{2}\,\frac{d^2g_{ij}}{ds^2}\,(\Delta s)^2 + \dots, \end{aligned} \tag{4.3}$$

where all the coefficients are to be evaluated at P.

Substituting (4.3) into $\bar{g}_{ij}\,v_1|^i\,v_1|^j = 1$ and $\bar{g}_{ij}\,v_2|^i\,v_2|^j = 1$, we obtain

$$\begin{aligned} &2g_{ij}\,v^i\,\frac{dv^j}{ds} + \frac{dg_{ij}}{ds}\,v^i\,v^j = 0, \\ &g_{ij}\,\frac{dv^i}{ds}\,\frac{dv^j}{ds} + g_{ij}\,v^i\,\frac{d^2v^j}{ds^2} + 2\,\frac{dg_{ij}}{ds}\,v^i\,\frac{dv^j}{ds} + \tfrac{1}{2}\,\frac{d^2g_{ij}}{ds^2}\,v^i\,v^j = 0, \\ &2g_{ij}\,v^i\,\Phi^j + \frac{dg_{ij}}{ds}\,v^i\,v^j = 0, \\ &g_{ij}\,\Phi^i\,\Phi^j + g_{ij}\,v^i\,\frac{d\Phi^j}{ds} + 2\,\frac{dg_{ij}}{ds}\,v^i\,\Phi^j + \tfrac{1}{2}\,\frac{d^2g_{ij}}{ds^2}\,v^i\,v^j = 0. \end{aligned} \tag{4.4}$$

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Let $\Delta\theta$ be the angle between $v_1|^i$ and $v_2|^i$. Then we have

$$\cos(\Delta\theta) = \bar{g}_{ij} v_1|^i v_2|^j. \tag{4.5}$$

By Taylor's theorem and equations (4.3), (4.4) we obtain

$$\begin{aligned} \cos(\Delta\theta) &= 1 - \tfrac{1}{2}(\Delta\theta)^2 + \dots, \\ \bar{g}_{ij} v_1|^i v_2|^j &= 1 - \tfrac{1}{2} g_{ij} v^i v^j (\Delta s)^2 + \dots . \end{aligned} \tag{4.6}$$

Substitution of (4.6) into (4.5) yields

$$\left(\frac{d\theta}{ds}\right)^2 = g_{ij} v^i v^j. \tag{4.7}$$

Hence we have

THEOREM 4.2. *The relative associate curvature of a vector along a curve C at a point P is numerically the arc-rate of change of the angle between the two vectors displaced locally and relative parallelly from the vector at P along C. Such arc-rate of change of angle is zero along C if and only if the vector suffers relative parallel displacement along C.*

When $v^i = dx^i/ds$, then equation (4.7) reduces to $d\theta/ds = \pm \bar{K}_g$. Hence we have another interesting geometric interpretation of the relative first curvature.

THEOREM 4.3. *If the unit tangent vector at a point of a curve undergoes relative parallel displacement along the curve, the arc-rate of change of the angle between the vector and the curve is numerically the relative first curvature of the curve at the point.*

The content of Theorem 4.1 can be derived as a consequence of Theorem 4.3 and reads as

THEOREM 4.4. *If the unit tangent vector at a point of a pseudogeodesic undergoes relative parallel displacement along the pseudogeodesic, the arc-rate of change of the angle between the vector and the pseudogeodesic is zero at the point.*

Let the coefficients in the second term of (4.1) be denoted by $L^i{}_{jk}$, that is

$$L^i{}_{jk} = \Gamma^i{}_{jk} - \Omega_{jk}\,\rho^i + \Omega_{jk}\,\rho_m \frac{dx^m}{ds}\frac{dx^i}{ds}.$$

It is evident that if $L^i{}_{jk}$ and $\bar{L}^h{}_{pq}$ are these coefficients in different coordinate systems x^i and $\bar{x}^i$ respectively, they satisfy the equations

$$\frac{\partial^2 x^i}{\partial\bar{x}^p\partial\bar{x}^q} + L^i{}_{jk}\frac{\partial x^j}{\partial\bar{x}^p}\frac{\partial x^k}{\partial\bar{x}^q} = \bar{L}^h{}_{pq}\frac{\partial x^i}{\partial\bar{x}^h} \tag{4.8}$$

exactly the same as the coefficients of connection except that the latter are functions of x's only (3, p. 3). Hence, if these coefficients $L^i{}_{jk}$ are called the *coefficients of relative connection*, the concepts of relative parallelism and pseudogeodesics become respectively those of parallelism and paths defined in terms of the relative connection (3, pp. 13, 57).

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