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RESEARCH ARTICLE

On divisorial stability of finite covers

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Abstract
Divisorial stability of a polarised variety is a stronger – but conjecturally equivalent – variant of uniform K-stability introduced by Boucksom–Jonsson. Whereas uniform K-stability is defined in terms of test configurations, divisorial stability is defined in terms of convex combinations of divisorial valuations on the variety.

We consider the behaviour of divisorial stability under finite group actions and prove that equivariant divisorial stability of a polarised variety is equivalent to log divisorial stability of its quotient. We use this and an interpolation technique to give a general construction of equivariantly divisorially stable polarised varieties.

Contents

1. Introduction

The theory of K-stability of Fano varieties has achieved its prominence due to its links both with Kähler geometry (through the existence of Kähler–Einstein metrics [12, 42]) and moduli theory (through the construction of moduli spaces of K-polystable Fano varieties; see [31, 43]). There are essentially two reasons why the algebro-geometric theory of K-stability of Fano varieties has been so successful: the first is the interplay with birational geometry and the minimal model programme (originating in [30, 36]), and the second is the reinterpretation of K-stability in terms of divisorial valuations on the Fano variety [20, 25].

K-stability is also of interest for general polarised varieties (projective varieties endowed with an ample line bundle), and in this situation, there is still a substantial literature linking K-stability with Kähler geometry through the existence of constant scalar curvature Kähler metrics (namely, the *Yau–Tian–Donaldson conjecture* [16, 42 44]). However, the algebro-geometric theory of K-stability of general polarised varieties is considerably less developed than its Fano counterpart and relatively little is known. Although one cannot expect birational geometry to play as significant a role in this generality, it is still reasonable to attempt to use valuative tools in studying K-stability of arbitrary polarised varieties. With this in mind, the first author and Legendre introduced a notion of *valuative stability* of a polarised variety [15], which should be *strictly weaker* than K-stability for general polarised varieties, although it is equivalent in the Fano situation.

The more powerful notion of *divisorial stability*, very recently introduced by Boucksom–Jonsson [10], associates numerical invariants to *convex combinations* of divisorial valuations. By their work, divisorial stability implies – and is conjecturally equivalent to – *uniform* K-stability, which in turn is conjecturally equivalent to the existence of constant scalar curvature Kähler metrics when the variety is smooth. In fact, the same conjecture that would lead to a resolution of the 'uniform version' of the Yau–Tian–Donaldson conjecture (through [5, 26]) would also imply that divisorial stability is equivalent to uniform K-stability [10]. There is already some evidence that divisorial stability is a more useful notion than uniform K-stability, through Boucksom–Jonsson's proof that divisorial stability is an *open* condition in the ample cone [10, Theorem A] (see Liu for prior work in the setting of valuative stability [27]).

Thus, it is hoped that divisorial stability will produce a richer theory of stability of polarised varieties, by analogy with the Fano situation. The goal of this paper is to showcase another situation in which divisorial stability appears more useful than the traditional approach. We denote by (X, L_X) and (Y, L_Y) normal polarised varieties, such that $\pi : (Y, L_Y) \to (X, L_X)$ is a Galois cover with Galois group G, by which we mean that G acts on (Y, L_Y) in such a way that its quotient by G is (X, L_X). In addition, let Δ_X and Δ_Y be effective $\mathbb{Q}$-divisors such that

$$K_Y + \Delta_Y = \pi^*(K_X + \Delta_X),$$

where we assume both sides are $\mathbb{Q}$-Cartier divisors.

Theorem 1.1. *$((Y, \Delta_Y), L_Y)$ is G-equivariantly divisorially stable if and only if $((X, \Delta_X); L_X)$ is log divisorially stable.*

The most applicable special case is when G is cyclic of degree m, Δ_Y is taken to be trivial and Δ_X is the integral divisor such that Riemann–Hurwitz produces $K_Y = \pi^*(K_X + (1 - 1/m)\Delta_X)$. Theorem 1.1 then gives the following corollary:

Corollary 1.2. *(Y, L_Y) is G-equivariantly divisorially stable if and only if $((X, (1 - 1/m)\Delta_X); L_X)$ is log divisorially stable.*

Analogous results holds for divisorial semistability. The proof compares the divisorial measures used to define divisorial stability of (Y, L_Y) to the corresponding objects on (X, L_X) and uses non-Archimedean geometry to compare various associated numerical invariants. The advantage of divisorial stability over K-stability is analogous to the advantage exploited by Boucksom–Jonsson in their work on openness of divisorial stability in the ample cone: the numerical invariants involved in the definition of divisorial stability involve an entropy (or log discrepancy) term that is *easier* to manage than the analogous quantity involved in K-stability, whereas the *energy* (or *norm*) terms become more complicated; much of the proof involves understanding the behaviour of these energy terms under finite covers. We emphasise again this key advantage of divisorial stability: although handling the energy

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terms becomes more involved than with the traditional approach, one should expect these terms to generally be more manageable (as is the case both in the present work and in Boucksom–Jonsson [10]). By contrast, the entropy term – behind much of the difficulty of K-stability – becomes considerably simpler to understand.

The corresponding result in the *Fano* case was proven in steps by several authors; the first author proved one direction [13] for cyclic groups G, with the other direction and various improvements being proven by Fujita [19, Corollary 1.7] and Liu–Zhu [32]. This result has found many applications to the construction of new examples of K-stable Fano varieties (beyond these three original papers, we give [28] as a typical application), and in the study of K-moduli of log Fano pairs and moduli spaces of K3 surfaces (through [1]). Whereas the technique of [13] uses the language of K-stability, the techniques of [19, 32] instead use divisorial valuations. It seems challenging, however, to adapt the techniques of [13] to prove an analogous result for general polarised varieties, as the proof given there relies on properties of K-stability of Fano varieties that do not hold more generally [30]. We thus emphasise that divisorial stability seems more suited to this problem, as Theorem 1.1 *exactly* generalises the Fano results to general polarised varieties. We also mention that the techniques we employ to prove Theorem 1.1 are quite distinct from those of [19, 32], since divisorial stability involves convex combinations of divisorial valuations, and the actual numerical invariants have a somewhat different flavour in the Fano situation.

We use an interpolation technique to produce examples.

Theorem 1.3. *Let (X, L_X) be a divisorially semistable normal polarised variety. There is a $k > 0$ such that if we let*

(i) *$\Delta_X \in |kL_X|$ be such that (X, Δ_X) is log canonical,*
(ii) *and let $\pi : Y \to X$ be the m-fold cover of X branched over Δ_X,*

then (Y, L_Y) is G-equivariantly divisorially stable, where G is the associated cyclic group of degree m and $L_Y = \pi^* L_X$.

The construction applies for any $m > 0$. The k needed depends explicitly on the geometry of (X, L_X); see Remark 4.2. The proof shows that $((X, \Delta_X); L_X)$ is automatically divisorially stable, meaning by interpolation, so is $((X, (1 - 1/m)\Delta_X); L_X)$. Hence, by Theorem 1.1, (Y, L_Y) is G-equivariantly divisorially stable. Although the hypotheses themselves are different, this result is analogous to [13, Corollary 1.2], where an interpolation strategy was used to give a sufficient condition for K-stability of finite covers of Fano varieties. This was the source of many of the examples of K-stable Fano varieties produced by the K-stability analogue of Theorem 1.1 in the Fano setting.

When the output (Y, L_Y) is smooth and the field we work over is $\mathbb{C}$, this is sufficient to produce constant scalar curvature Kähler metrics.

Corollary 1.4. *Under the same hypothesis as Theorem 1.3, provided (Y, L_Y) is smooth, $c_1(L_Y)$ admits a constant scalar curvature Kähler metric.*

This corollary relies on an equivariant version of the result of Boucksom–Jonsson relating divisorial stability to uniform K-stability on $\mathcal{E}^1$, and work of Li producing constant scalar curvature Kähler metrics from G-equivariant uniform K-stability on $\mathcal{E}^1$.

There are many analytic counterparts to the results mentioned above, all under smoothness assumptions. The usage of finite symmetry groups in the study of Kähler–Einstein metrics goes back at least to Siu [39], Nadel [34] and Tian [41], and general results more in the spirit of our work were proven by Arezzo–Ghigi–Pirola [3] and Li–Sun [29]. In the general constant scalar curvature setting, Aoi–Hashimoto–Zheng have proven one part of the analogue of Theorem 1.3 [4, Theorem 1.10] – namely, the existence of constant scalar curvature Kähler metrics with cone angle singularities along Δ_X for sufficiently large k – while Arezzo–Della Vedova–Shi have proven an analytic analogue of Theorem 1.3, producing constant scalar curvature Kähler metrics on suitable finite covers [2]. The existence of constant scalar curvature Kähler metrics with cone angle singularities for $k \gg 0$ is an analogue of results of Hashimoto and Zeng on twisted constant scalar curvature Kähler metrics, and we rely on an

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algebro-geometric counterpart of these results proven by the first author and Ross [17, Theorem 3.7]. Arezzo–Della Vedova–Shi use these results to produce new examples of constant scalar curvature Kähler metrics, and we refer to their work for a discussion of examples to which these sorts of results can be applied [2, Section 6] (though we emphasise that applications of Theorem 1.3 are currently limited as we currently know relatively few examples of divisorially semistable varieties).

In another direction, we note that Li has given examples of smooth polarised varieties which are uniformly K-stable on $\mathcal{E}^1$ using analytic techniques [26, Proposition 6.12] (hence divisorially stable by Boucksom–Jonsson); these results also apply to pairs, provided X is smooth and $(X, (1-1/m)\Delta_X)$ is log canonical (as Li's method is insensitive to singularites of the divisor provided they are log canonical). Taking Δ_X to be singular, the m-fold branched cover Y is singular and Theorem 1.1 implies (Y, L_Y) is G-equivariantly divisorially stable. This is a consequence of Li's work directly when Y is smooth, but is inaccessible using analytic techniques when Y is singular, meaning our result gives new examples of G-equivariantly stable varieties.

2. Divisorial stability of polarised varieties

We work over an algebraically closed field k of characteristic zero. We fix an n-dimensional normal projective variety X along with an effective $\mathbb{Q}$-divisor B such that $K_X + B$ is $\mathbb{Q}$-Cartier; we allow B to be trivial. We also fix an ample $\mathbb{Q}$-line bundle L on X.

2.1. Valuative stability

Although we are ultimately interested in *divisorial stability*, which is a notion of stability defined through *convex combinations* of divisorial valuations on X, the general theory is quite intricate and simplifies considerably for a *single* divisorial valuation. Thus, we begin by explaining the theory for a single valuation, as introduced by the first author and Legendre [15], generalising the work of Fujita and Li in the Fano setting [20, 25]. A reference explaining the background to the material presented here is [24].

Definition 2.1. A *prime divisor over* X is an irreducible prime divisor $F \subset Y$ for some projective variety Y which admits a birational morphism $\pi : Y \longrightarrow X$.

A prime divisor F over X equivalently induces a valuation $\operatorname{ord} F$ on the function field $k(X)$ of X.

Definition 2.2. A *divisorial valuation* is a valuation on $k(X)$ of the form $c \operatorname{ord} F$ for F a prime divisor over X and $c \in \mathbb{R} \geq 0$; we sometimes write this valuation as $v_{c \operatorname{ord} F}$.

By passing to a log resolution of singularities if necessary, we assume that the pair (Y, F) is log smooth. To such a divisorial valuation, we associate a numerical invariant called the *beta invariant* of F, defined through the following standard invariants in birational geometry.

Definition 2.3. Suppose L is a line bundle on X. We define the *volume* of L to be

$$\operatorname{Vol}(L) = \limsup_{r\to\infty} \frac{\dim H^0(X, rL)}{r^n/n!},$$

and we say that L is *big* if $\operatorname{Vol}(L) > 0$.

The volume satisfies the homogeneity property $\operatorname{Vol}(lL) = l^n \operatorname{Vol}(L)$, and hence, the definition extends to $\mathbb{Q}$-line bundles; it further extends to $\mathbb{R}$-line bundles by a continuity argument. We use two foundational results concerning the volume. First, the lim sup involved in the definition is actually a limit. Second, the volume is actually a continuously differentiable function on the cone of big ($\mathbb{R}$-)line bundles on X [7]. We extend this definition to take F into account by defining

$$\operatorname{Vol}(L - xF) = \operatorname{Vol}(\pi^* L - xF),$$

where the latter is calculated on Y.

We also require a measure of singularities, for which we use our hypothesis that $K_X + B$ is $\mathbb{Q}$-Cartier.

Definition 2.4. We define the *log discrepancy* of F to be

$$A_{(X,B)}(F) = \mathrm{ord}_F(K_Y - \pi^*(K_X + B)) + 1.$$

Here, we use that (Y, F) is log smooth; if not, one should work on a log resolution of singularities of (Y, F). The beta invariant is simply a combination of these invariants. Denote

$$S_L(F) = \frac{\int_0^\infty \mathrm{Vol}(L - xF)dx}{L^n}.$$

Definition 2.5. The *beta invariant* of F is defined to be

$$\beta(F) = A_{(X,B)}(F) + \nabla_{K_X+B} S_L(F),$$

where

$$\nabla_{K_X+B} S_L(F) = \frac{d}{dt}\bigg|_{t=0} S_{L+t(K_X+B)}(F).$$

Example 2.6. If $L = -K_X$ and $B = 0$, so that X is a Fano variety, this invariant reduces to

$$\beta(F) = A_X(F) - S_{-K_X}(F),$$

which is precisely the invariant introduced by Fujita and Li [20, 25]. In general, our presentation of $\beta(F)$ agrees with that of Boucksom–Jonsson; see [10, Theorem 2.18] for the equivalence with the original presentation [15].

Definition 2.7. We say that $((X, B); L)$ is

(i) *valuatively semistable* if for all prime divisors F over X, we have $\beta(F) \geq 0$;
(ii) *uniformly valuatively stable* if there exists an $\varepsilon > 0$ such that for all prime divisors F over X, we have $\beta(F) \geq \varepsilon S_L(F)$.

Strictly speaking, in [15], valuative stability required the divisors to be *dreamy*, which is a finite-generation hypothesis. As this plays no role in the present work – and in light of the work of Boucksom–Jonsson on divisorial stability [10], appears generally less relevant – we choose to omit this hypothesis (as in [27]).

Remark 2.8. These numerical invariants extend in a homogeneous way to divisorial valuations –namely, by defining $A_{(X,B)}(a \,\mathrm{ord}\, F) = aA_{(X,B)}(F)$, $S_L(a \,\mathrm{ord}\, F) = aS_L(F)$ and $\beta(a \,\mathrm{ord}\, F) = a\beta(F)$. In this way, uniform valuative stability with respect to prime divisors over X and with respect to divisorial valuations are equivalent.

2.2. Test configurations and K-stability

We next define test configurations and the associated Monge–Ampère energy (which will be used in the subsequent sections) and uniform K-stability (which will play a secondary role to divisorial stability).

Definition 2.9. A *test configuration* for $((X, B); L)$ is a variety $\mathcal{X}$ along with

(i) a $\mathbb{Q}$-Weil divisor $\mathcal{B} \subset \mathcal{X}$ and a $\mathbb{Q}$-line bundle $\mathcal{L}$;
(ii) a $\mathbb{C}^*$-action on $\mathcal{X}$ fixing $\mathcal{B}$ and lifting to $\mathcal{L}$;
(iii) a flat, $\mathbb{G}_m$-equivariant morphism $\pi : \mathcal{X} \to \mathbb{A}^1$ making $\mathcal{B} \to \mathbb{A}^1$ a flat morphism,

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such that each fibre $((\mathcal{X}_t, \mathcal{B}_t); \mathcal{L}_t)$ for $t \neq 0$ is isomorphic to $((X, B); L)$. We say that $((\mathcal{X}, \mathcal{B}); \mathcal{L})$ is *normal* if $\mathcal{X}$ is normal, *ample* if $\mathcal{L}$ is relatively ample, *semiample* if $\mathcal{L}$ is relatively semiample and *nef* if $\mathcal{L}$ is relatively nef.

The divisor $\mathcal{B} \subset \mathcal{X}$ is canonically defined by taking the $\mathbb{G}_m$-closure of $B \subset X \cong \mathcal{X}_1$. A test configuration admits a canonical compactification to a family over $\mathbb{P}^1$ by compactifying trivially at infinity, and we will use the same notation for the resulting family $((\mathcal{X}, \mathcal{B}); \mathcal{L}) \to \mathbb{P}^1$.

Definition 2.10. The *Monge–Ampère energy* of a nef test configuration $(\mathcal{X}, \mathcal{L})$ is defined to be

$$E(\mathcal{X}, \mathcal{L}) = \frac{\mathcal{L}^{n+1}}{(n+1)L^n},$$

where this intersection number is calculated on the compactified test configuration over $\mathbb{P}^1$. When we wish to emphasise the dependence on L, we denote this by $E_L(\mathcal{X}, \mathcal{L})$.

Note that this quantity is independent of $\mathcal{B}$, and hence, we omit $\mathcal{B}$ from the notation. To define further numerical invariants, it is useful to pass to a resolution of indeterminacy of the natural rational map $\mathcal{X} \dashrightarrow X \times \mathbb{P}^1$; we thus obtain a new test configuration with an equivariant morphism to $X \times \mathbb{P}^1$. We replace $\mathcal{X}$ by the associated resolution of indeterminacy, which we then say *dominates the trivial test configuration* $(X \times \mathbb{P}^1, L)$.

Definition 2.11. The *minimum norm* of a nef test configuration $(\mathcal{X}, \mathcal{L})$ is defined to be

$$\|(\mathcal{X}, \mathcal{L})\|_{\min} = \frac{\mathcal{L}^{n+1}}{(n+1)L^n} - \frac{\mathcal{L}^n.(\mathcal{L} - L)}{L^n},$$

where L is pulled back to $\mathcal{X}$ through the morphism $\mathcal{X} \to X$.

The minimum norm is called the 'non-Archimedean $I-J$-functional' in [9]; we follow the terminology of [14]. This quantity is again independent of $\mathcal{B}$. The *Mabuchi functional*, however, does actually depend on $\mathcal{B}$. In order to define this, for a test configuration dominating the trivial one, define the *entropy*

$$\mathrm{Ent}(\mathcal{X}, \mathcal{L}) = \frac{1}{L^n}\Big(\mathcal{L}^n.K_{(\mathcal{X},\mathcal{B})/(X,B)\times\mathbb{P}^1} + \mathcal{L}^n.(\mathcal{X}_0 - \mathcal{X}_{0,\mathrm{red}})\Big),$$

where

$$K_{(\mathcal{X},\mathcal{B})/(X,B)\times\mathbb{P}^1} = K_{\mathcal{X}} + \mathcal{B} - \pi^*(K_{X\times\mathbb{P}^1} + B)$$

is the relative canonical class and $\mathcal{X}_0, \mathcal{X}_{0,\mathrm{red}}$ denote the central fibre of $\mathcal{X}$ and the reduced the central fibre, respectively. To make sense of this definition, one uses that $\mathcal{X}$ is normal to ensure that $K_{\mathcal{X}}$ is a Weil divisor.

Definition 2.12. We define the *Mabuchi functional* on the set of normal, nef test configurations to take the value

$$M((\mathcal{X}, \mathcal{B}); \mathcal{L}) = \mathrm{Ent}(\mathcal{X}, \mathcal{L}) + \nabla_{K_X+B} E_L(\mathcal{X}, \mathcal{L}),$$

where $((\mathcal{X}, \mathcal{B}); \mathcal{L})$ is such a test configuration.

This is often called the *non-Archimedean* Mabuchi functional; as its Archimedean counterpart plays no role in the present work, we simplify the terminology. It agrees with the more traditional *Donaldson–Futaki invariant* of a normal, nef test configuration provided $\mathcal{X}_0$ is reduced; the associated notions of stability – which we next define – can be seen to be equivalent by a base-change argument [9, Proposition 7.15]. The directional derivative involved is defined by

$$\nabla_{K_X+B} E_L(\mathcal{X}, \mathcal{L}) = \frac{d}{dt}\Big|_{t=0} \frac{(\mathcal{L} + t(K_X + B))^{n+1}}{(n+1)(L + t(K_X + B))^n},$$

where we assume (as we may) that the test configuration dominates the trivial one and $K_X + B$ is also then used to denote its pullback to $\mathcal{X}$; this derivative can be computed explicitly to produce a version of the Mabuchi functional more commonly used in the literature.

Definition 2.13. We say that $((X, B); L)$ is

(i) *K-semistable* if for all normal, nef test configurations $((\mathcal{X}, \mathcal{B}); \mathcal{L})$ for $((X, B); L)$, we have $M(\mathcal{X}, \mathcal{L}) \geq 0$;
(ii) *uniformly K-stable* if there exists an $\varepsilon > 0$ such that for all normal, nef test configurations $((\mathcal{X}, \mathcal{B}); \mathcal{L})$ for $((X, B); L)$, we have $M((\mathcal{X}, \mathcal{B}); \mathcal{L}) \geq \varepsilon \|(\mathcal{X}, \mathcal{L})\|_{\min}$.

The relationship between uniform K-stability and K-stability is as follows: uniform K-stability with respect to test configurations *with irreducible central fibre* is equivalent to uniform valuative stability *with respect to dreamy prime divisors* [15, Theorem 1.1]. Roughly speaking, the central fibre of a test configuration with *irreducible central fibre* induces a prime divisor over X, and conversely, under a finite generation hypothesis, the reverse of this construction also succeeds; the beta invariant is defined in such a way that it equals the value of the Mabuchi functional at the associated test configuration. To obtain stronger results – giving a fuller valuative interpretation of K-stability, in particular allowing test configurations with reducible central fibres – one needs further tools from non-Archimedean geometry, which we now turn to.

2.3. Berkovich analytification

The appropriate way of viewing convex combinations of divisorial valuations is as a certain type of measure on the *Berkovich analytification* X^{an} of X, which we now define. As throughout, we assume that X is a normal projective variety defined over an algebraically closed field of characteristic zero; the boundary divisor B will be irrelevant in the present section. We refer to Reboulet [37, Section 2] or Boucksom–Jonsson [11] for further details and proofs of the results stated below.

For our purposes, it will be sufficient to define X^{an} as a topological space; we note that it naturally carries the richer structure of a locally ringed space. We endow the field k with the trivial absolute value.

Definition 2.14. As a set, we define the *Berkovich analytification* X^{an} of X to be the set of pairs $(V, |\cdot|_V)$, where V is an irreducible subvariety of X and $|\cdot|_V$ is an absolute value on the function field $k(V)$ extending the (trivial) absolute value on k.

As a topological space, we first consider an affine chart $U \subset X$, where we define a topology on U^{an} by requiring that for all $f \in \mathcal{O}_X(U)$, the evaluation map

$$f : U^{\mathrm{an}} \to \mathbb{R}$$

defined by $f(V, |\cdot|_V) = |f|_V$ be continuous. These topologies agree on intersections of affine charts of X, and hence glue to a topology on X^{an} which is compact and Hausdorff. The association $X \to X^{\mathrm{an}}$ is functorial, in the sense that a morphism $Y \to X$ of projective varieties induces a morphism of analytic spaces $Y^{\mathrm{an}} \to X^{\mathrm{an}}$.

If we take $V = X$, then we simply obtain the function field of X. The space $X^{\mathrm{val}} \subset X^{\mathrm{an}}$ of valuations on X is then an open dense subset of X^{an}, and X^{an} is thus a compactification of X^{val}. We further denote X^{div} the space of divisorial valuations on X, so that $X^{\mathrm{div}} \subset X^{\mathrm{val}} \subset X^{\mathrm{an}}$.

The $\mathbb{Q}$-line L bundle on X induces a $\mathbb{Q}$-line bundle L^{an} on X^{an}; rather than L^{an} itself, what will be important is the space of *non-Archimedean metrics* on L^{an}. We will give a shallow treatment of non-Archimedean metrics, omitting how to view Fubini–Study metrics as genuine metrics (in the sense of

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assigning a nonnegative number to a section at a point) and instead viewing them as certain *functions* on X^{an}. In Kähler geometry, the *quotient* of two Hermitian metrics can be viewed as a function, and our treatment is reasonable due to the presence of the *trivial metric* in the non-Archimedean setting of interest here. Here, the trivial metric φ_{triv} is induced by the trivial test configuration $(X \times \mathbb{A}^1, L)$, with X given the trivial $\mathbb{G}_m$-action.

To define a function on X^{an} associated to a test configuration, we may first assume that $(\mathcal{X}, \mathcal{L})$ dominates the trivial test configuration by passing to a $\mathbb{G}_m$-equivariant resolution of indeterminacy of the rational map $X \times \mathbb{A}^1 \dashrightarrow \mathcal{X}$ if necessary. We can thus write $\mathcal{L} - L = D$ (with L the pullback of L from $X \times \mathbb{A}^1$ to $\mathcal{X}$), where D is a $\mathbb{Q}$-Cartier divisor supported on $\mathcal{X}_0$. Writing $\mathcal{X}_0 = \sum_j b_j E_j$ as a cycle, so that the $E_j \subset \mathcal{X}_0$ are reduced and irreducible, this function is given as

$$\varphi_{(\mathcal{X},\mathcal{L})}(v_{a\,\mathrm{ord}\,E_j}) = b_j^{-1} a\,\mathrm{ord}_{E_j}(D), \tag{2.1}$$

with the function vanishing elsewhere. One checks that if $p : \mathcal{X}' \to \mathcal{X}$ is a $\mathbb{G}_m$-equivariant morphism with $(\mathcal{X}', p^*\mathcal{L})$ and $(\mathcal{X}, \mathcal{L})$ test configurations for (X, L), then $\varphi_{(\mathcal{X}', p^*\mathcal{L})} = \varphi_{(\mathcal{X},\mathcal{L})}$. We always identify $\varphi_{(\mathcal{X},\mathcal{L})}$ and $\varphi_{(\mathcal{X}',\mathcal{L}')}$ if the associated functions on X^{an} are equal. Finally, note that that $\varphi_{(\mathcal{X},\mathcal{L})}$ is supported on X^{val}.

Definition 2.15. We define a *Fubini–Study metric* to be a metric $\varphi_{(\mathcal{X},\mathcal{L})}$ associated to a nef test configuration $(\mathcal{X}, \mathcal{L})$. We denote by $\mathcal{H}^{\mathrm{NA}}(L^{\mathrm{an}})$ the set of Fubini–Study metrics on L^{an}.

When L is clear from context, we denote this simply by $\mathcal{H}^{\mathrm{NA}}$. We next define *flag ideals* which allow us to obtain a more concrete picture on the relationship between Fubini–Study metrics and test configurations. Studied in [9, 35], we refer to [11, Section 2.1] for further details.

Definition 2.16 [11, Section 2.1]. We define a *flag ideal* to be a vertical fractional ideal sheaf $\mathfrak{a}$ on $X \times \mathbb{A}^1$ (i.e., a $\mathbb{G}_m$-invariant, coherent fractional ideal sheaf on $X \times \mathbb{A}^1$ that is trivial on $X \times \mathbb{G}_m$).

Here, a vertical ideal sheaf by definition satisfies the condition that $\mathcal{O}_{X\times\mathbb{A}^1}/\mathfrak{a}$ is supported on $X \times \{0\}$. Fractional ideal sheaves are included as in the definition of a test configuration we allow $\mathcal{L}$ to be a $\mathbb{Q}$-line bundle. Any flag ideal admits a decomposition

$$\mathfrak{a} = \sum_{\lambda \in \mathbb{Z}} t^{-\lambda} \mathfrak{a}_\lambda,$$

where t is the coordinate of $\mathbb{A}^1$ and $\mathfrak{a}_\lambda \subset \mathcal{O}_X$ is a non-increasing sequence of integrally closed ideals on X with $\mathfrak{a}_\lambda = 0$ for $\lambda \gg 0$ and $\mathfrak{a}_\lambda = \mathcal{O}_X$ for $\lambda \ll 0$. In addition, every test configuration $\mathcal{X}$ dominating the trivial one is given as the blowup

$$\mathcal{X} = \mathrm{Bl}_{\mathfrak{a}}\, X \times \mathbb{P}^1,$$

where with D the exceptional divisor, we define $\mathcal{L} := L + D$. Under this correspondence, the function $\varphi_{(\mathcal{X},\mathcal{L})} = \varphi_{\mathfrak{a}}$ satisfies the property [11, Equation 2.4]

$$\varphi_{\mathfrak{a}\cdot\mathfrak{a}'} = \varphi_{\mathfrak{a}} + \varphi_{\mathfrak{a}'}, \tag{2.2}$$

where we use that the product of two flag ideals is a flag ideal.

Much as in Kähler geometry, it is also helpful to consider *singular* metrics.

Definition 2.17. We define a *plurisubharmonic metric* (or *psh metric*) on L^{an} to be a (pointwise) decreasing net of Fubini–Study metrics on L^{an}.

Example 2.18. For two Fubini–Study metrics $\varphi_{(\mathcal{X},\mathcal{L})}, \varphi_{(\mathcal{X}',\mathcal{L}')}$, the condition

$$\varphi_{(\mathcal{X}',\mathcal{L}')} \geq \varphi_{(\mathcal{X},\mathcal{L})}$$

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means that we can find a $\mathbb{G}_m$-equivariant birational model $\mathcal{Y}$ with morphisms to both $\mathcal{X}, \mathcal{X}'$ such that on $\mathcal{Y}$, the difference $\mathcal{L}' - \mathcal{L}$ of pullbacks to $\mathcal{Y}$ is effective.

Remark 2.19. By [11, Corollary 12.18 (iii)], psh metrics on L^{an} can also be viewed as (pointwise) decreasing limits of *sequences* of Fubini–Study metrics on X^{an}, allowing the avoidance of nets.

One should think that the theory of non-Archimedean geometry allows a language for discussing *sequences* of test configurations and, in particular, for discussing *compactness* properties for sequences of test configurations.

We will use the following extension of the Monge–Ampère energy of a test configuration.

Definition 2.20 [11, Sections 3, 7]. We define the *Monge–Ampère energy* of a Fubini–Study metric $\varphi_{(\mathcal{X},\mathcal{L})}$ to be $E(\varphi_{(\mathcal{X},\mathcal{L})}) = E(\mathcal{X},\mathcal{L})$. We extend this definition to arbitrary psh metrics ψ by setting

$$E(\psi) = \inf\{E(\varphi_{(\mathcal{X},\mathcal{L})}) : \varphi_{(\mathcal{X},\mathcal{L})} \geq \psi\}$$

and define a psh metric to be of *finite energy* if $E(\psi) > -\infty$. We denote by $\mathcal{E}^1(L^{\mathrm{an}})$ the space of finite energy psh metrics on L^{an}, or simply $\mathcal{E}^1$ when the polarisation L is clear from context.

We endow $\mathcal{E}^1(L^{\mathrm{an}})$ with the strong topology: the coarsest refinement of the weak topology (which requires convergence $\varphi_j \to \varphi$ if this holds pointwise) such that the Monge–Ampère energy $E : \mathcal{E}^1(L^{\mathrm{an}}) \to \mathbb{R}$ is continuous. With this topology, Fubini–Study metrics are dense in $\mathcal{E}^1(L^{\mathrm{an}})$.

Proposition 2.21 [11, Proposition 7.7 (i)]. *The Monge–Ampère energy is continuous along decreasing nets. In particular, if φ_k is a sequence of Fubini–Study metrics decreasing to φ, then $E(\varphi_k)$ converges to $E(\varphi)$.*

2.4. Measures on X^{an}

As we have seen, test configurations are analogous to Fubini–Study metrics in non-Archimedean geometry. In Kähler geometry, it is beneficial to consider volume forms and more general measures. The non-Archimedean construction of a measure associated to a metric is the following. Throughout, if $v = c\,\mathrm{ord}_F$ is a divisorial valuation, viewed as an element of $X^{\mathrm{div}} \subset X^{\mathrm{an}}$, we will denote by $\delta_{c\,\mathrm{ord}_F}$ the Dirac mass (or Dirac measure) supported at $v = c\,\mathrm{ord}_F$.

Definition 2.22 [11, Section 3.2]. Denote by $\mathcal{X}_0 = \sum_j b_j E_j$ the central fibre of a test configuration $(\mathcal{X},\mathcal{L})$ as a cycle, so that the E_j are reduced and irreducible. We define the *Monge–Ampère measure* $\mathrm{MA}(\varphi_{(\mathcal{X},\mathcal{L})})$ of $\varphi_{(\mathcal{X},\mathcal{L})}$ to be

$$\mathrm{MA}(\varphi_{(\mathcal{X},\mathcal{L})}) = \frac{1}{L^n}\sum_j b_j(\mathcal{L}^n.E_j)\delta_{b_j^{-1}\mathrm{ord}_{E_j}}.$$

In the following, we denote by $\mathcal{M}$ the set of Radon probability measures on X^{an} (i.e., the dual space $C^0(X^{\mathrm{an}})^\vee$, which we endow with the weak topology).

Proposition 2.23 [11, Proposition 7.19 (iv)]. *There is a unique extension of the Monge–Ampère measure from Fubini–Study metrics to general finite energy metrics defined in such a way that the map $\varphi \to \mathrm{MA}(\varphi)$ is continuous along decreasing nets.*

The inverse problem – associating a non-Archimedean metric to a measure – is the content of the *non-Archimedean Calabi–Yau theorem* (originating in [8]). We will require a general version of this, which involves *finite norm measures*.

Definition 2.24 [11, Definition 9.1]. The *norm* of a measure $\mu \in \mathcal{M}$ is defined as

$$\|\mu\|_L \coloneqq \sup_{\varphi\in\mathcal{E}^1}\left\{E(\varphi) - \int_{X^{\mathrm{an}}}\varphi\,\mathrm{d}\mu\right\} \in [0,+\infty].$$

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The space $\mathcal{M}^1 \subset \mathcal{M}$ of measures of *finite norm* is defined as the set

$$\mathcal{M}^1 := \{\mu \in \mathcal{M} \mid \|\mu\|_L < \infty\}.$$

The following then allows us to pass freely between measures and non-Archimedean metrics.

Theorem 2.25 [11, Theorem A]. *The Monge–Ampère operator defines a bijection*

$$\mathcal{E}^1(L^{\mathrm{an}})/\mathbb{R} \to \mathcal{M}^1,$$

where $\mathcal{E}^1(L^{\mathrm{an}})/\mathbb{R}$ denotes finite energy metrics modulo the addition of constants. Furthermore, given a measure μ, if $\mathrm{MA}(\varphi) = \mu \in \mathcal{M}^1$ and $\int_{X^{\mathrm{an}}} \varphi \, \mathrm{d}\mu = 0$, then

$$\|\mu\|_L = E(\varphi).$$

Remark 2.26. The supremum defining the norm of a measure can be taken over $\mathcal{H}^{\mathrm{NA}}(L^{\mathrm{an}})$; a benefit of considering the full space $\mathcal{E}^1(L^{\mathrm{an}})$ is that by Theorem 2.25, there is a $\varphi \in \mathcal{E}^1(L^{\mathrm{an}})$ actually achieving the supremum.

The bijection $\mathcal{E}^1(L^{\mathrm{an}})/\mathbb{R} \to \mathcal{M}^1$ – induced by the non-Archimedean Calabi–Yau theorem – can be upgraded to a homeomorphism if $\mathcal{M}^1$ is given the strong topology, though this will not be used in the present work. We will, however, use a further differentiability result in associating numerical invariants to measures:

Theorem 2.27 [10, Theorem A]. *Fix a measure $\mu \in \mathcal{M}^1$, and denote $\mathrm{Amp}_{\mathbb{Q}}(X)$ the space of ample $\mathbb{Q}$-divisors modulo numerical equivalence. Then the function $\mathrm{Amp}_{\mathbb{Q}}(X) \to \mathbb{R}$ defined by*

$$L \longrightarrow \|\mu\|_L$$

extends by continuity to a function on $\mathrm{Amp}_{\mathbb{R}}(X)$ which is continuously differentiable.

For an $\mathbb{R}$-divisor H, we denote

$$\nabla_H \|\mu\|_L := \frac{d}{dt}\bigg|_{t=0} \|\mu\|_{L+tH}$$

the resulting directional derivative.

A more well-behaved subspace of $\mathcal{M}^1$ will be sufficient for our purposes.

Definition 2.28. We define a *divisorial measure* on X^{an} to be a probability measure of the form

$$\mu = \sum_j a_j \delta_{v_j}$$

for some finite collection v_j of divisorial valuations on X, so in particular, $\sum_{j=0}^m a_j = 1$. We denote by $\mathcal{M}^{\mathrm{div}}$ the set of divisorial measures.

Example 2.29. Any divisorial valuation canonically induces a divisorial measure. Further, the Monge–Ampère measure of any Fubini–Study metric is a divisorial measure. Thus, divisorial measures can be viewed as a simultaneous generalisation of divisorial valuations and test configurations.

2.5. Uniform K-stability on $\mathcal{E}^1$

With the construction of the Monge–Ampère measure of a finite energy metric φ in hand, we may extend the uniform K-stability from test configurations to $\mathcal{E}^1$. First, it is easily checked that the value taken by

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the Mabuchi functional at a test configuration depends only on the associated Fubini–Study metric (and similarly the minimum norm has the same property); we denote the resulting functional

$$M : \mathcal{H}^{\mathrm{NA}} \to \mathbb{R}.$$

We extend the Mabuchi functional in the following manner. For this, we first recall that there is a natural way to extend the log discrepancy function $A_{(X,B)} : X^{\mathrm{div}} \to \mathbb{R}$ (which is nonnegative by definition if (X, B) has at worst log canonical singularities) to a function

$$A_{(X,B)} : X^{\mathrm{NA}} \to \mathbb{R},$$

using semicontinuity of A_X on X^{div} [10, Definition A.2].

Integration against the Monge–Ampère measure associated to $((\mathcal{X}, \mathcal{B}); \mathcal{L})$ produces [9, Corollary 7.18]

$$\int_{X^{\mathrm{an}}} A_{(X,B)} \mathrm{MA}(\varphi_{(\mathcal{X},\mathcal{L})}) = \mathrm{Ent}(\mathcal{X}, \mathcal{L});$$

we denote

$$\mathrm{Ent}(\varphi_{(\mathcal{X},\mathcal{L})}) = \mathrm{Ent}(\mathcal{X}, \mathcal{L}).$$

Second, the functional defined on $\mathcal{H}^{\mathrm{NA}}$ by

$$\varphi \to \nabla_{K_X+B} E_L(\varphi)$$

extends in a continuous manner to $\mathcal{E}^1$, essentially because it contains only Monge–Ampère energy (and 'mixed Monge–Ampère energy') terms [11, Theorem 7.14], producing a natural extension to a functional

$$M : \mathcal{E}^1 \to \mathbb{R}$$

taking the form

$$M(\varphi) = \mathrm{Ent}(\varphi) + \nabla_{K_X+B} E_{L_X}(\varphi).$$

The minimum norm similarly extends by continuity to a functional on $\mathcal{E}^1$.

Definition 2.30. We say that $((X, B); L)$ is *uniformly K-stable on* $\mathcal{E}^1$ if there is an $\varepsilon > 0$ such that for all $\varphi \in \mathcal{E}^1$, we have

$$M(\varphi) \geq \varepsilon \|\varphi\|_{\min}.$$

This condition is equivalent to Boucksom–Jonsson's notion of uniform K-stability with respect to filtrations and Li's notion of uniform K-stability with respect to models; see Li [26, Section 2.1.3] for a discussion and further details.

2.6. Divisorial stability

We are now in a position to associate numerical invariants to divisorial *measures* (rather than metrics), and hence to define divisorial stability, following Boucksom–Jonsson [10]. We begin with the entropy of $((X, B); L)$, which extends the log discrepancy of a single divisorial valuation to a general divisorial measure.

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Definition 2.31. We define the *entropy* $\mathrm{Ent}_{(X,B)} : \mathcal{M}^{\mathrm{div}} \to \mathbb{R}$ to be

$$\mathrm{Ent}_{(X,B)}(\mu) = \int_{X^{\mathrm{an}}} A_{(X,B)}\, \mathrm{d}\mu,$$

where μ is a divisorial measure.

Writing $\mu = \sum_j a_j \delta_{v_j}$, the entropy is given explicitly as

$$\mathrm{Ent}(\mu) = \sum_j a_j A_{(X,B)}(v_j).$$

Note that the entropy is independent of the ample line bundle L. This allows us to define the *beta invariant* of a divisorial measure on $((X,B);L)$.

Definition 2.32 [10, Definition 4.1]. The *beta invariant* of a divisorial measure $\mu \in \mathcal{M}^{\mathrm{div}}$ is defined to be

$$\beta_{((X,B);L)}(\mu) := \mathrm{Ent}_{(X,B)}(\mu) + \nabla_{K_X+B} \|\mu\|_L.$$

This allows us to define divisorial stability.

Definition 2.33 [10, Definition 4.3]. We say that $((X,B);L)$ is

(i) *divisorially semistable* if for all divisorial measures μ on X^{an}, we have $\beta(\mu) \geq 0$ on $\mathcal{M}^{\mathrm{div}}$;
(ii) *divisorially stable* if there exists an $\varepsilon > 0$ such that for all divisorial measures μ on X^{an}, we have $\beta(\mu) \geq \varepsilon \|\mu\|_L$.

We may extend the beta invariant of a divisorial measure to a general finite norm measure in a way analogous to the extension of the Mabuchi functional to $\mathcal{E}^1$: the entropy is defined in the same way for divisorial and finite norm measures, whereas the norm itself remains differentiable (in the polarisation) for a general finite energy measure [10, Theorem 2.15], meaning we can define

$$\beta(\mu) = \mathrm{Ent}_{(X,B)}(\mu) + \nabla_{K_X+B} \|\mu\|_L$$

for $\mu \in \mathcal{E}^1$. The resulting notion of stability is equivalent to divisorial stability by continuity of the various quantities in the measure.

Example 2.34. If $\varphi \in \mathcal{E}^1$, then Boucksom–Jonsson prove the key equality [10, Equation (4.5)]

$$M(\varphi) = \beta(\mathrm{MA}(\varphi)),$$

which implies that divisorial stability is *equivalent* to uniform K-stability on $\mathcal{E}^1$ (using also the non-Archimedean Calabi–Yau theorem); in particular, divisorial stability *implies* uniform K-stability. If instead $\mu = \delta_{v_F}$ for a divisorial valuation v_F on X associated to a prime divisor F over X, then $\beta(\mu) = \beta(F)$ with $\beta(F)$ the β-invariant of Definition 2.5, and relatedly, $S_L(F) = \|\mu\|_L$ [10, Theorem 2.18]. Thus, the β-invariant of finite norm measures (in particular, divisorial measures) can be seen as a simultaneous generalisation of the β-invariant of divisorial valuations and the Mabuchi functional on the set of Fubini–Study metrics (in particular, test configurations).

2.7. Equivariant divisorial stability

Consider now a finite group G acting on the projective variety X. Since G acts on X, it acts on the function field of X and hence on X^{div} by setting $(g(v))(f) = v(g^* f)$.

Definition 2.35. We say that a divisorial measure $\mu = \sum_j a_j \delta_{v_j}$ is *G-invariant* if for all $g \in G$,

$$\mu = \sum_j a_j \delta_{g(v_j)}.$$

We denote the space of G-invariant divisorial measures by $\mathcal{M}_Y^{\mathrm{div},G}$.

We will consider pushforwards of measures in Section 3.3, where we will show that this condition asks $g_*\mu = \mu$ for all $g \in G$.

We are now in a position to introduce the notion of *G-equivariant divisorial stability*.

Definition 2.36. We say that $((X, B); L)$ is

(i) *G-equivariantly divisorially semistable* if for all G-invariant divisorial measures μ on X^{an}, we have $\beta(\mu) \geq 0$ on $\mathcal{M}^{\mathrm{div}}$;
(ii) *G-equivariantly divisorially stable* if there exists an $\varepsilon > 0$ such that for all G-invariant divisorial measures μ on X^{an}, we have $\beta(\mu) \geq \varepsilon \|\mu\|$.

To compare with equivariant notions of uniform K-stability, we first make the following definition.

Remark 2.37. In Section 3.2, we define pullbacks of metrics under morphisms, which for $g : X \to X$, we denote $g^*\varphi$. Furthermore, in Corollary 3.12, we show that for a G-invariant divisorial measure μ, the sup defining the norm of μ can be taken over *G-invariant* psh metrics – namely,

$$\|\mu\|_L = \sup_{\varphi \in \mathcal{E}^{1,G}} \left\{ E(\varphi) - \int_{X^{\mathrm{an}}} \varphi \, \mathrm{d}\mu \right\}$$

with $\mathcal{E}^{1,G}(L_X^{\mathrm{an}})$ the space of G-invariant finite energy metrics, giving some justification for the definition.

Remark 2.38. We show in Theorem 4.6 that G-equivariant divisorial stability is equivalent to uniform K-stability on $\mathcal{E}^{1,G}$, primarily using the work of Boucksom–Jonsson described in Example 2.34 and some equivariant non-Archimedean geometry. We expect that, analogously to the Fano case [18, 46], G-equivariant divisorial stability and divisorial stability are actually equivalent.

3. Divisorial stability under finite covers

Our next aim is to prove Theorem 1.1, explaining the behaviour of divisorial stability under finite covers. The level of generality of Theorem 1.1 is an arbitrary Galois cover $\pi : Y \to X$ defined as the quotient under a group G, such that $L_Y = \pi^* L_X$ is ample and

$$K_Y + \Delta_Y = \pi^*(K_X + \Delta_X)$$

for effective $\mathbb{Q}$-divisors Δ_Y, Δ_X. To ease notation, we prove this result in the notationally simpler case when G is cyclic and B is an irreducible $\mathbb{Q}$-divisor on X such that by Riemann–Hurwitz,

$$K_Y = \pi^*\left(K_X + \left(1 - \frac{1}{m}\right)B\right).$$

This is the most important special case for applications; the proof in the general case is identical, but it requires an extra summation index at most steps.

More precisely, our setup is the following. We take a normal projective variety Y with a G-action, where G is a finite cyclic group of degree m and where K_Y is $\mathbb{Q}$-Cartier. We let $X = Y/G$ be the quotient of X by G, write $\pi : Y \to X$ for the resulting quotient map and let $B \subset X$ be the branch divisor.

It follows that $K_X + \left(1 - \frac{1}{m}\right)B$ is $\mathbb{Q}$-Cartier and satisfies

$$K_Y = \pi^*\left(K_X + \left(1 - \frac{1}{m}\right)B\right)$$

by Riemann–Hurwitz. We assume that G lifts to an action on an ample $\mathbb{Q}$-line bundle L_Y on Y, and we let L_X be its quotient, so that $\pi^* L_X = L_Y$.

3.1. Finite maps between analytifications

The map $\pi : Y \to X$ induces a map $\pi^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ defined by

$$\pi^{\mathrm{an}}(V, |\cdot|_V) = (\pi(V), |\cdot|_{\pi(V)});$$

we begin by giving a more explicit geometric description of this map on divisorial valuations. For a divisorial valuation, V is simply taken to be Y, so since π is surjective, we obtain a valuation on X from one on Y. Recall in general that the image of a valuation v on Y is defined for a rational function $f \in k(X)$ by setting

$$\pi(v)(f) = v(\pi^* f).$$

Proposition 3.1. *Let $u = c\,\mathrm{ord}_F \in Y^{\mathrm{div}}$ be a divisorial valuation on Y. Then F can be realised as a prime divisor on a birational model Y' of Y such that $Y' \to Y$ is G-equivariant. Further, denoting $X' = Y'/G$ and denoting D the image of F in X', then $\pi(u)$ is the divisorial valuation associated to $e_F c\,\mathrm{ord}_D$, where e_F denotes the ramification index of $Y' \to X'$ along F.*

Proof. We begin by replacing an arbitrary birational model Y' of Y with a birational model $Y'' \to Y$, which admits a lift of the G-action, in such a way that the morphism $Y'' \to Y$ is G-equivariant. It suffices to construct Y'' as a blowup of Y along a G-invariant subscheme of Y, since, in this case, the G-action lifts automatically by the universal property of blowups.

Since Y' is birational to Y, we may write $Y' = \mathrm{Bl}_{\mathcal{I}} Y$, where $\mathcal{I}$ is an ideal sheaf. We consider the orbit

$$\mathcal{I} \cdot g^{-1}\mathcal{I} \cdot \ldots \cdot (g^{m-1})^{-1} \cdot \mathcal{I} \subset Y$$

of $\mathcal{I}$, which is a *G-invariant* ideal sheaf (and where $g^{-1}\mathcal{I}$ denotes the inverse image of $\mathcal{I}$ under g). Letting $Y'' := \mathrm{Bl}_{\mathcal{I} \cdot g^{-1}\mathcal{I} \cdot \ldots \cdot (g^{m-1})^{-1} \cdot \mathcal{I}} Y$, by [33, Corollary 1] (namely, we use that the blowup of a product of ideal sheaves is the successive blowup of one factor and then the total transform of the other factor), we obtain birational morphisms $Y'' \to Y' \to Y$, and by construction, Y'' admits a G-action. Thus, we replace $F \subset Y'$ with its proper transform on Y'', which does not modify the divisorial valuation itself.

As we now assume Y' admits a G-action making the morphism $Y' \to Y$ a G-equivariant morphism, we may take the quotient Y'/G of Y' by G; we define $X' = Y'/G$. We then have a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\pi'} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\pi} & X, \end{array}$$

since $Y' \to X$ is G-invariant.

Setting $D = \pi'(F)$, it follows, for example, from [38, Exercise 2.2] that

$$\mathrm{ord}_F((\pi')^* f) = e_F\,\mathrm{ord}_D(f),$$

where e_F is the ramification index of $Y' \to X'$ along F and $f \in k(X)$. This completes the proof by the definition of the image of a valuation. $\square$

3.2. Pullbacks and pushforwards of metrics under finite covers

Our next goal is define pushforwards and pullbacks of metrics in order to relate G-invariant Fubini–Study metrics on Y to Fubini–Study metrics on X.

We recall the definition of the pullback of a psh metric, as defined by Boucksom–Jonsson [11, Proposition 3.6]. We begin with a Fubini–Study metric φ on L^{an} induced by a test configuration $(\mathcal{X}, \mathcal{L}_{\mathcal{X}})$ for (X, L_X). The $\mathbb{G}_m$-equivariant rational map $Y \times \mathbb{A}^1 \dashrightarrow \mathcal{X}$ induced by π admits a $\mathbb{G}_m$-equivariant resolution of indeterminacies, inducing a test configuration $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ for (Y, L_Y), where $\mathcal{L}_{\mathcal{Y}}$ is the pullback of $\mathcal{L}_{\mathcal{X}}$ through the morphism $\mathcal{Y} \to \mathcal{X}$; this test configuration dominates $Y \times \mathbb{A}^1$ by construction.

Definition 3.2. We define the *pullback* of the Fubini–Study metric φ on L_X^{an} associated to a test configuration $(\mathcal{X}, \mathcal{L})$ to be the Fubini–Study metric on L_Y^{an} induced by the test configuration $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$.

The pullback extends to arbitrary psh metrics by an approximation argument.

Definition 3.3. We say that a psh metric φ on L_X^{an} is *G-invariant* if $g^*\varphi = \varphi$ for all $g \in G$. We denote by $\mathcal{E}^{1,G}(L_X^{\mathrm{an}})$ the space of G-invariant finite energy metrics.

We next define pushforwards of G-invariant Fubini–Study metrics in a similar spirit. Let φ be a G-invariant Fubini–Study metric φ on L_Y^{an}, which hence corresponds to a test configuration $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$, which can be taken to dominate the trivial test configuration. We will show in Proposition 3.5 that $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ can be taken to be G-invariant, in the sense that it admits a G-action inducing the fixed action on Y and commuting with the $\mathbb{G}_m$-action. Then by [13, Lemma 3.1], taking the quotient of $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ by the G-action induces a test configuration $(\mathcal{X}, \mathcal{L}_{\mathcal{X}})$ for (X, L_X) such that $\pi^*\mathcal{L}_{\mathcal{X}} = \mathcal{L}_{\mathcal{Y}}$, where π is the quotient map; by construction, $(\mathcal{X}, \mathcal{L}_{\mathcal{X}})$ dominates the trivial test configuration provided $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ dominates the trivial test configuration.

Definition 3.4. We define the *pushforward* of a G-invariant Fubini–Study metric φ on L_Y^{an} corresponding to a G-invariant test configuration $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ to be the Fubini–Study metric on L_X^{an} associated to the test configuration obtained as the quotient of $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ by G.

We next prove that a G-invariant psh metric φ on L_Y^{an} (which is a decreasing limit of Fubini–Study metrics) is a decreasing limit of G-invariant Fubini–Study metrics φ_k and that these G-invariant Fubini–Study metrics can further be taken to be associated to G-invariant test configurations.

Proposition 3.5. *Every G-invariant psh metric can be realised as a decreasing limit of G-invariant Fubini–Study metrics on L_Y^{an} associated to explicitly defined G-invariant test configurations.*

Proof. Let φ be a G-invariant psh metric on Y^{an} with finite energy. By Remark 2.19, we may realise φ as a decreasing limit of Fubini–Study metrics φ_k associated to test configurations $(\mathcal{Y}_k, \mathcal{L}_k)$ for (Y, L_Y), which we may assume dominate the trivial test configuration. We define

$$\varphi_k^G = \sum_{g \in G} \frac{1}{|G|} g^* \varphi_k,$$

which is, by definition, a G-invariant function. This is a convex combination of psh metrics, and is hence itself psh [11, Theorem 4.7 (ii)]; we will see positivity more explicitly in what follows.

We will first show that $\varphi_k^G \to \varphi$. Since φ is the decreasing limit of the Fubini–Study metrics φ_k, for a point $y \in Y^{\mathrm{an}}$, $\varphi_k(g(y))$ decreases to $\varphi(g(y)) = \varphi(y)$, where we have used that φ is G-invariant.

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Hence, $\varphi_k(y)^G = \varphi_k^G(g(y))$ decreases to

$$\sum_{g\in G} \frac{1}{|G|} g^*\varphi = \varphi.$$

We next give an explicit description of φ_k^G as a Fubini–Study metric, for which we employ flag ideals. As φ_k is Fubini–Study, it corresponds to a nef test configuration $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$, and in addition, $\mathcal{Y} = \mathrm{Bl}_{\mathfrak{a}} X \times \mathbb{P}^1$ for a flag ideal $\mathfrak{a}$. Using a similar idea to the proof of Proposition 3.1, we define

$$\mathfrak{a}^G := \mathfrak{a} \cdot (g^{-1}\mathfrak{a}) \cdot \ldots \cdot ((g^{m-1})^{-1}\mathfrak{a}).$$

Notice that $\mathfrak{a}^G$ is a flag ideal on $Y \times \mathbb{P}^1$, and set

$$\mathcal{Y}^G = \mathrm{Bl}_{\mathfrak{a}^G} Y \times \mathbb{P}^1,$$

which admits a G-action by construction. By [11, Theorem 2.7, Proposition 3.6], each $g^*\varphi_k$ is Fubini–Study with associated flag ideal $g^*\varphi_k = \varphi_{g^{-1}\mathfrak{a}}$, where $g^{-1}\mathfrak{a}$ denotes the inverse image of $\mathfrak{a}$ under g. By Equation (2.2), the product of ideal sheaves defining $\mathfrak{a}^G$ corresponds precisely to the sum

$$\varphi_{\mathfrak{a}^G} = \sum_{g\in G} g^*\varphi_k.$$

Thus, φ_k^G is associated to the flag ideal $\mathfrak{a}^G$.

To understand the line bundle $\mathcal{L}^G$ on $\mathcal{Y}^G$ associated to the Fubini–Study metric φ_k^G, note first that $\mathcal{Y}^G$ admits a morphism to the test configuration associated to $g^{j*}\varphi_k$ for each j in the same way as the proof of Proposition 3.1. The pullback metric $g^*\varphi_{\mathcal{L}_k}$ can then be represented on $\mathcal{Y}^G$ itself through the pullback line bundle $g^*\mathcal{L}$ on $\mathcal{Y}^G$, since pullback of Fubini–Study metrics is defined through pulling back line bundles. Thus, φ_k^G corresponds to the test configuration $(\mathcal{Y}^G, \mathcal{L}^G)$, where

$$\mathcal{L}^G = \sum_{g\in G} \frac{1}{|G|} g^*\mathcal{L};$$

note $\mathcal{L}^G$ is relatively nef as $\mathcal{L}$ is so, and similarly, relatively semiample provided $\mathcal{L}$ is so (this can alternatively be obtained from [11, Proposition 2.25]). Since $\mathcal{L}$ can be viewed as a G-invariant $\mathbb{Q}$-Cartier divisor, it admits a lift of the G-action, meaning we have represented φ_k^G by a G-invariant test configuration $(\mathcal{Y}^G, \mathcal{L}^G)$.

Thus, any G-invariant psh metric is a decreasing limit of G-invariant Fubini–Study metrics induced by G-invariant test configurations, as claimed. □

We next relate pushforwards and pullbacks.

Proposition 3.6. *The pushforward and pullback define an energy-preserving bijection between the set of G-invariant Fubini–Study metrics on L_Y^{an} and Fubini–Study metrics on L_X^{an}.*

Proof. Consider by Proposition 3.5 a G-invariant Fubini–Study metric φ on L_Y^{an} which has an associated G-invariant test configuration $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$, so that by definition of the pushforward, $\pi_*\varphi$ is associated to the quotient test configuration $(\mathcal{X}, \mathcal{L}_{\mathcal{X}})$. To prove that the pushforward and pullback are mutual inverses, it thus suffices to prove that

$$\pi^*\pi_*\varphi = \varphi.$$

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By definition, the pullback $\pi^*\pi_*\varphi$ is the G-invariant Fubini–Study metric on Y^{an} corresponding to the G-invariant test configuration $(\mathcal{Y}', \mathcal{L}_{\mathcal{Y}'})$, where $\mathcal{Y}' = \mathcal{X} \times_{X\times\mathbb{P}^1} Y \times \mathbb{P}^1$ is the fibre product

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & Y \times \mathbb{P}^1 \\ \Big\downarrow{\scriptstyle p_{\mathcal{X}}} & & \Big\downarrow{\scriptstyle \pi} \\ \mathcal{X} & \longrightarrow & X \times \mathbb{P}^1; \end{array}$$

we set $\mathcal{L}_{\mathcal{Y}'} = p_{\mathcal{X}}^*\mathcal{L}_{\mathcal{X}}$. By the universal property of the fibre product $\mathcal{Y}'$, there is an induced morphism $\rho\colon \mathcal{Y} \to \mathcal{Y}'$, which then satisfies $\rho^*\mathcal{L}_{\mathcal{Y}'} = \mathcal{L}_{\mathcal{Y}}$. It follows that the non-Archimedean metrics associated to $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ and $(\mathcal{Y}', \mathcal{L}_{\mathcal{Y}'})$ are equal, proving that pullback and pushforward induce a bijection between G-invariant Fubini–Study metrics on L_Y^{an} and Fubini–Study metrics on L_X^{an}.

We finally prove that this bijection is energy-preserving. Denote by $(\mathcal{X}, \mathcal{L}_{\mathcal{X}})$ a test configuration associated to $\varphi_{(\mathcal{X},\mathcal{L}_{\mathcal{X}})}$, and denote $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ the test configuration associated to the pullback $\pi^*\varphi_{(\mathcal{X},\mathcal{L}_{\mathcal{X}})}$ defined through constructing an equivariant resolution of indeterminacy of $Y\times\mathbb{A}^1 \dashrightarrow \mathcal{X}$. Let $p : \mathcal{Y} \to \mathcal{X}$ be the resulting morphism. We calculate

$$\begin{aligned} E(\varphi_{(\mathcal{X},\mathcal{L}_{\mathcal{X}})}) &= \frac{(\mathcal{L}_{\mathcal{X}})^{n+1}}{(n+1)(L_X)^n}, \\ &= \frac{(p^*\mathcal{L}_{\mathcal{X}})^{n+1}}{(n+1)(\pi^*L_X)^n}, \\ &= \frac{(\mathcal{L}_{\mathcal{Y}})^{n+1}}{(n+1)(L_Y)^n}, \\ &= E(\varphi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})}), \end{aligned}$$

which shows that $E(\varphi_{(\mathcal{X},\mathcal{L}_{\mathcal{X}})}) = E(\varphi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})})$. A similar calculation shows for a G-invariant Fubini–Study metric $\varphi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})}$ on L_Y^{an} that

$$E(\varphi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})}) = E(\pi_*\varphi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})}),$$

proving the result. □

Remark 3.7. By continuity of the Monge–Ampère energy stated as Proposition 2.21, the pullback of general psh metrics also preserves the Monge–Ampère energy.

3.3. *Pullbacks and pushforwards of measures under finite covers*

We next relate divisorial measures on Y to those on X. We begin by recalling the explicit construction of divisorial valuations on X from those on Y. Given a prime divisor $F \subset Y' \to Y$ over Y, by Proposition 3.1, we may assume that Y' admits a G-action, meaning we may form the quotient $X' = Y'/G$. We denote by $\pi(F)$ the prime divisor over X given by the image of F under the morphism $Y' \to X'$. Proposition 3.1 then shows that the image of the divisorial valuation $c\,\mathrm{ord}_F$ under the map $Y^{\mathrm{val}} \to X^{\mathrm{val}}$ is the divisorial valuation $e_F c\,\mathrm{ord}_{\pi(F)}$, where e_F is the ramification index of $Y' \to X'$ along F.

Let $D = \pi(F)$ be a prime divisor over Y. Then π^*D, the pullback cycle, takes the form

$$\pi^*D = \sum_{F_j \in \pi^{-1}(D)} e_{F_j} F_j,$$

with e_{F_j} the ramification index along F_j. The ramification indices are equal for all such F_j, so in this expression, $e_{F_j} = e_{F_l}$ for all j, l. In addition, the divisors F_j and F_l belong to the same G-orbit, in the sense that for all j, l, there exists a $g \in G$ such that $F_j = g(F_l)$.

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Restating Definition 2.35 through the explicit interpretation of the image of a divisorial valuation, a divisorial measure

$$\mu = \sum_i a_i \delta_{c_i \operatorname{ord}_{F_i}}$$

is G-invariant if $a_i = a_l$ and $c_i = c_l$ for all i, l such that F_i and F_l lie in the same G-orbit, or equivalently, such that $\pi(F_i) = \pi(F_l) = D$ for a prime divisor D over X. We will use the following notation for G-invariant divisorial measures on Y^{an}:

$$\mathcal{M}_Y^{\mathrm{div},G} \ni \mu = \sum_{D/X} a_D \left(\sum_{F_j \in \pi^{-1}(D)} \delta_{c_D \operatorname{ord}_{F_j}} \right). \tag{3.1}$$

Here, the first sum $\sum_{D/X}$ is taken over all prime divisors D over X and is finite since there is a finite number of nonzero a_D, while the second sum is taken over all divisors F_j over Y in the preimage of D. The coefficients a_D are arbitrary nonnegative coefficients such that $\int_{Y^{\mathrm{an}}} \mathrm{d}\mu = 1$, so that the measure is a probability measure.

We next consider pushforwards and pullbacks of measures. For a divsiorial measure, the pushforward is given explicitly by the following expression.

Lemma 3.8. *If*

$$\mu = \sum_i a_i \delta_{c_i \operatorname{ord}_{F_i}} \in \mathcal{M}_Y^{\mathrm{div}}$$

is a divisorial measure on Y^{an}, then

$$\pi_*\mu = \sum_i a_i \delta_{e_{F_i} c_i \operatorname{ord}_{\pi(F_i)}} \in \mathcal{M}_X^{\mathrm{div}}$$

is a divisorial measure on X^{an}, where e_{F_i} is the ramification index along F_i.

Proof. For a single Dirac mass δ_u supported at a point $u \in Y^{\mathrm{div}}$, for $U \subset X^{\mathrm{an}}$, we have

$$(\pi_*\delta_u)(U) = \delta_{\pi(u)}$$

by definition of the pushfoward. Thus, $\pi_*\delta_u \in \mathcal{M}_X^{\mathrm{div}}$ is a divisorial measure since Proposition 3.1 implies that $\pi(u)$ is itself a divisorial valuation. Writing $u = c \operatorname{ord}_F$, by Proposition 3.1, its image is given explicitly by

$$\pi(c \operatorname{ord}_F) = e_F c \operatorname{ord}_{\pi(F)} .$$

The general case is identical. □

Although there is a canonical pushforward, we must define pullbacks explicitly. For a single divisorial valuation $a \operatorname{ord}_D$, a divisorial valuation has image $a \operatorname{ord}_D$ in X^{an} if and only if it takes the form $e_F^{-1} a \operatorname{ord}(F)$, where $\pi(F) = D$ and e_F is the ramification index along F, since

$$\pi(e_F^{-1} c \operatorname{ord}(F)) = c \operatorname{ord}_D$$

by Proposition 3.1. Writing

$$\pi^* D = \sum_{F_j \in \pi^{-1}(D)} e_F F_j,$$

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where, as before, the ramification indices $e_F := e_{F_j}$ are equal for each j, we define

$$\pi^* \delta_{a \operatorname{ord}_D} = \sum_{F_j \in \pi^{-1}(D)} \frac{e_F}{|G|} \delta_{e_F^{-1} c \operatorname{ord}(F_j)}.$$

Note that $\pi^* \delta_{a \operatorname{ord}_D}$ is still a probability measure since by the orbit-stabiliser theorem,

$$\sum_{F_j \in \pi^{-1}(D)} \frac{e_F}{|G|} = 1.$$

We now define pullbacks of general divisorial measures in a similar way, essentially extending linearly.

Definition 3.9. We define the *pullback* of a divisorial measure

$$\nu = \sum_{D/X} a_D \delta_{c_D \operatorname{ord}_D} \in \mathcal{M}_X^{\mathrm{div}}$$

by

$$\pi^* \nu = \sum_{D/X} a_D \left(\sum_{F_j \in \pi^{-1}(D)} \frac{e_{F_j}}{|G|} \delta_{e_{F_j}^{-1} c_D \operatorname{ord}(F_j)} \right).$$

As before, the sum D/X denotes a finite sum of prime divisors over X. As in the case when μ is a Dirac mass at a single divisorial valuation, it follows from the orbit-stabiliser theorem that $\pi_* \mu$ is a probability measure.

Proposition 3.10. *The above pushforward and pullback constructions are mutual inverses between the spaces of G-invariant divisorial measures on Y^{an} and divisorial measures on X^{an}.*

In particular, pushforward and pullback induce an isomorphism

$$\mathcal{M}_Y^{\mathrm{div},G} \cong \mathcal{M}_X^{\mathrm{div}}.$$

Proof. We consider a G-invariant divisorial measure $\mu \in \mathcal{M}_Y^{\mathrm{div},G}$ and begin by showing that

$$\pi^* \pi_* \mu = \mu.$$

Continuing the notation used in Equation (3.1), we denote

$$\mu = \sum_{D/X} a_D \left(\sum_{F_j \in \pi^{-1}(D)} \delta_{c_D \operatorname{ord}_{F_j}} \right),$$

where the first sum is taken over a finite sum of prime divisors D over X and the a_D are coefficients such that $\sum_{D/X} a_D = 1$, and the second sum is taken over all divisors F_j in the preimage of D.

We calculate

$$\begin{aligned} \pi_* \mu &= \sum_{D/X} a_D \left(\sum_{F_j \in \pi^{-1}(D)} \delta_{e_{F_j} c_D \operatorname{ord}_D} \right) \\ &= \sum_{D/X} a_D \delta_{e_{F_j} c_D \operatorname{ord}_D} \left(\sum_{F_j \in \pi^{-1}(D)} 1 \right), \end{aligned}$$

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where e_{F_j} is the common ramification index of the F_j, so $\pi^*D = e_{F_j}\sum_j F_j$. Then,

$$\pi^*\pi_*\mu = \sum_{D/X} a_D\left(\sum_{F_l\in\pi^{-1}(D)} \delta_{c_D \operatorname{ord}_{F_l}} \frac{e_{F_l}}{|G|}\left(\sum_{F_j\in\pi^{-1}(D)} 1\right)\right),$$
$$= \sum_{D/X} a_D \sum_{F_l\in\pi^{-1}(D)} \delta_{c_D \operatorname{ord}_{F_l}},$$

where we use that

$$\frac{e_{F_l}}{|G|}\sum_{F_j\in\pi^{-1}(D)} 1 = 1$$

by the orbit-stabiliser theorem and the fact that $e_{F_l} = e_{F_j}$ for all l, j. Thus,

$$\pi^*\pi_*\mu = \mu,$$

as claimed.

In the reverse direction, let $\nu \in \mathcal{M}_X^{\mathrm{div}}$ take the form

$$\nu = \sum_{D/X} a_D \delta_{c_D \operatorname{ord}_D},$$

so that by definition,

$$\pi^*\nu = \sum_{D/X} a_D\left(\sum_{F_j\in\pi^{-1}(D)} \frac{e_{F_j}}{|G|}\delta_{e_{F_j}^{-1} c_D \operatorname{ord}_{F_j}}\right).$$

The pushforward of this measure is then given by

$$\pi_*\pi^*\nu = \sum_{D/X} a_D\left(\sum_{F_j\in\pi^{-1}(D)} \frac{e_{F_j}}{|G|}\delta_{c_D \operatorname{ord}_D}\right),$$
$$= \sum_{D/X} a_D\delta_{c_D \operatorname{ord}_D} \sum_{F_j\in\pi^{-1}(D)} \frac{e_{F_j}}{|G|},$$
$$= \sum_{D/X} a_D\delta_{c_D \operatorname{ord}_D},$$
$$= \nu,$$

where we again use the orbit-stabiliser theorem. This completes the proof. □

We end this section by showing that G-invariant measures of finite norm are given as Monge–Ampère measures of G-invariant psh metrics when G is a finite group.

Proposition 3.11. *Let $G \subset \operatorname{Aut}(Y, L_Y)$ be a finite group, and suppose $\mu \in \mathcal{M}_Y^{\mathrm{div},G}$ is a G-invariant divisorial measure. Then the solution $\varphi \in \mathcal{E}^1(L_Y^{\mathrm{an}})$ of the Monge–Ampère equation*

$$\mathrm{MA}(\varphi) = \mu$$

is a G-invariant metric.

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Proof. Letting $g \in G \subset \mathrm{Aut}(Y, L_Y)$ be such that $g_*\mu = \mu$, we must show that $g^*\varphi = \varphi$. We first claim that it is enough to show that

$$\mathrm{MA}(\varphi) = g_*\mathrm{MA}(g^*\varphi). \tag{3.2}$$

Indeed, by Proposition 3.10,

$$g^*g_*\mathrm{MA}(g^*\varphi) = \mathrm{MA}(g^*\varphi),$$

so by Equation (3.2),

$$\begin{aligned}\mathrm{MA}(g^*\varphi) &= g^*\mathrm{MA}(\varphi),\\ &= g^*\mu,\\ &= \mu,\end{aligned}$$

where we use that μ is G-invariant. Thus, φ and $g^*\varphi$ both solve the Monge–Ampère equation for the measure μ, meaning they must be equal *up to the addition of a constant*, by uniqueness of solutions of the Monge–Ampère equation – namely, Theorem 2.25. Since, for example, φ and $g^*\varphi$ have the same supremum, they must genuinely be equal.

It therefore suffices to prove that for $g \in G \subset \mathrm{Aut}(Y, L_Y)$,

$$\mathrm{MA}(\varphi) = g_*\mathrm{MA}(g^*\varphi).$$

By Remark 2.19, we may realise $\varphi \in \mathcal{E}^1(L_Y^{\mathrm{an}})$ as a decreasing limit of Fubini–Study metrics φ_k, associated to test configurations $(\mathcal{Y}_k, \mathcal{L}_{\mathcal{Y}_k})$, which we may assume dominate the trivial test configuration. As such, we will prove the equality of Equation (3.2) for Fubini–Study metrics, and the result will follow in general by continuity of the Monge–Ampère operator.

Thus, let $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ be a test configuration on (Y, L_Y) dominating the trivial test configuration, associated to a Fubini–Study metric $\psi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})}$, and denote the central fibre of $\mathcal{Y}$ by

$$\mathcal{Y}_0 = \sum_j b_j E_j,$$

where the E_j are reduced and irreducible. By definition of the Monge–Ampère operator,

$$\mathrm{MA}(\psi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})}) = \frac{1}{L_Y^n}\sum_j b_j(\mathcal{L}_{\mathcal{Y}}^n.E_j)\delta_{b_j^{-1}\mathrm{ord}_{E_j}},$$

where $\delta_{b_j^{-1}\mathrm{ord}_{E_j}}$ is the Dirac mass at the divisorial valuation $b_j^{-1}\mathrm{ord}_{E_j} \in Y^{\mathrm{div}} \subset Y^{\mathrm{an}}$.

The morphism g induces by pullback a test configuration $(\mathcal{Y}', \mathcal{L}_{\mathcal{Y}'})$ for (Y, L_Y), where by Proposition 3.5, we may (and do) assume that $\mathcal{Y}' \cong \mathcal{Y}$, but where the line bundle $\mathcal{L}_{\mathcal{Y}'} = g^*\mathcal{L}_{\mathcal{Y}}$ may not agree with $\mathcal{L}_{\mathcal{Y}}$ (indeed, we wish to show $g^*\psi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})} = \psi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})}$, which precisely asks that $g^*\mathcal{L}_{\mathcal{Y}} = \mathcal{L}_{\mathcal{Y}}$). Since $g : \mathcal{Y} \to \mathcal{Y}$ is an isomorphism, for each j, the pullback g^*E_j (which is simply the preimage $g^{-1}(E_j)$) is a single irreducible component of $\mathcal{Y}_0$, and further, if $g^*E_j = E_l$, then $b_j = b_l$; we use these properties frequently in what follows.

The push-pull formula gives

$$\mathcal{L}_{\mathcal{Y}'}^n.g^*E_j = \mathcal{L}_{\mathcal{Y}}^n.E_j,$$

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which in turn produces

$$\begin{aligned}\mathrm{MA}(g^*\psi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})}) &= \frac{1}{L_Y^n}\sum_j b_j(\mathcal{L}_{\mathcal{Y}'}^n.g^*E_j)\delta_{b_j^{-1}\operatorname{ord}_{g^*E_j}},\\ &= \frac{1}{L_Y^n}\sum_j b_j(\mathcal{L}_{\mathcal{Y}}^n.E_j)\delta_{b_j^{-1}\operatorname{ord}_{g^*E_j}}.\end{aligned}$$

By Lemma 3.8, using that g is an $g : \mathcal{Y} \to \mathcal{Y}$ is an isomorphism and hence is unramified,

$$\begin{aligned}g_*\mathrm{MA}(g^*\psi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})}) &= \frac{1}{L_Y^n}\sum_j b_j(\mathcal{L}_{\mathcal{Y}}^n.E_j)\delta_{b_j^{-1}\operatorname{ord}_{g(g^*(E_j))}},\\ &= \frac{1}{L_Y^n}\sum_j b_j(\mathcal{L}_{\mathcal{Y}}^n.E_j)\delta_{b_j^{-1}\operatorname{ord}_{E_j}},\\ &= \mathrm{MA}(\psi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})}),\end{aligned}$$

as required. □

A consequence, using the fact that the norm of a measure is computed as the energy of its Monge–Ampère-inverse by Theorem 2.25, is the following:

Corollary 3.12. *Let $\mu \in \mathcal{M}_Y^{\mathrm{div},G}$ be a G-invariant divisorial measure on Y^{an}. Then*

$$\|\mu\|_L = \sup_{\varphi\in\mathcal{E}^{1,G}}\left\{E(\varphi) - \int_{Y^{\mathrm{an}}}\varphi\,\mathrm{d}\mu\right\}.$$

3.4. *Proof of the main theorem*

We next turn to comparing numerical invariants of divisorial measures under finite covers and, in particular, to the proof of Theorem 1.1. We begin with the energies of the measures, and throughout this section, we include subscripts in various numerical invariants to emphasise whether we are calculating quantities on X or on Y.

Proposition 3.13. *Let $\mu \in \mathcal{M}_Y^{\mathrm{div},G}$ be a divisorial measure on X. Then*

$$\|\mu\|_{L_Y} = \|\pi_*\mu\|_{L_X}.$$

Proof. By Corollary 3.12, we must show that

$$\sup_{\varphi\in\mathcal{E}_Y^{1,G}}\left\{E(\varphi) - \int_{Y^{\mathrm{an}}}\varphi\,\mathrm{d}\mu\right\} = \sup_{\psi\in\mathcal{E}_X^{1}}\left\{E(\psi) - \int_{X^{\mathrm{an}}}\psi\pi_*(\mathrm{d}\mu)\right\}.$$

We may take the supremum defining $\|\mu\|_{L_Y}$ with respect to G-invariant Fubini–Study metrics on L_Y^{an} by Proposition 3.5 and the continuity property of the Monge–Ampère energy stated in Proposition 2.21. Similarly, the norm $\|\pi_*\mu\|_{L_X}$ can be computed as a supremum over Fubini–Study metrics on L_X^{an}.

Taking an arbitrary $\psi \in \mathcal{H}^{\mathrm{NA}}(L_X^{\mathrm{an}})$, by Proposition 3.6, its pullback $\pi^*\psi$ is a G-invariant Fubini–Study metric satisfying $E(\psi) = E(\pi^*\psi)$; Proposition 3.6 also implies any G-invariant Fubini–Study metric $\varphi \in \mathcal{H}^{\mathrm{NA}}(L_Y^{\mathrm{an}})$ can be realised in this way. Then

$$\int_{Y^{\mathrm{an}}}\pi^*\psi\,\mathrm{d}\mu = \int_{X^{\mathrm{an}}}\psi\pi_*(\mathrm{d}\mu)$$

by definition of the pushforward measure, completing the proof. □

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Corollary 3.14. *Let $\nu \in \mathcal{M}_X^{\mathrm{div}}$ be a divisorial measure on Y. Then*

$$\|\nu\|_{L_X} = \|\pi^*\nu\|_{L_Y}.$$

Proof. From Proposition 3.10, the pullback $\pi^*\nu \in \mathcal{M}_Y^{\mathrm{div},G}$ satisfies $\pi_*\pi^*\nu = \nu$, so by Proposition 3.13,

$$\begin{aligned} \|\pi^*\nu\|_{L_Y} &= \|\pi_*\pi^*\nu\|_{L_X}, \\ &= \|\nu\|_{L_X}, \end{aligned}$$

as required. □

The following is a trivial consequence.

Corollary 3.15. *Let H be an $\mathbb{R}$-divisor on X. Then for $\mu \in \mathcal{M}_X^{\mathrm{div}}$,*

$$\nabla_H \|\mu\|_{L_X} = \nabla_{\pi^*H} \|\pi^*\mu\|_{\pi^*L_X}.$$

Similarly, for $\nu \in \mathcal{M}_Y^{\mathrm{div},G}$,

$$\nabla_H \|\pi_*\nu\|_{L_X} = \nabla_{\pi^*H} \|\nu\|_{\pi^*L_X}.$$

We are now in a position to prove our main result. We recall the setup, which is a cyclic Galois cover $\pi\colon (Y, L_Y) \to (X, L_X)$ of normal projective varieties of degree m with branch divisor $B \subset X$. We assume $\pi^*L_X = L_Y$ and that $K_X + (1 - 1/m)B$ is $\mathbb{Q}$-Cartier, so that by Riemann–Hurwitz,

$$K_Y = \pi^*(K_X + (1 - 1/m)B).$$

Theorem 3.16. *(Y, L_Y) is G-equivariantly divisorially (semi-)stable if and only if $((X, (1-1/m)B); L_X)$ is divisorially (semi-)stable.*

Proof. We prove the result for semistability; the proof for stability is identical. First, supposing that $((X, B); L_X)$ is divisorially semistable, we aim to show that $\beta(\mu) \geq 0$ for any $\mu \in \mathcal{M}_Y^{\mathrm{div},G}$. As in the proof of Proposition 3.1, we may assume that each of the divisors comprising μ lives on a model Y' of Y, which admits a G-action, making $Y' \to Y$ a G-equivariant morphism, so that taking quotients produces a commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "\pi'"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "\pi"] & X. \end{tikzcd}$$

Denote

$$\mu = \sum_i a_i \delta_{c_i \,\mathrm{ord}_{F_i}} \in \mathcal{M}_Y^{\mathrm{div},G},$$

where the F_i are each prime divisors on Y'. The prime divisors F_i then have image which we denote $\pi'(F_i) = D_i$ in X'; we let e_{F_i} denote the ramification index along F_i.

We next calculate the pushforward measure, which by Proposition 3.1 and Lemma 3.8 is given by

$$\begin{aligned} \pi_*\mu &= \sum_i a_i \delta_{\pi(c_i \,\mathrm{ord}_{F_i})}, \\ &= \sum_i a_i \delta_{e_{F_i} c_i \,\mathrm{ord}_{D_i}}. \end{aligned}$$

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We can now use [22, Proof of Proposition 5.20] to conclude that the discrepancies satisfy

$$A_Y(F_i) = e_{F_i} A_{(X,B)}(D_i),$$

implying that the entropies satisfy

$$\begin{aligned}\mathrm{Ent}_{(X,B)}(\pi_*\mu) &= \sum_i a_i e_{F_i} c_i A_{(X,B)}(D_i),\\ &= \sum_i a_i c_i A_Y(F_i),\\ &= \mathrm{Ent}_Y(\mu),\end{aligned}$$

where we use the definition of the entropy of a general divisorial valuation given in Remark 2.8.

Corollary 3.15 proves a similar inequality for the derivatives of the energies of measures involved in the definitions of $\beta(\mu)$ and $\beta(\pi_*\mu)$, where we use the Riemann–Hurwitz formula $K_Y = \pi^*(K_X + (1 - 1/m)B)$, proving equality of the two beta invariants.

Since any divisorial measure on X^{an} is of the form $\pi_*\mu$ for $\mu \in \mathcal{M}_Y^{\mathrm{div},G}$ by Proposition 3.10, the other direction also follows. □

4. Interpolation and applications

We next give a general situation in which one can apply Theorem 1.1 to construct G-equivariantly divisorially stable varieties and, in particular, prove Theorem 1.3 and Corollary 1.4.

4.1. *Divisorial stability in the asymptotic regime*

We consider a normal projective variety X endowed with an ample $\mathbb{Q}$-line bundle L_X. Recall that given an effective $\mathbb{Q}$-divisor B on X, we say that (X, B) is *log canonical* if $K_X + B$ is $\mathbb{Q}$-Cartier and $A_{(X,B)} \geq 0$ on X^{div}.

Theorem 4.1. *There is a $k > 0$ such that, for any $B \in |kL|$ such that (X, B) is log canonical, $((X, B); L)$ is log divisorially stable.*

Proof. Since (X, B) is log canonical, for any divisorial measure μ on X^{an}, the entropy satisfies

$$\mathrm{Ent}_{(X,B)}(\mu) \geq 0.$$

Thus, to prove the result, it suffices to prove that there is a $k > 0$ and an $\varepsilon > 0$ such that for all divisorial measures μ, we have

$$\nabla_{K_X+B}\|\mu\|_{L_X} \geq \varepsilon\|\mu\|_{L_X}.$$

We next reduce to an analogous claim regarding metrics instead of measures. Recall from Example 2.34 that if $\varphi \in \mathcal{E}^1$, then

$$M(\varphi) = \beta(\mathrm{MA}(\varphi)).$$

Boucksom–Jonsson's proof of this statement, in fact, proves that

$$\nabla_{K_X+B} E_{L_X}(\varphi) = \nabla_{K_X+B}\|\mathrm{MA}(\varphi)\|_{L_X}.$$

Thus, by Theorem 2.25 – namely, the non-Archimedean Calabi–Yau theorem – if we can prove that there is a $k > 0$ and an $\varepsilon > 0$ such that for all $\varphi \in \mathcal{E}^1$

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$$\nabla_{K_X+B} E_{L_X}(\varphi) \geq \varepsilon \|\varphi\|_{\min},$$

the proof is concluded. But this inequality on $\mathcal{H}^{\mathrm{NA}}$ is (modulo our non-Archimedean language) proven as part of [17, Proof of Theorem 3.7], where we use that $B \in |kL_X|$. The corresponding inequality on $\mathcal{E}^1$ follows by continuity of the extension of mixed Monge–Ampère energies from $\mathcal{H}^{\mathrm{NA}}$ to $\mathcal{E}^1$. □

Remark 4.2. As explained in the introduction, this result is the (divisorial) 'log' version of its (metric) 'twisted' counterpart [17, Theorem 3.7], which in turn is the algebro-geometric counterpart of analytic results of Hashimoto [21] and Zeng [45] regarding twisted constant scalar curvature Kähler metrics and Aoi–Hashimoto–Zheng regarding conical constant scalar curvature Kähler metrics [4, Theorem 1.10] when working over $\mathbb{C}$. Our proof is completely algebro-geometric, and the k needed is explicit: for a $\mathbb{Q}$-line bundle H, setting

$$\mu(X, L_X) = \frac{-K_X.L_X^{n-1}}{L_X^n},$$

a sufficient condition is that

$$\frac{n}{n+1}\mu(X, L_X)L_X - K_X + \frac{k}{2n(n+1)}L_X$$

be nef (which is certainly true for $k \gg 0$). This requirement can be sharpened using analytic techniques when X is smooth and X and Y are defined over $\mathbb{C}$; see Remark 4.9.

We next note the following interpolation result for divisorial stability, analogous to (for example) [13, Lemma 2.6], which concerns log K-stability. In what follows, we assume that both K_X and B are $\mathbb{Q}$-Cartier, so that $K_X + cB$ is $\mathbb{Q}$-Cartier for all $c \in \mathbb{Q}$.

Lemma 4.3. *Suppose (X, L_X) is divisorially semistable and $((X, B); L_X)$ is log divisorially stable. Then $((X, cB); L_X)$ is log divisorially stable for all* $0 < c \leq 1$.

Proof. We denote the beta invariant of a divisorial measure μ defined with respect to $((X, cB); L_X)$ and (X, L_X) by $\beta_{((X,cB);L_X)}$ and $\beta_{(X,L_X)}$, respectively, for clarity. For a divisorial measure μ, we then wish to compare $\beta_{((X,cB);L_X)}(\mu)$ and $\beta_{(X,L_X)}(\mu)$, and we begin with their respective entropy terms $\mathrm{Ent}_{(X,cB)}(\mu)$ and $\mathrm{Ent}_X(\mu)$.

Write $\mu = \sum_j a_j \delta_{b_j \operatorname{ord} F_j}$ for a finite collection of divisorial valuations $b_j \operatorname{ord} F_j$ on X. For a single divisorial valuation $b_j \operatorname{ord} F_j$ with $F_j \subset Y_j$ and $\pi_j : Y_j \to X$ the associated birational morphism, we note

$$\begin{aligned}
A_{(X,cB)}(b_j \operatorname{ord} F_j) &= b_j A_{(X,cB)}(F_j),\\
&= b_j(\operatorname{ord}_{F_j}(K_{Y_j} - \pi^*(K_X + cB) + 1),\\
&= b_j(\operatorname{ord}_{F_j}(K_{Y_j} - \pi^* K_X + 1)) - b_j c \operatorname{ord}_{F_j}(\pi^* B),\\
&= A_X(b_j \operatorname{ord}_{F_j}) - c\Big(b_j \operatorname{ord}_{F_j}(\pi^* B)\Big).
\end{aligned}$$

By linearity, we thus obtain

$$\mathrm{Ent}_{(X,cB)}(\mu) - \mathrm{Ent}_X(\mu) = -c\left(\sum_j b_j \operatorname{ord}_{F_j}(\pi^* B)\right).$$

We next consider the dependence on c of the remaining term $\nabla_{K_X+cB}\|\mu\|_{L_X}$ comprising $\beta_{((X,cB);L_X)}(\mu)$, which by linearity satisfies (using that B is $\mathbb{Q}$-Cartier)

$$\nabla_{K_X+cB}\|\mu\|_{L_X} = \nabla_{K_X}\|\mu\|_{L_X} + c\nabla_B\|\mu\|_{L_X}.$$

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Thus, if we define, for a divisorial measure $\mu = \sum_j a_j \delta_{b_j \operatorname{ord} F_j}$, a functional

$$\tilde{\beta}(\mu) : \mathcal{M}^{\mathrm{div}} \to \mathbb{R}$$

by

$$\tilde{\beta}(\mu) = \nabla_B \|\mu\|_{L_X} - \sum_j b_j \operatorname{ord}_{F_j}(\pi^* B),$$

then

$$\beta_{((X,cB);L_X)}(\mu) = \beta_{(X,L_X)}(\mu) + c\tilde{\beta}(\mu).$$

By hypothesis, there is an $\varepsilon > 0$ such that for all divisorial measures μ,

$$\beta_{(X,L_X)}(\mu) \geq 0 \text{ and } \beta_{((X,B);L_X)} \geq \varepsilon \|\mu\|_{L_X}.$$

Writing

$$\beta_{((X,cB);L_X)}(\mu) = c\big(\beta_{(X,L_X)}(\mu) + \tilde{\beta}(\mu)\big) + (1-c)\beta_{(X,L_X)}(\mu),$$

we next recall that by hypothesis,

$$\beta_{(X,L_X)}(\mu) + \tilde{\beta}(\mu) \geq \varepsilon \|\mu\|_{L_X}.$$

Thus, since $0 < c \leq 1$, we obtain

$$\beta_{((X,cB);L_X)}(\mu) \geq c\varepsilon \|\mu\|_{L_X},$$

proving log divisorial stability. □

The following is then an automatic consequence of these two results.

Corollary 4.4. *Suppose (X, L_X) is divisorially semistable. Then there is a $k > 0$ such that, for any $B \in |kL|$ such that (X, B) is log canonical and any $m > 1$, $((X, (1 - 1/m)B); L)$ is log divisorially stable.*

In particular, Theorem 1.1 produces the following, which proves Theorem 1.3. We continue the notation of Corollary 4.4 and let G be the cyclic group of order m.

Corollary 4.5. *Under the hypotheses of Corollary 4.4, suppose $\pi : Y \to X$ is the m-fold branched cover over B, and set $L_Y = \pi^* L_X$. Then (Y, L_Y) is G-equivariantly divisorially stable.*

We refer to Kollár for the construction of m-fold branched coverings over Cartier divisors [23, Section 2.11].

4.2. G-equivariant divisorial and uniform K-stability

We end by using Corollary 4.5 to produce constant scalar curvature Kähler metrics, for which we need Y to be smooth. We need the following equivariant analogue of Boucksom–Jonsson's work relating divisorial and uniform K-stability, proved using their results.

Theorem 4.6. *A polarised variety (Y, L_Y) is G-equivariantly divisorially stable if and only if it is uniformly K-stable on $\mathcal{E}^{1,G}$.*

Proof. Let $\mu \in \mathcal{M}^{1,G}$, and let $\varphi \in \mathcal{E}^{1,G}$ be such that

$$\mathrm{MA}(\varphi) = \mu,$$

where G-invariance of such a φ follows from Proposition 3.11. By Proposition 3.5, there is a sequence $\varphi_k \in \mathcal{H}^{\mathrm{NA},G}$ of G-invariant Fubini–Study metrics decreasing to φ, and in particular, $\mathrm{MA}(\varphi_k)$ converges to $\mathrm{MA}(\varphi) = \mu$ by continuity of the Monge–Ampère operator. Thus, we have produced a sequence of G-invariant divisorial measures converging to μ. It follows that G-equivariant divisorial stability is equivalent to the existence of an $\varepsilon > 0$ such that for all $\mu \in \mathcal{M}^{1,G}$,

$$\beta(\mu) \geq \varepsilon \|\mu\|_{L_Y},$$

by continuity of the quantities involved.

We next show that the latter condition is equivalent to uniform K-stability on $\mathcal{E}^{1,G}$. As the content of their proof that divisorial stability is equivalent to uniform K-stability on $\mathcal{E}^1$ (along with the non-Archimedean Calabi–Yau theorem), Boucksom–Jonsson prove that for any $\varphi \in \mathcal{E}^1$, we have equality

$$M(\varphi) = \beta(\mathrm{MA}(\varphi)); \tag{4.1}$$

see Example 2.34. But Theorem 3.11 implies that the Monge–Ampère operator induces a bijection between finite norm *G-invariant* measures and finite energy *G-invariant* metrics (modulo addition of constants), meaning that Equation (4.1) proves the equivalence of the conditions

$$\beta(\mu) \geq \varepsilon \|\mu\|_{L_Y} \text{ for all } \mu \in \mathcal{M}^{1,G}$$

and

$$M(\varphi) \geq \varepsilon \|\varphi\|_{\min} \text{ for all } \varphi \in \mathcal{E}^{1,G}$$

and hence proves the result. □

We now take the field k over which the varieties X and Y are defined to be $\mathbb{C}$. We use the following important result of Li [26, Theorem 1.10].

Theorem 4.7. *Suppose Y is smooth. If (Y, L_Y) is uniformly K-stable on $\mathcal{E}^{1,G}$, then $c_1(L)$ admits a constant scalar curvature Kähler metric.*

From this, we obtain the following consequence of Corollary 4.5, which proves Corollary 1.4.

Theorem 4.8. *Let (X, L_X) be a divisorially semistable smooth polarised variety. There is a $k > 0$ such that if we let*

(i) *$B \in |kL_X|$ be such that (X, B) is log canonical,*
(ii) *and let $\pi : Y \longrightarrow X$ be the m-fold cover of X branched over B,*
(iii) *and assume Y is smooth,*

then Y admits a constant scalar curvature Kähler metric in $c_1(L_Y)$, where G is the associated cyclic group of degree m and $L_Y = \pi^* L_X$.

Remark 4.9. As mentioned in the introduction, the existence of such metrics in this situation is also consequence of work of Arezzo–Della Vedova–Shi [2], provided one replaces the assumption that (X, L_X) be divisorially semistable with the assumption that the (Archimedean) Mabuchi functional be bounded below on the space of Kähler metrics in $c_1(L_X)$ (which holds, in particular, when $c_1(L_X)$ admits a cscK metric). In fact, the k they obtain as sufficient is slightly more general than our work, essentially as they use analytic techniques and work of Song–Weinkove [40], whereas our work relies on the entirely algebro-geometric [17, Theorem 3.7] which is slightly weaker. We refer to [2, Section 6] for many examples of applications.

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