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LIFTING ISOMORPHISMS OF MODULES

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Throughout this note, let R be a discrete valuation ring with prime element π, residue class field $\bar{R}$, and quotient field K. Let Λ be an R-order in a finite dimensional K-algebra A. A Λ-*lattice* is an R-free finitely generated left Λ-module. For $k > 0$, we set

$$R_k = R/\pi^k R, \quad \Lambda_k = \Lambda/\pi^k \Lambda, \quad M_k = M/\pi^k M,$$

where M is any Λ-lattice. Obviously, for Λ-lattices M and N,

$$M \cong N \text{ as } \Lambda\text{-lattices} \Rightarrow M_k \cong N_k \text{ as } \Lambda_k\text{-modules for each } k > 0.$$

Maranda [**1**] and D. G. Higman [**3**] considered the reverse implication, and proved

THEOREM. *Let Λ be an R-order in a separable K-algebra A. Then there exists a positive integer k (which depends on Λ) with the following property: for each pair of Λ-lattices M and N,*

$$M_k \cong N_k \textit{ as } \Lambda_k\textit{-modules} \Rightarrow M \cong N \textit{ as } \Lambda\textit{-lattices}.$$

Indeed, it suffices to choose k so that

$$\pi^{k-1} \cdot H^1(\Lambda, \operatorname{Hom}_R (M, N)) = 0.$$

Maranda proved this result for the special case where Λ is the integral group ring RG of a finite group G. The general case was treated by D. G. Higman, who showed the existence of a nonzero ideal $i(\Lambda)$ of R such that

$$i(\Lambda) \cdot H^1(\Lambda, \mathrm{T}) = 0$$

for all Λ-bimodules T. This readily implies that the integer k (occurring above) can be chosen independently of M and N. Higman's construction depends on the fact that if A is any separable K-algebra, then $H^1(A, Y) = 0$ for all A-bimodules Y. For details of the proof, see Curtis-Reiner [**2**, §§ 75, 76].

The aim of this note is to prove an analogue of the Maranda-Higman Theorem, in which the K-algebra A need not be separable, nor even semi-simple. (As a matter of fact, the proof below applies equally well to the situation in which Λ is an R-algebra, finitely generated as an R-module, and does not require that Λ have a unity element. However, this seemingly more general case readily reduces to the case where Λ is an R-order in a K-algebra A.)

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THEOREM. *Let Λ be an R-order in an arbitrary finite dimensional K-algebra A, and let M, N be Λ-lattices. If $M_k \cong N_k$ as Λ_k-modules for each $k > 0$, then $M \cong N$ as Λ-modules.*

Proof. We set $\mathrm{T} = \mathrm{Hom}_R(M, N)$, a Λ-bimodule. A *derivation* $D : \Lambda \to \mathrm{T}$ is an R-homomorphism such that

$$D(xy) = xD(y) + D(x)y \quad \text{for all } x, y \in \Lambda.$$

Let Der (Λ, T) be the R-module consisting of all derivations from Λ to T; it is clearly finitely generated over R.

For each $k > 0$, we are given a Λ_k-isomorphism $\varphi_k : M_k \cong N_k$. Since M is R-projective, we can find a map $\theta_k \in \mathrm{Hom}_R(M, N)$ making the following diagram commute:

$$\begin{tikzcd} M \arrow[r, "\theta_k"] \arrow[d] & N \arrow[d] \\ M_k \arrow[r, "\varphi_k"] & N_k . \end{tikzcd}$$

Then $N = \theta_k(M) + \pi^k N$, so θ_k is surjective by Nakayama's Lemma. But M and N have the same R-rank, namely the R_k-rank of M_k; thus θ_k must be an R-isomorphism. The commutativity of the diagram implies that for each $x \in \Lambda$,

$$\theta_k(xm) - x\theta_k(m) \in \pi^k N \quad \text{for all } m \in M.$$

Hence for each $m \in M$, we may write

$$\theta_k(xm) - x\theta_k(m) = \pi^k n \quad \text{for some } n \in N,$$

and m uniquely determines n because N is R-free. We may therefore define an R-homomorphism $D_k : \Lambda \to \mathrm{T}$ by setting

$$D_k(x) = \pi^{-k}(\theta_k x - x\theta_k), \quad x \in \Lambda.$$

It is easily checked that D_k is a derivation.

Now consider the R-submodule $\mathscr{D}$ of Der (Λ, T) generated by $D_1, D_2, \ldots$. Since Der (Λ, T) is finitely generated as R-module, and R is noetherian, it follows that $\mathscr{D}$ is also finitely generated over R. Hence there exists a positive integer k such that D_k is an R-linear combination of $D_1, \ldots, D_{k-1}$, say

$$D_k = \alpha_1 D_1 + \ldots + \alpha_{k-1} D_{k-1}, \quad \text{with each } \alpha_i \in R.$$

Using the definition of the D's, this gives

$$\pi^{-k}(\theta_k x - x\theta_k) = \sum_1^{k-1} \alpha_i \pi^{-i}(\theta_i x - x\theta_i) \quad \text{for each } x \in \Lambda.$$

Multiplying by π^k and setting

$$\theta' = \theta_k - \sum_1^{k-1} \alpha_i \pi^{k-i}\theta_i,$$

we obtain $\theta' x = x\theta'$ for each $x \in \Lambda$. Thus $\theta' \in \mathrm{Hom}_\Lambda(M, N)$, and $\theta' \equiv \theta_k (\mathrm{mod}\ \pi)$. But then θ' is an R-isomorphism of M onto N, since θ_k is such an isomorphism. This shows that $M \cong N$ as Λ-modules, and completes the proof.

The preceding result shows that two Λ-lattices M, N are isomorphic if and only if $M_k \cong N_k$ for each k. There does not seem to be any obvious way to find a single choice for k, depending on Λ, M and N, such that $M_k \cong N_k$ implies that $M \cong N$.

We may also show that decomposability of lattices can be handled in the same manner, provided that R is complete. Let us prove

THEOREM. *Let R be a complete discrete valuation ring, and let Λ be an R-order in an arbitrary finite dimensional K-algebra A. Let M be a Λ-lattice such that M_k is Λ_k-decomposable for each $k > 0$. Then M is decomposable as a Λ-lattice.*

Proof. For each $k > 0$, let $\varphi_k : M_k \to M_k$ be a nontrivial idempotent in the endomorphism ring $\mathrm{End}_{\Lambda_k}(M_k)$. As above, we may choose $\theta_k \in \mathrm{End}_R(M)$ so that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\theta_k} & M \\ \downarrow & & \downarrow \\ M_k & \xrightarrow{\varphi_k} & M_k \end{array}$$

is commutative. The proof of the preceding theorem shows that for some $n > 0$, there exists a map $\theta' \in \mathrm{End}_\Lambda(M)$ such that $\theta' \equiv \theta_n\ (\mathrm{mod}\ \pi)$. Hence both θ' and θ_n are liftings of $\varphi_n\ (\mathrm{mod}\ \pi)$, and therefore

$$\{(\theta')^2 - \theta'\} M \subseteq \pi M.$$

We now set $E = \mathrm{End}_\Lambda(M)$, $\bar{E} = E/\pi E$, and let ψ be the image of θ' in $\bar{E}$. The preceding inclusion shows that $(\theta')^2 - \theta'$ lies in πE, whence $\psi^2 = \psi$ in $\bar{E}$. But then ψ is a nontrivial idempotent, since ψ coincides with $\varphi_n\ (\mathrm{mod}\ \pi)$. On the other hand, the method of "lifting idempotents" then implies that there exists a nontrivial idempotent $\mu \in E$ whose image in $\bar{E}$ is ψ. Therefore $M = \mu M \oplus (1 - \mu)M$ gives a nontrivial decomposition of M, and the result follows.

For the case of orders in separable algebras, the above technique is due to Heller, and improves the method originally used by Maranda. For references, see Curtis-Reiner [**2**, § 76].

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