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AUTOMORPHIC PSEUDODIFFERENTIAL OPERATORS, POINCARÉ SERIES AND EISENSTEIN SERIES

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We construct Poincaré series and Eisenstein series for automorphic pseudodifferential operators, and show that the space of automorphic pseudodifferential operators associated to cusp forms is generated by Poincaré series. We also obtain explicit formulas for such Poincaré series and Eisenstein series.

1. Introduction

Pseudodifferential operators generalise usual differential operators by including fractional powers of linear ordinary differential operators, and they play an essential role in handling a certain class of integrable nonlinear partial differential equations such as the Korteweg-de Vries (KdV) equation or the Kadomtsev-Petviashvili (KP) equation (see for example, [**2**]). Over the years numerous papers have been devoted to the study of pseudodifferential operators in connection with various areas of pure and applied mathematics.

One of the important tools in number theory is the theory of automorphic forms, which range from the classical holomorphic automorphic forms of one variable, also known as modular forms, to the more general automorphic forms on Lie groups or algebraic groups. Poincaré series and Eisenstein series are basic examples of holomorphic modular forms. Modular forms that vanish at the cusps are called cusp forms, and it is well-known that the space of cusp forms is generated by Poincare series. Connections between modular forms and pseudodifferential operators have been investigated in a number of papers (see for example, [**4**]). In [**1**] Cohen, Manin and Zagier studied pseudodifferential operators that can be obtained as the image of holomorphic modular forms for a Fuchsian group of the first kind under a certain canonically constructed linear map, which they called automorphic pseudodifferential operators (see also [**5**]).

In this paper, we construct Poincaré series and Eisenstein series for automorphic pseudodifferential operators, and show that the space of automorphic pseudodifferential operators associated to cusp forms is generated by Poincaré series. We also obtain explicit formulas for such Poincaré series and Eisenstein series.

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2. Automorphic pseudodifferential operators

In this section, we review some of the results on automorphic pseudodifferential operators obtained by Cohen, Manin and Zagier in [**1**]. Let z be a local coordinate for the complex plane $\mathbb{C}$, and let ∂ denote the differential operator d/dz. If R is a ring of functions on $\mathbb{C}$, we denote by $\Psi\text{DO}(R)$ the space of pseudodifferential operators over R, that is,

$$\Psi\text{DO}(R) = \left\{ \sum_{l \in \mathbb{Z}} h_l \partial^l \,\Big|\, h_l \in R,\ h_l = 0 \text{ for } l \gg 0 \right\}$$

(see for example, [**2**]). For each integer n we set

$$\Psi\text{DO}(R)_n = \left\{ \sum_{l=0}^{\infty} h_l \partial^{n-l} \,\Big|\, h_l \in R \right\}.$$

Then the map

$$\Psi\text{DO}(R)_n \to R, \quad \sum_{l=0}^{\infty} h_l \partial^{n-l} \mapsto h_0$$

induces a short exact sequence

$$0 \to \Psi\text{DO}(R)_{n-1} \to \Psi\text{DO}(R)_n \to R \to 0.$$

Given an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ we consider the corresponding coordinate change obtained by the associated linear fractional transformation

$$\widetilde{z} = gz = \frac{az+b}{cz+d}.$$

Under this coordinate change, the differential operator ∂ is transformed to

$$\widetilde{\partial} = (d\widetilde{z}/dz)^{-1}\partial = (cz+d)^2\partial,$$

and the corresponding transformation of the operator ∂^n is given by

$$\widetilde{\partial}^n = \left[(cz+d)^2\partial\right]^n = \sum_{l=0}^{\infty} l! \binom{n}{l} \binom{n-1}{l} c^l (cz+d)^{2n-l} \partial^{n-l}.$$

Thus if we define the associated action on R by

$$f \mapsto f|_{-2n} g = (cz+d)^{2n} f\left(\frac{az+b}{cz+d}\right),$$

then we obtain the corresponding action

$$\sum_{l=0}^{\infty} h_l \partial^{n-l} \mapsto \left(\sum_{l=0}^{\infty} h_l \partial^{n-l}\right) \circ g$$

of $g \in SL(2, \mathbb{C})$ on the space $\Psi\mathrm{DO}(R)_n$ that is equivariant with each map in the above short exact sequence.

For the rest of this section, let R be the ring of holomorphic functions on the Poincaré upper half plane $\mathcal{H}$ that are bounded by a power of $(|z|^2 + 1)/\mathrm{Im}(z)$. Let $\Gamma \subset SL(2, \mathbb{R})$ be a Fuchsian group of the first kind. If ν is an integer, then a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is called an automorphic form (or a modular form) for Γ of weight ν if it is holomorphic at each cusp and satisfies the condition

$$f(z) = (f|\gamma)(z) = (cz + d)^{-\nu} f\left(\frac{az + b}{cz + d}\right)$$

for all $z \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then $M_\nu(\Gamma)$ is finite-dimensional for each $\nu \in \mathbb{Z}$ and is zero for $\nu < 0$, and we obtain a sequence of the form

$$0 \to \Psi\mathrm{DO}(R)^{\Gamma}_{-k-1} \to \Psi\mathrm{DO}(R)^{\Gamma}_{-k} \to M_{2k}(\Gamma) \to 0,$$

which is exact except possibly at the last arrow; here the superscript Γ denotes the set of Γ-invariant elements. In [1] Cohen, Manin and Zagier proved that the above sequence is exact. More precisely, they obtained the following result.

Theorem 2.1. *Given an integer $k \geqslant 1$, define the operators $\mathcal{L}_k : R \to \Psi DO(R)_{-k}$ and $\mathcal{L}_{-k} : R \to \Psi DO(R)_k$ by*

$$\mathcal{L}_k(f) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+k)!(n+k-1)!}{n!(n+2k-1)!} f^{(n)} \partial^{-k-n},$$

$$\mathcal{L}_{-k}(f) = \sum_{n=0}^{k-1} \frac{(2k-n)!}{n!(k-n)!(k-n-1)!} f^{(n)} \partial^{k-n},$$

and set $\mathcal{L}_0(f) = f$, where $f^{(n)}$ denotes the n-th derivative of f. Then we have

$$\mathcal{L}_k(f|_{2k} g) = \mathcal{L}_k(f) \circ g$$

for all $g \in SL(2, \mathbb{C})$ and $k \in \mathbb{Z}$.

Proof: See [**1**, Proposition 1]. ∎

From Theorem 2.1 it follows that $\mathcal{L}_k(f) \in \Psi\mathrm{DO}(R)^{\Gamma}_{-k}$ if $f \in M_{2k}(\Gamma)$. Hence the above sequence is an exact sequence that splits canonically, and, using the linear map $\mathcal{L}_k$, each modular form of weight $2k$ can be lifted to the space of pseudodifferential operators.

Definition 2.2: Elements of the space $\Psi\mathrm{DO}(R)^{\Gamma}_{-k}$ of Γ-invariant elements of $\Psi\mathrm{DO}(R)_{-k}$ are called *automorphic pseudodifferential operators for Γ of weight $2k$.*

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3. Poincaré Series and Eisenstein series

In this section, we construct Poincaré and Eisenstein series for automorphic pseudodifferential operators. Let $\mathcal{H}$ be the Poincaré upper half plane as in Section 2 on which the group $SL(2,\mathbb{R})$ acts by linear fractional transformations. We denote by $j : SL(2,\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ the automorphy factor given by

$$j(\gamma, z) = cz + d$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and $z \in \mathcal{H}$. Let $\Gamma \subset SL(2,\mathbb{R})$ be a Fuchsian group of the first kind, and let x be a cusp of Γ. Then there is an element $\sigma \in SL(2,\mathbb{R})$ such that $\sigma x = \infty$. We set $\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\}$, and let h be a positive real number such that

$$\sigma \Gamma_x \sigma^{-1} \cdot \{\pm 1\} = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^n \,\middle|\, n \in \mathbb{Z} \right\}.$$

For integers $\nu \geqslant 3$ and $m \geqslant 0$ we set

$$P^{\Gamma}_{\nu,m}(z) = \sum_{\gamma \in \Gamma_x \backslash \Gamma} j(\sigma\gamma, z)^{-\nu} \mathbf{e}^{m/h}(\sigma\gamma z)$$

for $z \in \mathcal{H}$, where $\mathbf{e}^{m/h}(\cdot) = \exp\bigl(2\pi i m(\cdot)/h\bigr)$. Then it is known that the series $P^{\Gamma}_{\nu,m}(z)$ converges absolutely and uniformly on any compact subset of $\mathcal{H}$ (see for example, [**3**, Section 2.6]). It is an automorphic form for Γ of weight ν, and is called the Poincaré series for $n \geqslant 1$ and the Eisenstein series for $n = 0$.

Given integers $k \geqslant 2$ and $m \geqslant 0$, let $\mathcal{L}_k : R \to \Psi\mathrm{DO}(R)_{-k}$ be the linear operator described in Theorem 2.1, and set

$$P^{\Gamma,\Psi}_{k,m} = \sum_{\gamma \in \Gamma_x \backslash \Gamma} \mathcal{L}_k(\varphi_m) \circ \gamma,$$

where $\varphi_m(z) = j(\sigma, z)^{-2k} \mathbf{e}^{m/h}(\sigma z)$ for each $z \in \mathcal{H}$ and the action of γ on $\mathcal{L}_k(\varphi_m)$ is the corresponding coordinate change operation described in Section 2.

Definition 3.1:

(i) $P^{\Gamma,\Psi}_{k,m}$ is called the *Poincaré series for automorphic pseudodifferential operators* if $m \geqslant 1$.

(ii) $P^{\Gamma,\Psi}_{k,0}$ is called the *Eisenstein series for automorphic pseudodifferential operators.*

If h is as above, a holomorphic modular form for Γ of weight $\nu \geqslant 0$ has a Fourier expansion of the form

$$\sum_{l=0}^{\infty} c_l \cdot \exp(2\pi i l z/h)$$

for $z \in \mathcal{H}$ at each cusp of Γ. If $c_0 = 0$ for each cusp of Γ, then the modular form is called a cusp form (see for example, [3]). We denote by $S_\nu(\Gamma)$ the space of cusp forms for Γ of weight ν.

DEFINITION 3.2: An automorphic pseudodifferential operator in $\Psi\mathrm{DO}(R)^\Gamma_{-k}$ is said to be *cuspidal* if it belong to the image of $S_{2k}(\Gamma)$ under $\mathcal{L}_k$. Thus $\mathcal{L}_k(S_{2k}(\Gamma))$ is the space of all cuspidal automorphic pseudodifferential operators for Γ of weight $2k$.

THEOREM 3.3.

(i) *For integers $k \geqslant 2$ and $m \geqslant 0$ we have $P^{\Gamma,\Psi}_{k,m} = \mathcal{L}_k(P^\Gamma_{2k,m})$.*

(ii) *The space $\mathcal{L}_k(S_{2k}(\Gamma))$ of all cuspidal automorphic pseudodifferential operators for Γ of weight $2k$ is generated by the set of Poincaré series $P^{\Gamma,\Psi}_{k,m}$ for $m \geqslant 1$.*

PROOF: (i) Since $j(\sigma\gamma, z) = j(\sigma, \gamma z) j(\gamma, z)$ for all $\sigma, \gamma \in \Gamma$ and $z \in \mathcal{H}$, we have

$$\begin{aligned} P^\Gamma_{2k,m}(z) &= \sum_{\gamma \in \Gamma_x \backslash \Gamma} j(\sigma\gamma, z)^{-2k} \mathrm{e}^{m/h}(\sigma\gamma z) = \sum_{\gamma \in \Gamma_x \backslash \Gamma} j(\gamma, z)^{-2k}\Big(j(\sigma, \gamma z)^{-2k} \mathrm{e}^{m/h}(\sigma\gamma z)\Big) \\ &= \sum_{\gamma \in \Gamma_x \backslash \Gamma} j(\gamma, z)^{-2k} \varphi_m(\gamma z) = \sum_{\gamma \in \Gamma_x \backslash \Gamma} (\varphi_m|_{2k}\gamma)(z). \end{aligned}$$

By Theorem 2.1 we have $\mathcal{L}_k(f|_{2k}\gamma) = \mathcal{L}_k(f) \circ \gamma$ for all $\gamma \in \Gamma$ and $f \in S_{2k}(\Gamma)$. Thus it follows that

$$\mathcal{L}_k(P^\Gamma_{2k,m}) = \mathcal{L}_k\Big(\sum_{\gamma \in \Gamma_x \backslash \Gamma} (\varphi_m|_{2k}\gamma)\Big) = \sum_{\gamma \in \Gamma_x \backslash \Gamma} \mathcal{L}_k(\varphi_m) \circ \gamma = P^{\Gamma,\Psi}_{k,m}.$$

(ii) It is well-known that for each $m \geqslant 1$ the Poincaré series $P^\Gamma_{2k,m}$ is a cusp form for Γ of weight $2k$, and the space $S_{2k}(\Gamma)$ of cusp forms for Γ of weight $2k$ is generated by the set

$$\{P^\Gamma_{2k,m} \mid m \geqslant 1\}$$

(see [3, Section 2.6]). Since $\mathcal{L}_k$ is a linear operator, it follows that the space $\mathcal{L}_k(S_{2k}(\Gamma))$ is generated by $\mathcal{L}_k(P^\Gamma_{2k,m}) = P^{\Gamma,\Psi}_{k,m}$ for $m \geqslant 1$. ∎

4. EXPLICIT FORMULAS

Let $\Gamma \subset SL(2, \mathbb{R})$ be a Fuchsian group of the first kind, and let k, m be integers with $k \geqslant 2$, $m \geqslant 0$. In this section we give an explicit description of the associated Poincaré series (for $m \geqslant 1$) or Eisenstein series $P^{\Gamma,\Psi}_{k,m}$ (for $m = 0$) for automorphic pseudodifferential operators in $\Psi\mathrm{DO}(R)^\Gamma_{-k}$.

Lemma 4.1. *Let $f : \mathcal{H} \to \mathbb{C}$ be the function given by $f(z) = e^{\lambda gz}$ for $z \in \mathcal{H}$, where $\lambda \in \mathbb{C}$ and*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}), \quad gz = \frac{az+b}{cz+d}.$$

If n is a positive integer, then we have

$$f^{(n)}(z) = \sum_{l=0}^{n-1} (-1)^l l! \binom{n}{l} \binom{n-1}{l} c^l \lambda^{n-l} j^{l-2n} f(z),$$

where $j = cz + d$ for each $z \in \mathcal{H}$.

Proof: We shall use induction on n. Using $\det g = ad - bc = 1$, we have

$$f'(z) = \lambda \frac{ad - bc}{(cz+d)^2} f(z) = \lambda j^{-2} f(z);$$

hence the given formula is true for $n = 1$. Now we assume that it is true for $n = m$. Then for $n = m + 1$ we have

$$\begin{aligned}
f^{(m+1)} = (f^{(m)})' &= \sum_{l=0}^{m-1} (-1)^l l! \binom{m}{l} \binom{m-1}{l} c^l \lambda^{m-l} \left(c(l-2m) j^{l-2m-1} f + j^{l-2m} \lambda j^{-2} f\right) \\
&= \sum_{l=0}^{m-1} (-1)^l l! \binom{m}{l} \binom{m-1}{l} \left((l-2m) c^{l+1} \lambda^{m-l} j^{l-2m-1} + c^l \lambda^{m-l+1} j^{l-2m-2}\right) f \\
&= \sum_{l=1}^{m} (-1)^{l-1} (l-1)! \binom{m}{l-1} \binom{m-1}{l-1} (l-2m-1) c^l \lambda^{m-l+1} j^{l-2m-2} f \\
&\qquad + \sum_{l=0}^{m-1} (-1)^l l! \binom{m}{l} \binom{m-1}{l} c^l \lambda^{m-l+1} j^{l-2m-2} f \\
&= (-1)^{m-1} (m-1)! \binom{m}{m-1} \binom{m-1}{m-1} (-m-1) c^m \lambda j^{-m-2} f \\
&\quad + (-1)^0 0! \binom{m}{0} \binom{m-1}{0} c^0 \lambda^{m+1} j^{-2m-2} f \\
&\quad + \sum_{l=1}^{m-1} (-1)^l l! c^l \lambda^{m-l+1} j^{l-2m-2} f \left[\frac{2m-l+1}{l} \binom{m}{l-1} \binom{m-1}{l-1} + \binom{m}{l} \binom{m-1}{l} \right].
\end{aligned}$$

However, we have

$$(-1)^{m-1} (m-1)! \binom{m}{m-1} \binom{m-1}{m-1} (-m-1) = (-1)^m m! \binom{m+1}{m} \binom{m}{m},$$

$$\binom{m}{0} \binom{m-1}{0} = \binom{m+1}{0} \binom{m}{0},$$

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and

$$\begin{aligned}\frac{2m-l+1}{l}\binom{m}{l-1}\binom{m-1}{l-1}+\binom{m}{l}\binom{m-1}{l}\\ &=\left[\frac{m+1}{l}+\frac{m-l}{l}\right]\binom{m}{l-1}\binom{m-1}{l-1}+\binom{m}{l}\binom{m-1}{l}\\ &=\binom{m+1}{l}\binom{m-1}{l-l}+\binom{m}{l-1}\binom{m-1}{l}+\binom{m}{l}\binom{m-1}{l}\\ &=\binom{m+1}{l}\binom{m-1}{l-l}+\binom{m-1}{l}\binom{m+1}{l}=\binom{m+1}{l}\binom{m}{l}.\end{aligned}$$

Therefore we obtain

$$f^{(m+1)}=\sum_{l=0}^{m}(-1)^l l!\binom{m+1}{l}\binom{m}{l}c^l\lambda^{m-l+1}j^{l-2m-2}f,$$

which is the desired formula for $n=m+1$; hence the lemma follows. ∎

Now an explicit formula for the Poincaré series or the Eisenstein series is given in the next theorem.

THEOREM 4.2. *Let k and m be nonnegative integers with $k \geqslant 2$, then the Poincaré series $P_{k,m}^{\Gamma,\Psi}$ (or the Eisenstein series for $m=0$) is given by*

$$\begin{aligned}P_{k,m}^{\Gamma,\Psi}=\sum_{n=0}^{\infty}\sum_{\gamma\in\Gamma_z\backslash\Gamma}\frac{(n+k)!(n+k-1)!}{n!(n+2k-1)!}c^n j^{-2k-n}\\ \times\left[\frac{(2k+n-1)!}{(2k-1)!}+\sum_{\nu=0}^{n-1}\sum_{l=0}^{n-\nu-1}\binom{n}{\nu}\binom{n-\nu}{l}\binom{n-\nu-1}{l}\right.\\ \left.\times\frac{l!(2k+\nu-1)!}{(2k-1)!}\left(\frac{2\pi im}{h}\right)^{n-\nu-l}(-c)^{\nu+l-n}j^{\nu+l-n}\right]e^{m/h}\left(\frac{az+b}{cz+d}\right)\partial^{-k-n}\end{aligned}$$

for $z\in\mathcal{H}$, where $j=j(\sigma\gamma,z)=cz+d$ for $\sigma\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{R})$.

PROOF: By Theorem 3.3(i) we have

$$P_{k,m}^{\Gamma,\Psi}=\mathcal{L}_k(P_{2k,m}^{\Gamma})=\sum_{n=0}^{\infty}\sum_{\gamma\in\Gamma_z\backslash\Gamma}(-1)^n\frac{(n+k)!(n+k-1)!}{n!(n+2k-1)!}\psi^{(n)}\partial^{-k-n},$$

where $\psi(z)=j^{-2k}e^{m/h}(\sigma\gamma z)$ with $j=j(\sigma\gamma,z)$ for $z\in\mathcal{H}$. Using the Leibniz formula we obtain

$$\psi^{(n)}(z)=\sum_{\nu=0}^{n}\binom{n}{\nu}(j^{-2k})^{(\nu)}\left(e^{m/h}(\sigma\gamma z)\right)^{(n-\nu)}.$$

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For $\sigma\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, we have

$$(j^{-2k})^{(\nu)} = (-1)^{\nu} \frac{(2k+\nu-1)!}{(2k-1)!} c^{\nu} j^{-2k-\nu}.$$

On the other hand, using Lemma 4.1, for $n - \nu \geqslant 1$ we get

$$\left(\mathrm{e}^{m/h}(\sigma\gamma z)\right)^{(n-\nu)} = \sum_{l=0}^{n-\nu-1} (-1)^l l! \binom{n-\nu}{l} \binom{n-\nu-1}{l} c^l \left(\frac{2\pi i m}{h}\right)^{n-\nu-l} j^{l-2n+2\nu} \mathrm{e}^{m/h}(\sigma\gamma z).$$

Hence we have

$$\begin{aligned} P_{k,m}^{\Gamma,\Psi} = \sum_{n=0}^{\infty} \sum_{\gamma \in \Gamma_z \backslash \Gamma} (-1)^n \frac{(n+k)!(n+k-1)!}{n!(n+2k-1)!} \Bigg[(-1)^n \frac{(2k+n-1)!}{(2k-1)!} c^n j^{-2k-n} \\ + \sum_{\nu=0}^{n} \binom{n}{\nu} (-1)^{\nu} \frac{(2k+\nu-1)!}{(2k-1)!} c^{\nu} j^{-2k-\nu} \sum_{l=0}^{n-\nu-1} (-1)^l l! \binom{n-\nu}{l} \binom{n-\nu-1}{l} \\ \times c^l \left(\frac{2\pi i m}{h}\right)^{n-\nu-l} j^{l-2n+2\nu} \Bigg] \mathrm{e}^{m/h}\left(\frac{az+b}{cz+d}\right) \partial^{-k-n}. \end{aligned}$$

The desired formula follows by rearranging this expression. ∎

References

[1] P. Cohen, Y. Manin and D. Zagier, 'Automorphic pseudodifferential operators', *Progr. Nonlinear Differential Equations Appl.* **26** (1995), 17–47.

[2] L. Dickey, *Soliton equations and Hamiltonian systems* (World Scientific, Singapore, 1991).

[3] T. Miyake, *Modular forms* (Springer-Verlag, Berlin, Heidelberg, New York, 1989).

[4] L. Takhtajan, 'Modular forms as τ-functions for certain integrable reductions of the Yang-Mills equations', in *Progress in Math.* **115** (Birkhäuser, Boston, 1993), pp. 115–129.

[5] D. Zagier, 'Modular forms and differential operators', *Proc. Indian Acad. Sci. Math. Sci.* **104** (1994), 57–75.

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