Can. J. Math., Vol. XXVI, No. 6, 1974, pp. 1321–1340

SOME EXTENSIONS OF A THEOREM OF HARDY, LITTLEWOOD AND PÓLYA AND THEIR APPLICATIONS

KONG-MING CHONG

Introduction. In [**6**], by means of convex functions $\Phi : R \to R$, Hardy, Littlewood and Pólya proved a theorem characterizing the strong spectral order relation $\prec$ for any two measurable functions which are defined on a finite interval and which they implicitly assumed to be essentially bounded (cf. [**6**, the approximation lemma on p. 150 and Theorem 9 on p. 151 of their paper]; see also L. Mirsky [**10**, pp. 328–329] and H. D. Brunk [**1**, Theorem A, p. 820]). Later in [**8**], Ky Fan and G. G. Lorentz proved a theorem for two systems of decreasing bounded functions f_i, g_i on the unit interval [0, 1] satisfying $f_i \prec g_i$, $i = 1, 2, \ldots, n$, and obtained Hardy, Littlewood and Pólya's result as a particular case (see [**8**, Section 3, p. 630]). Recently, W. A. J. Luxemburg [**9**, Theorems (13.3), (13.5) and (13.6), pp. 125–127] gave a generalization of Hardy, Littlewood and Pólya's Theorem for functions belonging to a universally rearrangement invariant Banach function space normed by a saturated Fatou norm. In his paper [**9**, p. 124], he pointed out that the type of convex functions Φ considered earlier by Hardy, Littlewood and Pólya in [**6**] satisfies

$$\left| \liminf_{u \to -\infty} \frac{\Phi(u)}{u} \right| < \infty .$$

It is easy to verify that the original proof of Hardy, Littlewood and Pólya as given in [**6**] is also immediately applicable to proving the theorem for L^1 functions defined on a finite measure space and for all convex functions Φ which approach linearity asymptotically at both ends of the real line (e.g. the convex functions Φ_n defined by

$$\Phi_n(u) = +(1 + u^{2n})^{1/2n}$$

where $u \in R$ and $n \in N$, are of this type).

For L^1 functions f, g defined on a finite measure space, the conditions imposed earlier on the convex functions Φ (i.e. the linearity of Φ at infinity) are clearly too restrictive, for even some of the most common convex functions Φ such as $\Phi(u) = u^{2n}$, $u \in R$ (with n = a positive integer), and $\Phi(u) = \exp|u|$, $u \in R$, do not satisfy these conditions. In this paper, using some new characterizations of the spectral order relations $\prec$ and $\ll$, i.e. Theorem 1.6

Received April 26, 1973. This research constitutes part of the author's doctoral dissertation written under the guidance of Professor N. M. Rice at Queen's University. The author was supported by the Canadian Commonwealth Scholarship.

https://doi.org/10.4153/CJM-1974-126-1 Published online by Cambridge University Press

and Corollary 1.8 below, we are able to show that all these restrictions, i.e. the essential boundedness condition for the functions f and g and the linearity of the convex functions Φ at infinity in case $f, g \in L^1 - L^\infty$, can be removed without affecting the validity of the conclusion of the theorem. Moreover, our method enables us to discuss the case of equality in our extension of the Hardy, Littlewood and Pólya's Theorem (and hence to obtain for the first time conditions for equality in the classical Hardy, Littlewood and Pólya's Theorem) and to prove some very useful new theorems concerning the weaker relation $\ll$ (see Theorems 2.1 & 2.3).

1. Preliminaries. Denote by $M(X, \mu)$ the set of all extended real-valued measurable functions on a measure space (X, Λ, μ). Let $f \in M(X, \mu)$, $g \in M(X', \mu')$, where $\mu(X) = \mu'(X')$. Then f and g are said to be *equimeasurable* (written $f \sim g$) if

$$(1) \qquad \mu(f^{-1}[I]) = \mu'(g^{-1}[I])$$

for all bounded closed subintervals I of the extended real line $\bar{R}$; here I may be the singleton set $\{-\infty\}$ or $\{+\infty\}$.

Each $f \in M(X, \mu)$ induces on the Borel sets of $\bar{R}$ a unique Borel measure μ_f (called the *μ-spectral measure of f* or the *spectral measure of f with respect to μ*) which is defined by

$$(2) \qquad \mu_f(B) = \mu(f^{-1}(B))$$

for each Borel set $B \subset \bar{R}$. Observe that μ_f is the Lebesgue-Stieltjes measure on R generated by the right continuous, non-decreasing function $-D_f$, where $D_f : \bar{R} \to [0, \mu(X)]$ (called the *distribution function* of f) is defined by

$$(3) \qquad D_f(t) = \mu(\{f > t\})$$

for all $t \in \bar{R}$.

By (1), it is straightforward to verify that

$$(4) \qquad \Phi(f)\chi_{f^{-1}[B]} \sim \Phi(g)\chi_{g^{-1}[B]}$$

for all real Borel measurable functions Φ on $\bar{R}$ and for all Borel subsets B of $\bar{R}$, whenever $f \sim g$. Moreover, one can easily see that

$$(5) \qquad \int_X f d\mu = \int_{X'} g d\mu'$$

if $f \in M(X, \mu)$ and $g \in M(X', \mu')$ are equimeasurable.

If $f \in M(X, \mu)$ and if I denotes the identity map of the extended real line $\bar{R}$, then, by (1) and (2), it is obvious that

$$(6) \qquad f \sim I$$

whenever $\bar{R}$ is provided with the μ-spectral measure μ_f of f. Consequently, (4), (5) and (6) imply that

$$(7)\qquad \int_X \Phi(f)d\mu = \int_{\bar{R}} \Phi d\mu_f = -\int_{-\infty}^{\infty} \Phi(t)dD_f(t).$$

We need the following 'integration by parts' formula for the extension of the Hardy-Littlewood-Pólya rearrangement theorem to be given in the next section.

PROPOSITION 1.1. *Suppose f is a measurable function defined on a measure space (X, Λ, μ). Suppose $\Phi : R^+ \to R$ (respectively $\Phi : R \to R$) is a left continuous, non-decreasing function which is bounded from below, i.e., $\Phi(0^+) = \lim_{t\to 0^+} \Phi(t)$ (respectively $\Phi(-\infty) = \lim_{t\to-\infty} \Phi(t)$) is finite. Then*

$$(8)\qquad \int_X \Phi(f^+)d\mu \equiv \int_0^{\infty} D_f(t)d\Phi(t) + \Phi(0)\mu(X)$$

(*respectively*

$$(9)\qquad \int_X \Phi(f)d\mu \equiv \int_{-\infty}^{\infty} D_f(t)d\Phi(t) + \Phi(-\infty)\mu(X))$$

provided that $\Phi(0) = 0$ (respectively $\Phi(-\infty) = 0$) if $\mu(X)$ is infinite, in which case the term $\Phi(0)\mu(X)$ (respectively $\Phi(-\infty)\mu(X)$) does not appear.

Proof. This follows from (7) and Lebesgue's Monotone Convergence Theorem using integration by parts. (We refer to [**2**, Theorem 3.3, pp. 34–36] for the details of proof).

COROLLARY 1.2. *If $f \in M(X, \mu)$ where $\mu(X)$ is finite or infinite, then*

$$(10)\qquad \int_X (f-u)^+d\mu \equiv \int_u^{\infty} D_f(t)dt$$

for all $u \in R$.

Proof. Let $\Phi(t) = t$. Then Proposition 1.1 implies

$$\int_X (f-u)^+d\mu = \int_0^{\infty} D_{f-u}(t)dt = \int_0^{\infty} D_f(t+u)dt = \int_u^{\infty} D_f(t)dt.$$

COROLLARY 1.3. *If $f \in M(X, \mu)$ where $\mu(X)$ is finite, then*

$$(11)\qquad \int_X (f-u)^-d\mu \equiv \int_{-\infty}^{u} [a - D_f(t)]dt$$

for all $u \in R$.

Proof. This follows from Proposition 1.1 as in Corollary 1.2.

The following proposition proves another 'integration by parts' formula which is essential to establishing conditions for equality in certain rearrangement inequalities involving convex functions (see Theorem 2.3 below).

https://doi.org/10.4153/CJM-1974-126-1 Published online by Cambridge University Press

PROPOSITION 1.4. *Suppose* $f \in M(X, \mu)$, *where* $\mu(X) = a$ *is finite. Suppose* $\Phi : R \to R$ *is a non-decreasing convex function with derivative* $\Phi' = \phi$ *m-a.e. If* $\Phi(f) \in L^1(X, \mu)$ *and if*

$$F_1(t) = \int_{-\infty}^{t} [a - D_f(s)]ds, \quad F_2(t) = -\int_{t}^{\infty} D_f(s)ds,$$

then

$$(12) \quad \int_X \Phi(f)d\mu$$
$$= \phi(u)\int_X (f-u)d\mu + \int_{-\infty}^{u} F_1(t)d\phi(t) - \int_{u}^{\infty} F_2(t)d\phi(t) + a\Phi(u)$$

for all $u \in R$ *such that* ϕ *is continuous at* u.

Proof. By (7), Corollaries 1.2 and 1.3, the result follows using integration by parts and Lebesgue's Monotone and Dominated Convergence Theorem repeatedly. (For details, see [**2**, Theorem 3.11, pp.41–43].)

If f is any measurable (respectively non-negative integrable) function defined on a finite (respectively infinite) measure space (X, Λ, μ), then there exists a unique right continuous decreasing function δ_f on the interval $[0, \mu(X)]$, called the *decreasing rearrangement* of f, such that δ_f and f are equimeasurable. In fact,

$$(13) \quad \delta_f(t) = \inf\{s \in R : D_f(s) \leqq t\}$$

where $t \in [0, \mu(X)]$.

In what follows, we shall denote the Lebesgue measure on R by m.

If $f, g \in M(X, \mu) \cup M(X', \mu')$ and $f^+, g^+ \in L^1(X, \mu) \cup L^1(X', \mu')$, where $\mu(X) = \mu'(X') = a < \infty$, then we write $f \ll g$ whenever

$$(14) \quad \int_0^t \delta_f dm \leqq \int_0^t \delta_g dm, \qquad t \in [0, a]$$

and $f \prec g$ whenever $f \ll g$ and

$$(15) \quad \int_0^a \delta_f dm = \int_0^a \delta_g dm.$$

If $a = \mu(X) = \mu'(X')$ is infinite, then the 'spectral' order relations $\prec$ and $\ll$ are defined for non-negative integrable functions $f, g \in L^1(X, \mu) \cup L^1(X', \mu')$ analogously.

We shall now give a new characterization of these so-called Hardy, Littlewood and Pólya spectral order relations which were originally defined in terms of the decreasing rearrangements of the functions involved. First, we prove a lemma.

LEMMA 1.5. *If $f^+ \in L^1(X, \mu)$ where $\mu(X) = a$ is finite, then*

$$\int_0^t (\delta_f - u)dm \leqq \int_u^\infty D_f dm$$

for all $t \in [0, a]$, $u \in R$ and equality holds if and only if t and u are related by the inequality

$$D_f(u) \leqq t \leqq D_f(u^-).$$

The same is true for positive integrable functions defined on an infinite measure space.

Proof. We need only prove the lemma for the case $u = 0$. The general case then follows on replacing f by $f - u$.

Now

$$\int_0^t \delta_f dm \leqq \int_0^t (\delta_f)^+ dm \leqq \int_0^a \delta_{f^+} dm = \int_0^\infty D_f dm$$

by Corollary 1.2.

To prove the conditions for equality, we first assume that

$$m\{\delta_f > 0\} = D_f(0) \leqq t \leqq D_f(0^-) = m\{\delta_f \geqq 0\}.$$

Then it is not hard to see that $\{\delta_f > 0\} \subset [0, t]$ and $[0, t) \subset \{\delta_f \geqq 0\}$ and in this case,

$$\int_0^\infty D_f dm = \int_0^a \delta_f{}^+ dm = \int_{\{\delta_f \geqq 0\}} \delta_f dm = \int_{[0,t)} \delta_f dm = \int_0^t \delta_f dm.$$

On the other hand, if $t < D_f(0)$ or $t > D_f(0^-)$ then it is not hard to see that in either case

$$\int_0^t \delta_f dm < \int_0^\infty D_f dm$$

be virtue of the fact that $[0, D_f(0)) \subset \{\delta_f > 0\}$ and $(D_f(0^-), a] \subset \{\delta_f < 0\}$ which can be easily verified.

THEOREM 1.6.

(i) $f \ll g$ *if and only if* $\int_u^\infty D_f dm \leqq \int_u^\infty D_g dm$ *for all* $u \in R$.

(ii) $f \prec g$ *if and only if* $\int f = \int g$ *and* $\int_u^\infty D_f dm \leqq \int_u^\infty D_g dm$ *for all* $u \in R$.

Proof. Suppose $f \ll g$. Then for any $u \in R$, choose t such that $D_f(u) \leqq t \leqq D_f(u^-)$. Using Lemma 1.5, we have

$$\int_u^\infty D_f dm = \int_0^t (\delta_f - u)dm \leqq \int_0^t (\delta_g - u)\, dm \leqq \int_u^\infty D_g dm.$$

Conversely, suppose

$$\int_u^\infty D_f dm \leqq \int_u^\infty D_g dm$$

for all $u \in R$. For each $t \in (0, a)$, let $u = \delta_g(t)$. Then it is easy to see that $D_g(u) \leqq t \leqq D_g(u^-)$. Again, by Lemma 1.5, we have

$$\int_0^t (\delta_f - u)dm \leqq \int_u^\infty D_f dm \leqq \int_u^\infty D_g dm = \int_0^t (\delta_f - u)dm.$$

COROLLARY 1.7 (Hardy, Littlewood and Pólya [**6**, p. 152]). *$f \prec g$ if and only if $\int f = \int g$ and $\int (f - u)^+ \leqq \int (g - u)^+$ for all $u \in R$.*

Proof. This is obvious by virtue of Corollary 1.2 and Theorem 1.6.

COROLLARY 1.8. *If f^- is integrable, then $f \prec g$ if and only if $\int f = \int g$ and either*

$$\int (f - u)^- \leqq \int (g - u)^- \quad or \quad \int_{-\infty}^u (a - D_f)dm \leqq \int_{-\infty}^u (a - D_g)dm$$

for all $u \in R$.

Proof. Since $\int f = \int g$, the result follows from Corollary 1.7 and the fact that $\int f - u = \int (f - u)^+ - \int (f - u)^- = \int (g - u)^+ - \int (g - u)^- = \int (g - u)$.

COROLLARY 1.9 (Luxemburg [**9**, Lemma (6.2)(i)). *If f^- is integrable, then $f \prec g$ if and only if $-f \prec -g$.*

Proof. The result follows immediately from Corollaries 1.7 and 1.8.

COROLLARY 1.10. *If $f \ll g$, then $f^+ \ll g^+$. If $f \prec g \in L^1$, then $f^+ \ll g^+$ and $f^- \ll g^-$.*

Proof. The first part follows immediately from Theorem 1.6(i) and the fact that

$$D_{f^+}(t) = \begin{cases} D_f(t), & t \geqq 0 \\ \mu(X), & t < 0 \end{cases}$$

if $f \in M(X, \mu)$ with $f^+ \in L^1(X, \mu)$. The rest is an easy consequence of the first part and Corollary 1.9.

COROLLARY 1.11. *If $f_n \downarrow f$, $g_n \downarrow g$ as $n \to \infty$, and if $f_n \ll g_n$ (respectively $f_n \prec g_n$) for $n = 1, 2, \ldots$, then $f \ll g$ (respectively $f \prec g$) provided that g^+ is integrable.*

Proof. Since $f_n \downarrow f$ as $n \to \infty$ implies $(f_n - u)^+ \downarrow (f - u)^+$ as $n \to \infty$ for each $u \in R$, $\int (f_n - u)^+ \leqq \int (g_n - u)^+$ implies $\int (f - u)^+ \leqq \int (g - u)^+$ by Levi's Monotone Convergence Theorem. The result thus follows.

Remarks. (i) By (7), (10) and Theorem 1.6, we can characterize the relations $\ll$ and $\prec$ in terms of spectral measures as follows. For each $u \in R$, let $\Phi_u : R \to$

R be defined by $\Phi_u(t) = (t - u)^+$, $t \in R$. Then

$$f \ll g \Leftrightarrow \int_{\bar{R}} \Phi_u d\mu_f \leqq \int_{\bar{R}} \Phi_u d\mu_g$$

for all $u \in R$ and

$$f \prec g \Leftrightarrow f \ll g \text{ and } \int f = \int g.$$

This characterization explains why we adopt the terminology "spectral" order relations for $\ll$ and $\prec$.

(ii) The proof given in Corollary 1.9 is different from and is, in fact, simpler than that given by Luxemburg in [**9**, Lemma (6.2)(i)].

(iii) Corollary 1.10 is due to Day [**5**], but the proof given by him only shows that $f \ll g$ implies $\int f^+ \leqq \int g^+$ (see [**5**, Lemma (8.2)(vii)]). A complete proof for Corollary 1.10 was first given in Chong and Rice [**4**, Proposition (10.2)(ix)]; the proof we give here is new.

2. Some extensions of a theorem of Hardy, Littlewood and Pólya. In order to extend Hardy, Littlewood and Pólya's Theorem to include integrable functions defined on a finite measure space, we need to establish the following stronger version of the theorem, showing that it is also true for non-negative integrable functions defined on general measure spaces. This stronger theorem turns out to be the key theorem of the whole section, from which all others follow.

THEOREM 2.1. *Suppose (X, Λ, μ) and (X', Λ', μ') are measure spaces with equal total measures, finite or infinite. Suppose $f \in L^1(X, \mu)$ and $g \in L^1(X', \mu')$ are non-negative. Then $f \ll g$ if and only if*

$$(16) \quad \int_X \Phi(f) d\mu \leqq \int_{X'} \Phi(g) d\mu'$$

for all non-negative increasing convex functions $\Phi : R^+ \to R^+$ such that $\Phi(0) = 0$.

If $f \ll g$ and if, in addition, Φ is strictly convex, then equality holds in (16), *i.e.*

$$(17) \quad \int_X \Phi(f) d\mu = \int_{X'} \Phi(g) d\mu'$$

if and only if $f \sim g$.

If $\mu(X) = \mu'(X') < \infty$, then the restriction that both f and g be non-negative may be removed, provided that $\Phi : R \to R^+$ is a non-negative, increasing convex function satisfying

$$(18) \quad \lim_{u \to -\infty} \Phi(u) = 0.$$

Proof. For the first part of the theorem, condition (16) is clearly sufficient by virtue of Corollary 1.7 since the function $\Phi_u : R^+ \to R^+$, $u \in R^+$, defined by

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$\Phi_u(t) = (t - u)^+$ for $t \in R^+$ is non-negative, increasing convex and satisfying $\Phi_u(0) = 0$.

To prove that condition (16) is necessary, for each $t \in R^+$, let

$$G(t) = -\int_t^\infty D_g dm, \quad F(t) = -\int_t^\infty D_f dm.$$

Then $f \ll g$ if and only if $G(t) \leqq F(t)$ for all $t \in R^+$.

Since $\Phi : R^+ \to R^+$ is convex, its derivative Φ' exists (except possibly at a countable number of points) and is increasing on R^+. We can, therefore, assume that $\Phi' = \phi$ a.e. for some increasing function $\phi : R^+ \to R$ which is right continuous.

If $f \ll g$, assume that $\int \Phi(g) d\mu' < \infty$ (i.e., $\Phi(g) \in L^1(X', \mu')$ since $\Phi(g) \geqq 0$); otherwise, there is nothing to prove, for (16) is then trivially satisfied. First we claim that $\Phi(f)$ is also integrable. By Proposition 1.1,

$$\int_{X'} \Phi(g) d\mu' \equiv \int_0^\infty D_g(t) d\Phi(t) = \int_0^\infty \phi(t) dG(t),$$

so $\Phi(g) \in L^1(X', \mu')$ if and only if ϕ is integrable with respect to G over $[0, \infty]$. Using integration by parts, we have, for each $u \in R^+$,

$$\begin{aligned}\int_0^u \phi(t) dG(t) &= \int_0^u [-G(t)] d\phi(t) + \phi(u^+) G(u^+) - \phi(0^-) G(0^-) \\ &= \phi(u) G(u) - \phi(0^-) G(0) - \int_0^u G(t) d\phi(t),\end{aligned}$$

but (cf. [**2**, Lemma (3.1), p. 30]) $\lim_{u \to \infty} \phi(u) G(u) = 0$ whenever

$$-\infty < \int_0^\infty \phi(t) dG(t) < \infty$$

and so, as $u \to \infty$, Lebesgue's Monotone Convergence Theorem implies that

$$\int_0^\infty \phi(t) dG(t) = -\int_0^\infty G(t) d\phi(t) - \phi(0^-) G(0). \tag{19}$$

Thus

$$-\infty < \int_0^\infty G(t) d\phi(t) < \infty.$$

Since $f \ll g$ if and only if $G \leqq F$ and since $F \leqq 0$, we have

$$-\infty < \int_0^\infty G(t) d\phi(t) \leqslant \int_0^\infty F(t) d\phi(t) \leqslant 0.$$

Now following exactly the same arguments as before, we obtain

$$\int_0^\infty \phi(t) dF(t) + \int_0^\infty F(t) d\phi(t) = -\phi(0^-) F(0) \tag{20}$$

which, *a fortiori*, implies that

$$-\infty < \int_0^\infty \phi(t)dF(t) < \infty.$$

But, by Proposition 1.1,

$$\int_X \Phi(f)d\mu \equiv \int_0^\infty D_f(t)d\Phi(t) = \int_0^\infty \phi(t)dF(t).$$

We therefore conclude that $\Phi(f) \in L^1(X, \mu)$.

To prove (16), we have, using (19) and (20),

$$\begin{aligned}\int_{X'} \Phi(g)d\mu' - \int_X \Phi(f)d\mu &= \int_0^\infty \phi(t)dG(t) - \int_0^\infty \phi(t)dF(t) \\ &= \phi(0^-)[F(0) - G(0)] \\ &\quad + \int_0^\infty [F(t) - G(t)]d\phi(t)\end{aligned}$$

which is non-negative since Φ is convex increasing (a fact which implies that ϕ is non-negative increasing) and since $f \ll g$ if and only if $F \geqq G$.

Now suppose that Φ is strictly convex and increasing and that $f \ll g$. Then, clearly, (17) holds whenever $f \sim g$. Conversely, suppose (17) holds. Since Φ is strictly convex, its derivative ϕ is strictly increasing. Let m_ϕ be the Lebesgue-Stieltjes measure on R^+ generated by ϕ. Then (17) implies that

$$\phi(0^-)[F(0) - G(0)] + \int_0^\infty [F(t) - G(t)]dm_\phi = 0$$

which, in turn, implies that both $\phi(0^-)[F(0) - G(0)] = 0$ and

$$\int_0^\infty [F(t) - G(t)]dm_\phi = 0 \tag{21}$$

hold, since the left hand sides of both expressions are non-negative. Again, since $G \leqq F$, (21) implies $F - G = 0$ m_ϕ-a.e. on R^+. Now, we claim that $F = G$ everywhere on R^+. Suppose by contradiction that $F(t_0) \neq G(t_0)$ for some $t_0 \in R^+$. Then there exists an interval $(a, b]$ such that $t_0 \in (a, b]$ and $F \neq G$ on $(a, b]$, since both F and G are, by definition, continuous on R^+. But then $m_\phi((a, b]) = \phi(b) - \phi(a) > 0$ since ϕ is strictly increasing and $a < b$ and thus contradicting the fact that $F = G$ m_ϕ-a.e. on R^+. Hence $F = G$, i.e.,

$$-\int_u^\infty D_f dm = -\int_u^\infty D_g dm$$

for all $u \in R$, implying that $f \ll g$ and $g \ll f$ or $f \sim g$.

Finally, if $\mu(X) = \mu'(X') < \infty$, then exactly the same argument as before will apply to prove the last assertion of the theorem.

Remark. In Theorem 2.1, the last assertion (concerning the case that both $\mu(X)$ and $\mu'(X')$ are finite and equal) remains valid if Φ is only assumed to be bounded from below (cf. Proposition 1.1), i.e. $\lim_{u\to-\infty}\Phi(u)$ exists and is finite.

COROLLARY 2.2. *Suppose $f \in L^1(X, \mu)$ and $g \in L^1(X', \mu')$ where $\mu(X) = \mu'(X')$, finite or infinite. If $|f| \ll |g|$, then*

$$\Phi(|f|) \ll \Phi(|g|) \tag{22}$$

for all increasing convex functions $\Phi : R^+ \to R^+$ such that $\Phi(0) = 0$ and $\Phi(|g|) \in L^1(X', \mu')$. In particular, if $g \in L^p(X', \mu')$ for some $p \geqq 1$, then $f \in L^p(X, \mu)$ and $c|f|^p \ll c|g|^p$ for all non-negative real numbers c.

Moreover, if $|f| \ll |g|$ and if $\Phi : R^+ \to R^+$ is strictly convex, increasing and satisfying $\Phi(0) = 0$ and $\Phi(|g|) \in L^1(X', \mu')$, then

$$\Phi(|f|) \prec \Phi(|g|) \tag{23}$$

if and only if $\Phi(|f|) \sim \Phi(|g|)$ or, equivalently, $|f| \sim |g|$.

Proof. Let $\Psi : R^+ \to R^+$ be any increasing convex function satisfying $\Psi(0) = 0$. Then, clearly, the composition $\Psi \circ \Phi : R^+ \to R^+$ is again increasing convex and satisfying $\Psi \circ \Phi(0) = 0$ and thus, by Theorem 2.1,

$$\int_X \Psi \circ \Phi(|f|)d\mu \leqq \int_{X'} \Psi \circ \Phi(|g|)d\mu'$$

for all increasing convex functions $\Psi : R^+ \to R^+$ such that $\Psi(0) = 0$. Hence, by Theorem 2.1 again, we conclude that (22) holds.

The last assertion follows directly from Theorem 2.1 and the first part of the corollary.

For integrable functions defined on finite measure spaces, we have the following important consequence of Theorem 2.1. This theorem turns out to be an equivalent form of our extension of Hardy, Littlewood and Pólya's Theorem to be given in Theorem 2.5 (see [**2**, Section 33, pp. 277-287]).

THEOREM 2.3. *Suppose (X, Λ, μ) and (X', Λ', μ') are finite measure spaces such that $\mu(X) = \mu'(X')$. If $f \in L^1(X, \mu)$ and $g \in L^1(X', \mu')$, then $f \ll g$ if and only if*

$$\int_X \Phi(f)d\mu \leqq \int_{X'} \Phi(g)d\mu' \tag{24}$$

for all increasing convex functions $\Phi : R \to R$.

If $f \ll g$, where $f \in L^1(X, \mu)$ and $g \in L^1(X', \mu')$, and if Φ is strictly convex such that $\Phi(f) \in L^1(X, \mu)$, then

$$\int_X \Phi(f)d\mu = \int_{X'} \Phi(g)d\mu' \tag{25}$$

if and only if $f \sim g$.

Proof. For the first part of the theorem, the sufficiency of condition (24) follows as in Theorem 2.1.

To prove that condition (24) is necessary, let $\Phi : R \to R$ be any increasing convex function. We first consider the case that Φ is bounded from below, i.e. $\lim_{u\to-\infty} \Phi(u)$ exists and is finite. Let $c = \lim_{u\to-\infty} \Phi(u)$. Then the function $\Phi_c = \Phi - c$ is non-negative, increasing convex and satisfies $\lim_{u\to-\infty} \Phi_c(u) = 0$. Thus, we can apply Theorem 2.1 to obtain

$$\int_X \Phi_c(f)d\mu \leqq \int_{X'} \Phi_c(g)d\mu'$$

whenever $f \ll g$, but this implies that

$$\int_X [\Phi(f) - c]d\mu \leqq \int_{X'} [\Phi(g) - c]d\mu'$$

or that (24) holds since $\mu(X) = \mu'(X') < \infty$.

Next assume that Φ is unbounded from below, i.e., $\lim_{u\to-\infty} \Phi(u) = -\infty$. We also assume that $\int\Phi(f)d\mu \neq -\infty$ and $\int\Phi(g)d\mu' \neq \infty$, i.e. both $\Phi^-(f)$ and $\Phi^+(g)$ are integrable; otherwise, there is nothing to prove. Now for each positive integer n, the function $(\Phi + n)^+$ is non-negative, increasing, convex and satisfies $\lim_{u\to-\infty} (\Phi + n)^+(u) = 0$. Thus Theorem 2.1 implies $\int(\Phi + n)^+(f)d\mu \leqq \int(\Phi + n)^+(g)d\mu'$ whenever $f \ll g$. Let $\Phi_n = (\Phi + n)^+ - n$. Then it is plain that

$$\int_X \Phi_n(f)d\mu \leqq \int_{X'} \Phi_n(g)d\mu'.$$

(In fact, this is also an immediate consequence of the first case considered above, since $\lim_{u\to-\infty} \Phi_n(u) = -n$, which is finite.)

Clearly,

$$\int_X \Phi^+(f)d\mu \leqq \int_{X'} \Phi^+(g)d\mu'$$

since $\Phi^+ = \Phi_0$. Thus $\Phi^+(g) \in L^1(X', \mu')$ implies that $\Phi^+(f) \in L^1(X, \mu)$, and so $\Phi(f) \in L^1(X, \mu)$. Since $\Phi_n(f) \downarrow \Phi(f)$ and $\Phi_n(g) \downarrow \Phi(g)$ as $n \to \infty$, by Levi's Monotone Convergence Theorem, we conclude that (24) holds.

Finally, suppose $f \ll g$ and Φ is a strictly convex increasing function such that $\Phi(f) \in L^1(X, \mu)$. Then, clearly, (25) holds whenever $f \sim g$. Conversely, suppose (25) holds. If Φ is bounded from below, i.e. if $c = \lim_{u\to-\infty} \Phi(u)$ is finite, then the function $\Phi_c = \Phi - c$ is non-negative, increasing, strictly convex and satisfies

$$\lim_{u\to-\infty} \Phi(u) = 0 \text{ and } \int_X \Phi_c(f)d\mu = \int_{X'} \Phi_c(g)d\mu'.$$

Hence f and g are equimeasurable, by Theorem 2.1.

Suppose now that Φ is unbounded from below, i.e., $\lim_{u\to-\infty}\Phi(u) = -\infty$. As in the proof of Theorem 2.1, let ϕ be the (right continuous and strictly increasing) derivative of Φ and let m_ϕ be the Lebesgue-Stieltjes measure on R generated by ϕ. Let $\mu(X) = a = \mu'(X')$. For $t \in R$, let

$$F_1(t) = \int_{-\infty}^{t} (a - D_f)dm, \quad F_2(t) = -\int_{t}^{\infty} D_f dm,$$

$$G_1(t) = \int_{-\infty}^{t} (a - D_f)dm, \quad G_2(t) = -\int_{t}^{\infty} D_g dm.$$

Then, by Proposition 1.4, since both $\Phi(f)$ and $\Phi(g)$ are integrable and $\int\Phi(f)d\mu = \int\Phi(g)d\mu'$, we have, for each point u of continuity of ϕ,

$$\begin{aligned}
&\phi(u)\left[\int (f-u)d\mu\right] + \int_{-\infty}^{u} F_1(t)d\phi(t) - \int_{u}^{\infty} F_2(t)d\phi(t) + a\Phi(u)\\
&= \int \Phi(f)d\mu\\
&= \int \Phi(g)d\mu'\\
&= \phi(u)\left[\int (g-u)d\mu'\right] + \int_{-\infty}^{u} G_1(t)d\phi(t) - \int_{u}^{\infty} G_2(t)d\phi(t) + a\Phi(u),
\end{aligned}$$

which implies that

$$\phi(u)\left[\int g d\mu' - \int f d\mu\right] + \int_{-\infty}^{u} (G_1 - F_1)dm_\phi + \int_{u}^{\infty} (F_2 - G_2)dm_\phi = 0,$$

except possibly for those u belonging to an at most countable subset of R, i.e. the points of discontinuity of ϕ.

Now, since both $\Phi(f)$ and $\Phi(g)$ are integrable, it is clear the F_1 and G_1 are integrable with respect to m_ϕ over $[-\infty, u]$ for any point u of continuity of ϕ. Thus Lebesgue's Dominated Convergence Theorem implies that

$$\int_{-\infty}^{u} (G_1 - F_1)dm_\phi \to 0$$

as $u \to -\infty$ and, since $f \ll g$ if and only if $G_2 \leqq F_2$, Lebesgue's Monotone Convergence Theorem implies

$$\int_{u}^{\infty} (F_2 - G_2)dm_\phi \to \int_{-\infty}^{\infty} (F_2 - G_2)dm_\phi$$

as $u \to -\infty$. Furthermore, since Φ is (strictly) convex increasing, its derivative ϕ is non-negative and (strictly) increasing and thus $\lim_{u\to-\infty}\phi(u) \geqq 0$. In view of the above, we have

$$\left[\lim_{u\to-\infty}\phi(u)\right]\left[\int g d\mu' - \int f d\mu\right] + \int_{-\infty}^{\infty} (F_2 - G_2)dm_\phi = 0$$

which, therefore, implies that

$$\left[\lim_{u\to-\infty} \phi(u)\right]\left[\int gd\mu' - \int fd\mu\right] = 0, \int_{-\infty}^{\infty} (F_2 - G_2)dm_\phi = 0$$

since both are non-negative. If we now proceed in exactly the same way as we did in the proof of Theorem 2.1, we arrive at the conclusion that f and g are equimeasurable.

COROLLARY 2.4. *Suppose* $f \in L^1(X, \mu)$ *and* $g \in L^1(X', \mu')$ *where* $\mu(X) = \mu'(X') < \infty$. *If* $f \ll g$, *then* $\Phi(f) \ll \Phi(g)$ *for all increasing convex functions* $\Phi : R \to R$ *such that* $\Phi^+(g) \in L^1(X', \mu')$ *in which case,* $\Phi^+(f)$ *is also integrable.*

In particular, if $g \in L^p(X', \mu')$ *for some* $p \geqq 1$, *then* $f \in L^p(X, \mu)$ *and* $r|f|^p + s \ll r|g|^p + s$ *for all* $r \geqq 0$, $s \in R$, *whenever* $|f| \ll |g|$.

Moreover, if $f \ll g$ *and if* $\Phi : R \to R$ *is increasing, strictly convex and satisfying* $\Phi(g) \in L^1(X', \mu')$ *then* $\Phi(f) \prec \Phi(g)$ *if and only if* $\Phi(f) \sim \Phi(g)$ *or, equivalently,* f *and* g *are equimeasurable.*

Proof. This follows from Theorem 2.3 in exactly the same way that Corollary 2.2 is obtained from Theorem 2.1.

As a direct consequence of Theorem 2.3 (and hence of Theorem 2.1), we can now extend Hardy, Littlewood and Pólya's Theorem to include unbounded L^1 functions defined on any finite measure space as follows. (In a subsequent paper, we prove that Theorem 2.3 can also be obtained from Theorem 2.5, showing that the two theorems are in fact equivalent.)

THEOREM 2.5. *If* $f \in L^1(X, \mu)$, $g \in L^1(X', \mu')$ *where* $\mu(X)$ *and* $\mu'(X')$ *are finite and equal, then* $f \prec g$ *if and only if*

$$(26) \quad \int_X \Phi(f)d\mu \leqq \int_{X'} \Phi(g)d\mu'$$

for all convex functions $\Phi : R \to R$.

If $f \prec g$ *and if* Φ *is strictly convex such that* $\Phi(f) \in L^1(X, \mu)$, *then*

$$(27) \quad \int_X \Phi(f)d\mu = \int_{X'} \Phi(g)d\mu'$$

if and only if f *and* g *are equimeasurable.*

Proof. For the first part of the theorem, the condition is clearly sufficient by virtue of Corollary 1.7, since the functions $t \mapsto t$, $t \mapsto -t$ and $t \mapsto (t - u)^+$, where $u \in R$, are all convex in $t \in R$.

Conversely, assume $f \prec g$. If $\Phi : R \to R$ is decreasing convex, then the function $\Psi : R \to R$ defined by $\Psi(t) = \Phi(-t)$ for $t \in R$ is clearly increasing convex. By Corollary 1.9, $f \prec g$ if and only if $-f \prec -g$ and thus by Theorem

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2.3, we have

$$\int_X \Psi(-f)d\mu \leqq \int_{X'} \Psi(-g)d\mu'$$

i.e. (26) holds.

Next, if $\Phi : R \to R$ is increasing convex, then, clearly, the result follows again from Theorem 2.3.

Finally, if the convex function $\Phi : R \to R$ is not of the type we have just considered, then there exists a point $c \in R$, where $\Phi(c) \neq -\infty$, such that Φ is non-decreasing on $[c, \infty)$ and non-increasing on $(-\infty, c)$. Now let $\Phi_c : R \to R$ be defined by $\Phi_c(t) = \Phi(c + t)$ where $t \in R$. Then Φ_c is also convex and Φ_c is non-decreasing over $[0, \infty)$ and non-increasing over $(-\infty, 0)$. Let $f_c = f - c$ and $g_c = g - c$. Then $f \prec g$ implies $f_c \prec g_c$ which in turn implies $f_c^+ \ll g_c^+$ and $f_c^- \ll g_c^-$, by Corollary 1.10. Since the functions $t \mapsto \Phi_c(t)$ and $t \mapsto \Phi_c(-t)$ are increasing convex on $[0, \infty)$, we have

$$\int_X \Phi_c(f_c^+)d\mu \leqq \int_{X'} \Phi_c(g_c^+)d\mu'$$

and

$$\int_X \Phi_c(-f_c^-)d\mu \leqq \int_{X'} \Phi_c(-g_c^-)d\mu'$$

by Theorem 2.1 or Theorem 2.3. As $\Phi \geqq \Phi(c) = \Phi_c(0)$ on R and $\Phi(f) \in L^1(X, \mu)$, it is clear that $\Phi(c)$ is finite since

$$-\infty < \int_X \Phi(c)d\mu \leqq \int_X \Phi(f)d\mu < \infty.$$

Now,

$$\begin{aligned}\int_X \Phi(f)d\mu &= \int_X \Phi_c(f - c)d\mu = \int_X \Phi_c(f_c)d\mu \\ &= \int_X [\Phi_c(f_c^+) + \Phi_c(-f_c^-) - \Phi_c(0)]d\mu \\ &\leqq \int_{X'} [\Phi_c(g_c^+) + \Phi_c(-g_c^-) - \Phi_c(0)]d\mu' \\ &= \int_{X'} \Phi_c(g_c)d\mu' \\ &= \int_{X'} \Phi(g)d\mu'.\end{aligned}$$

For the last part of the theorem, suppose $f \prec g$ and Φ is strictly convex such that $\Phi(f) \in L^1(X, \mu)$. Then, clearly, (27) holds whenever f and g are equimeasurable. Conversely, suppose (27) holds. First of all, if Φ is increasing, then $f \prec g$ and $\int \Phi(f)d\mu = \int \Phi(g)d\mu'$ imply that f and g are equimeasurable, by

Theorem 2.3. Next if Φ is non-increasing and strictly convex, then the map $t \mapsto \Phi(-t)$ is increasing and strictly convex in $t \in R$. Since $f \prec g$ implies $-f \prec -g$, by Corollary 1.9, the equality

$$\int_X \Phi[-(-f)]d\mu = \int_{X'} \Phi[-(-g)]d\mu'$$

implies $-f \sim -g$, again by Theorem 2.3, and so f and g are equimeasurable. Finally, if the strictly convex function Φ is not of the type we have just considered, then there exists a point $c \in R$ such that $\Phi(c) \not\equiv -\infty$ and Φ is non-decreasing on $[c, \infty)$ and non-increasing on $(-\infty, c)$. Let the functions Φ_c, f_c and g_c be defined as in the proof of the first part of the theorem. Then, as before,

$$\begin{aligned}\int_X \Phi(f)d\mu &= \int_X [\Phi_c(f_c^+) + \Phi_c(-f_c^-) - \Phi(c)]d\mu \\ &\leqq \int_{X'} [\Phi_c(g_c^+) + \Phi_c(-g_c^-) - \Phi(c)]d\mu' \\ &= \int_{X'} \Phi(g)d\mu'.\end{aligned}$$

Thus the equality

$$\int_X \Phi(f)d\mu = \int_{X'} \Phi(g)d\mu$$

and the inequalities

$$\int_X \Phi_c(f_c^+)d\mu \leqq \int_{X'} \Phi_c(g_c^+)d\mu\,, \quad \int_X \Phi_c(-f_c^-)d\mu \leqq \int_{X'} \Phi_c(-g_c^-)d\mu'$$

force the last two inequalities to be equalities. Since the functions Φ_c and $t \mapsto \Phi_c(-t)$ are strictly convex increasing on $[0, \infty)$ and since $f_c^+ \ll g_c^+$ and $f_c^- \ll g_c^-$, the equalities

$$\int_X \Phi_c(f_c^+)d\mu = \int_{X'} \Phi_c(g_c^+)d\mu', \quad \int_X \Phi_c(-f_c^-)d\mu = \int_{X'} \Phi_c(-g_c^-)d\mu'$$

imply $f_c^+ \sim g_c^+$ and $f_c^- \sim g_c^-$, by Theorem 2.3. It is easy to see that $f_c^+ \sim g_c^+$ and $-f_c^- \sim -g_c^-$ imply $f_c^+ - f_c^- \sim g_c^+ - g_c^-$, i.e., $f_c \sim g_c$ or $f - c \sim g - c$. Hence $f \sim g$.

Remark. For functions defined on a finite discrete measure space with atoms of equal measures, the case of equality in Theorem 2.5 is an improvement of that given in Theorem 108 of Hardy, Littlewood and Pólya [**7**, p. 89] where they imposed the stronger restriction that the second derivative Φ'' of Φ exists and is strictly positive (cf. [**7**, Theorem 95, p. 77]).

COROLLARY 2.6. *Suppose* $f \in L^1(X, \mu)$ *and* $g \in L^1(X', \mu')$ *where* $\mu(X)$ *and* $\mu'(X')$ *are finite and equal.*

If $f \prec g$, *then* $\Phi(f) \ll \Phi(g)$ *for all convex functions* $\Phi : R \to R$ *such that* $\Phi^+(g) \in L^1(X', \mu')$ *and in that case,* $\Phi^+(f)$ *is also integrable.*

In particular, if $f \prec g$ *and if* $g \in L^{2n}(X', \mu')$ *for some integer* $n \in N$, *then* $f \in L^{2n}(X, \mu)$ *and* $rf^{2n} + s \ll rg^{2n} + s$ *for all* $r \geqq 0$, $s \in R$.

Moreover, if $f \prec g$ *and if* $\Phi : R \to R$ *is a strictly convex function such that* $\Phi(g) \in L^1(X', \mu')$, *then* $\Phi(f) \prec \Phi(g)$ *if and only if* $\Phi(f) \sim \Phi(g)$ *or, equivalently,* f *and* g *are equimeasurable.*

Proof. Using the fact that (the composition with) an increasing convex function of a convex function on R is again a convex function on R, the corollary then follows from Theorems 2.3 and 2.5 in the same way that Corollary 2.2 is obtained from Theorem 2.1.

COROLLARY 2.7 (W. A. J. Luxemburg [**9**]). *If* $f \in L^1(X, \mu)$ *and* $g \in L^1(X', \mu')$ *where* $\mu(X)$ *and* $\mu'(X')$ *are finite and equal, then* $|f| \ll |g|$ *whenever* $f \prec g$.

Proof. Since the function $t \mapsto |t|$ is convex in $t \in R$, the result is an immediate consequence of Corollary 2.6.

Remark. The proof given in Corollary 2.7 is much simpler than that given by Luxemburg in [**9**, Theorem (9.5)]. For an alternative proof, see Chong [**2**, Theorem (4.3)(xviii), p. 48].

Theorem 2.3 can be extended for not necessarily integrable functions f and g as follows.

THEOREM 2.8. *Suppose* (X, Λ, μ) *and* (X', Λ', μ') *are finite measure spaces such that* $\mu(X) = \mu'(X')$.

If $f \in M(X, \mu)$ *and* $g \in M(X', \mu')$ *with* $g^+ \in L^1(X', \mu')$, *then* $f \ll g$ *if and only if* $\Phi(f) \ll \Phi(g)$ *for all non-decreasing convex functions* $\Phi : R \to R$ *such that* $\Phi^+(g) \in L^1(X', \mu')$, *or, equivalently,*

$$\int_X \Phi(f)d\mu \leqq \int_{X'} \Phi(g)d\mu'$$

for all non-decreasing convex functions $\Phi : R \to R$.

Proof. The condition is clearly sufficient.

To prove that the condition is necessary, assume that $\Phi^+(g) \in L^1(X', \mu')$. Without loss of generality, we may assume $\Phi(0) = 0$ (otherwise, let $\Phi_0 = \Phi - \Phi(0)$, then Φ_0 is non-decreasing convex and satisfying $\Phi_0(0) = 0$). It is easily seen that $\Phi^+(h) = \Phi(h^+)$ for all $h \in M(X, \mu) \cup M(X', \mu')$. Let $f_n = (f + n)^+ - n$, $g_n = (g + n)^+ - n$ where $n \geqq 0$ is an integer. Then $f_n, g_n \in L^1$ for each $n \geqq 0$. Since $\Phi^+(g) \in L^1(X', \mu')$, we also have $\Phi^+(g_n) \in L^1(X', \mu')$ for all $n \geqq 0$, i.e., $\Phi(g_n^+) \in L^1(X', \mu')$. Thus, by Corollary 2.4, $\Phi(f_n) \ll \Phi(g_n)$ for each $n \geqq 0$. But, clearly, $\Phi(f_n) \downarrow \Phi(f)$, $\Phi(g_n) \downarrow \Phi(g)$ and so $\Phi(f) \ll \Phi(g)$, by Corollary 1.11. Hence

$$\int_X \Phi(f)d\mu \leqq \int_{X'} \Phi(g)d\mu'.$$

3. Some applications. We shall now give some applications of the foregoing Hardy-Littlewood-Pólya-type rearrangement theorems to rearrangement inequalities, i.e. inequalities involving the equimeasurable rearrangements of functions. In subsequent papers, we give more applications of these Hardy-Littlewood-Pólya-type rearrangement theorems, showing that they serve as a unifying thread connecting many well-known inequalities.

First of all, we apply them to what we call *spectral inequalities*. By a *spectral inequality*, we mean an expression of the form $f \prec g$ or $f \ll g$, or to be precise, the former is called a *strong spectral inequality* and the latter a *weak spectral inequality*. A spectral inequality $f \prec g$ (respectively $f \ll g$) is said to be *strictly strong* (respectively *strictly weak*) if *f and g are not equimeasurable* (respectively if *f and g do not have equal total integrals*).

Spectral inequalities are essential tools for the study of rearrangement inequalities, for, in most cases, spectral inequalities turn out to be the "generators" for rearrangement inequalities (see [**2**, Chapters V and VI, pp. 156-238]).

The following theorem, which is related to Theorem 2.3 regarding the case of equality, proves that a strictly weak spectral inequality is preserved under composition with strictly increasing convex functions, provided that the composite functions are integrable. In some instances (see [**2**, Theorem (19.1), p. 164], for example), this result enables us to discuss the case of equality for a wider class of convex functions than the one given in Theorem 2.3.

THEOREM 3.1. *Suppose $f \in L^1(X, \mu)$ and $g \in L^1(X', \mu')$ where $\mu(X)$ and $\mu'(X')$ are finite and equal. If $f \ll g$ and if*

$$\int_X \Phi(f)d\mu = \int_{X'} \Phi(g)d\mu'$$

for some convex strictly increasing function (in particular, for some strictly convex increasing function) $\Phi : R \to R$ such that $\Phi(g)$ is integrable, then

$$\int_X fd\mu = \int_{X'} gd\mu'$$

i.e., $f \prec g$.

Proof. Since Φ is strictly increasing, the inverse Φ^{-1} of Φ exists and is increasing. It is easy to see that Φ^{-1} is concave since Φ is convex. Thus $-\Phi^{-1}$ is a decreasing convex function. But Corollary 2.4 and the hypothesis imply that $\Phi(f) \prec \Phi(g)$ and so $-\Phi^{-1} \circ \Phi(f) \ll -\Phi^{-1} \circ \Phi(g)$ by Corollary 2.6

since $-\Phi^{-1}$ is convex. Thus $-f \ll -g$ and so

$$\int_X f d\mu \geqq \int_{X'} g d\mu'$$

whence the result follows.

COROLLARY 3.2. *Suppose $f \in L^1(X, \mu)$, $g \in L^1(X', \mu')$ where $\mu(X) = \mu'(X') < \infty$. If $f \ll g$ and if*

$$\int_X f d\mu < \int_{X'} g d\mu',$$

then

$$\int_X \Phi(f) d\mu < \int_{X'} \Phi(g) d\mu'$$

for all strictly increasing convex functions (in particular, for all increasing strictly convex functions) $\Phi : R \to R$ such that both $\Phi(f)$ and $\Phi(g)$ are integrable.

Using spectral inequalities, we can summarize the results obtained in Corollaries 2.2, 2.4, 2.6, Theorem 2.8 and Theorem 3.1 as follows.

THEOREM 3.3. *For measurable functions defined on finite measure spaces with integrable positive parts (respectively for non-negative integrable functions defined on infinite measure spaces), weak spectral inequalities are preserved under composition with non-decreasing convex functions on R (respectively non-negative non-decreasing convex functions on R which vanish at the origin) provided that the composite functions have integrable positive parts.*

For integrable functions defined on finite measure spaces, the following hold:

(i) *Strictly weak spectral inequalities are preserved under composition with strictly increasing convex functions, provided that the composite functions are also integrable.*

(*ii*) *Strong (respectively strictly strong) spectral inequalities become weak (respectively strictly weak) spectral inequalities under the compositions with convex (respectively strictly convex) functions on R provided that the composite functions have integrable positive parts (respectively the composite functions are integrable).*

Proof. This follows directly from Theorems 2.1, 2.3, 2.5, 2.8 and 3.1.

In what follows, we show that the classical Jensen's Inequality is a rearrangement inequality by proving that it is, in fact, a particular case of our extended form of Hardy, Littlewood and Pólya's Theorem, i.e. Theorem 2.5. Moreover, the condition for equality which we obtain for the classical Jensen's Inequality is an improvement of that given by Hardy, Littlewood and Pólya [**7**, Theorem 204, p. 151], where they assumed the existence and strict positivity of the second derivative of the convex function concerned.

The spectral inequality obtained in the following lemma extends a well-known spectral inequality in [**9**, Lemma (6.2)(iii), p. 96], but the proof given here is different and, in fact, much simpler; it also proves more, since it shows that the spectral inequality in [**9**, Lemma (6.2)(iii)] can be extended to include measurable functions with integrable positive parts.

LEMMA 3.4. *Suppose $f \in M(X, \Lambda, \mu)$ where $\mu(X) < \infty$. If $f^+ \in L^1(X, \Lambda, \mu)$ and if $E \in \Lambda$, then*

$$\left[\frac{1}{\mu(E)} \int_E f d\mu\right] \chi_E \prec f \chi_E$$

and, in particular,

$$\left[\frac{1}{\mu(X)} \int_X f d\mu\right] \chi_X \prec f.$$

Proof. For any $t \in R$, clearly, $(f - t) \leqq (f - t)^+$ and so

$$\begin{aligned}\int_X \left[\left(\frac{1}{\mu(E)} \int_E f d\mu\right) \chi_E - t\right]^+ d\mu &= \int_X \left\{\left[\left(\frac{1}{\mu(E)} \int_E f d\mu\right) - t\right]^+ \chi_E + (-t)^+ \chi_{E^c}\right\} d\mu \\ &= \left[\int_E (f - t) d\mu\right]^+ + (-t)^+ \mu(E^c) \\ &\leqq \int_E (f - t)^+ d\mu + (-t)^+ \mu(E^c) = \int_X (f\chi_E - t)^+ d\mu\end{aligned}$$

whence the result follows by virtue of Corollary 1.7.

THEOREM 3.5 (Jensen). *If $f \in L^1(X, \mu)$, where $\mu(X) < \infty$, and if $\Phi : R \to R$ is convex, then*

$$\Phi\left[\frac{1}{\mu(X)} \int_X f d\mu\right] \leqq \frac{1}{\mu(X)} \int_X \Phi(f) d\mu. \tag{28}$$

If Φ is strictly convex, in particular, if Φ has strictly positive second derivative, then equality holds if and only if f is a constant μ-a.e.

Proof. Clearly, inequality (28) is an immediate consequence of Theorem 2.3 by virtue of Lemma 3.4.

The case concerning equality is simple, owing to the fact that any function equimeasurable with a constant must itself be a constant almost everywhere.

As a concluding remark, in view of Theorem 2.8 and Lemma 3.4, it is evident that Jensen's Inequality (28) is true for all $f \in M(X, \mu)$ with integrable positive parts and for all non-decreasing convex functions $\Phi : R \to R$.

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University of Malaya,
Kuala Lumpur 22-11, *Malaysia.*

https://doi.org/10.4153/CJM-1974-126-1 Published online by Cambridge University Press