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# ON A REGULARITY PROPERTY AND A PRIORI ESTIMATES FOR SOLUTIONS OF NONLINEAR PARABOLIC VARIATIONAL INEQUALITIES

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**Abstract.** In this paper we consider the following nonlinear parabolic variational inequality; $u(t) \in D(\Phi)$ for all $t \in I$, $(u_t(t), u(t)-v)+\langle\Delta_p u(t), u(t)-v\rangle+\Phi(u(t)) - \Phi(v) \leqq (f(t), u(t) - v)$ for all $v \in D(\Phi)$ a.e. $t \in I$, $u(x, 0) = u_0(x)$, where $\Delta_p$ is the so-called $p$-Laplace operator and $\Phi$ is a proper, lower semicontinuous functional. We have obtained two results concerning to solutions of this problem. Firstly, we prove a few regularity properties of solutions. Secondly, we show the continuous dependence of solutions on given data $u_0$ and $f$.

## §1. Introduction

Let $\Omega$ be a bounded domain in $R^n$ with coordinates $x = (x_1, \dots, x_n)$. The boundary $\Gamma$ of $\Omega$ is assumed to be of class $C^1$. For any positive number $T$ we denote the open interval $(0, T)$ by $I$ and the cylinder $\Omega \times I$ by $G$, i.e. $G = \{(x, t); x \in \Omega,\ t \in I\}$. The usual Sobolev space $W^{1,p}(\Omega)$ is defined as follows: $W^{1,p}(\Omega) = \{v \in L^p(\Omega); D_j v \in L^p(\Omega),\ j = 1, \dots, n\}$ with the norm $|v|_{1,p} = (|v|_p^p + |Dv|_p^p)^{1/p}$ $(1 \leqq p < \infty)$. Here $D_j v = \partial v/\partial x_j$, $Dv = (D_1 v, D_2 v, \dots, D_n v)$ and $|v|_p = \|v\|_{L^p(\Omega)}$.

Let $S$ be a compact $C^1$ manifold of $(n-2)$-dimension contained in $\Gamma$. We assume that $S$ divides $\Gamma$ into two relatively open subsets $\Gamma_1$, $\Gamma_2$ such that $\Gamma_1 \neq \emptyset$, more precisely, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup S$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. Next let $C^\infty_{(0)}(\bar\Omega)$ be the family of all infinitely differentiable functions in $\bar\Omega$ vanishing in each neighborhood of $\overline{\Gamma_1}$. The completion of $C^\infty_{(0)}(\bar\Omega)$ with respect to the norm $|\ \ |_{1,p}$ is denoted by $V$. We denote the norm in $V$ by $|\ \ |_V$, the dual space of $V$ by $V'$, the pairing between $V$ and $V'$ by $\langle\ ,\ \rangle$ and the inner product of $L^2(\Omega)$ by $(\ ,\ )$.

For any Banach space $X$ and any $s$ $(1 \leqq s < \infty)$ let us denote by $L^s(I, X)$ the space of equivalent classes of functions $v(t)$ from $I$ to $X$, which are $L^s$-integrable on $I$. It is a Banach space with the norm $\|v\|_{L^s(I,X)} =$

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$(\int_I |v(t)|_X^s dt)^{1/s}$, where $|\quad|_X$ is the norm in $X$. In the case of $s = \infty$, $L^\infty(I, X)$ means the set of all measurable functions $v; I \to X$, satisfying $\|v\|_{L^\infty(I,X)} = \operatorname{ess.sup}_I |v(s)|_X < \infty$.

A functional $\Phi; X \to (-\infty, \infty]$ is said to be proper if $\Phi \not\equiv \infty$. And a proper functional $\Phi$ is said to be lower semicontinuous if

$$\Phi(v_0) \leqq \liminf_{v \to v_0} \Phi(v) \text{ for any } v_0 \in X.$$

For a proper lower semicontinuous functional $\Phi$ on $X$ we set $D(\Phi) = \{v \in X; \Phi(v) < \infty\}$.

We consider the nonlinear parabolic variational inequality

$$(1.1)\quad \begin{cases} u(t) \in D(\Phi) \quad \text{for all} \quad t \in I, \\ (u_t(t), u(t) - v) + \langle \Delta_p u(t), u(t) - v\rangle + \Phi(u(t)) - \Phi(v) \\ \qquad\qquad \leqq (f(t), u(t) - v) \quad \text{for all } v \in D(\Phi) \text{ a.e. } t \in I, \\ u(x, 0) = u_0(x), \end{cases}$$

where $\langle \Delta_p u, v\rangle = \sum_{j=1}^n (|Du|^{p-2} D_j u, D_j v)$. In the above $\Phi$ is supposed to be proper, convex lower semicontinuous on $V$. Throughout this paper we assume that $1 < p \leqq 2$, if $n = 1, 2$, and $2n/(n+2) < p \leqq 2$, if $3 \leqq n$.

In this paper we denote by the same $C$ any positive constant which does not depend on $u$, $u_0$ and $f$.

The first aim of this paper is to show the following

Theorem A. *It is assumed that the inequality*

$$(1.2)\qquad \langle \Delta_p u_0, u_0 - v\rangle + \Phi(u_0) - \Phi(v) \leqq (f(0), u_0 - v) \quad \textit{for any} \quad v \in D(\Phi)$$

*holds, where* $u_0 \in D(\Phi)$, $f \in L^2(I, L^2(\Omega))$ *and* $f_t \in L^2(I, L^2(\Omega))$. *Then, for any solution* $u$ *of* (1.1) *it holds that* $(Du)_t \in L^2(I, L^p(\Omega))$ *and*

$$\|(Du)_t\|_{L^2(I, L^p(\Omega))} \leqq \sigma^{1/p}(u_0, f)$$

*where* $\sigma(u_0, f) = C(|Du_0|_p^p + \|f_t\|^2_{L^2(I, L^2(\Omega))})$.

It is easy to deduce the follwoing corollary from Theorem A:

Corollary. *Under the same assumptions as in Theorem A it holds that* $(Du)_t \in L^p(G)$ *and*

$$\|(Du)_t\|_{L^p(G)} \leqq \sigma(u_0, f).$$

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Secondly, we give a priori estimates which assure the continuous dependence of solutions of (1.1) on the given data $u_0$ and $f$.

THEOREM B. *Let $u_1$ (resp. $u_2$) be any solution of* (1.1) *with* $u_1(x,0) = u_{1,0}$ (*resp.* $u_2(x,0) = u_{2,0}$) *and* $f = f_1$ (*resp.* $f_2$). *Let the condition* (1.2) *be satisfied for $u_{i,0}$ and $f_i$, $i = 1, 2$. Under the condition that $u_{1,0}$, $u_{2,0} \in D(\Phi)$, $f_1$, $f_2 \in L^2(I, L^2(\Omega))$ and $(f_1)_t$, $(f_2)_t \in L^2(I, L^2(\Omega))$ the followings hold: for any $t \in I$*

$$(1) \qquad |(u_1 - u_2)(t)|_2 \leqq C(|u_{1,0} - u_{2,0}|_2 + \|f_1 - f_2\|_{L^2(I,L^2(\Omega))}),$$

*and*

$$(2) \qquad \|D(u_1 - u_2)\|_{L^2(I,L^p(\Omega))} \leqq (\sigma(u_{1,0}, f_1) + \sigma(u_{2,0}, f_2))^{(2-p)/2p} \cdot (|u_{1,0} - u_{2,0}|_2 + \|f_1 - f_2\|_{L^2(I,L^2(\Omega))}).$$

Under the same conditions on $p$ as this paper we have proved some decay properties of solutions for (1.1) in [N5], where we have replaced $\Phi$ by the indicator function $I_K$ of a closed onvex subset $K$ of $V$ and $I$ by $R_1^+ = (0, \infty)$.

In [N2] we considered the nonlinear parabolic variational inequality

$$(1.3) \qquad \begin{cases} u(t) \in D(\Phi) \quad \text{for all} \quad t \in I, \\ (u_t(t), u(t) - v) + \langle A(t)u(t), u(t) - v\rangle + \Phi(u(t)) - \Phi(v) \\ \qquad\qquad \leqq (f(t), u(t) - v) \quad \text{for all } v \in D(\Phi) \text{ a.e. } t \in I, \\ u(x, 0) = u_0(x). \end{cases}$$

In the above the operator $A(t); V \to V'$ is defined in such a way that

$$(1.4) \qquad \langle A(t)v, w\rangle = \sum_{j=1}^{n} (a_j(\,.\,, t, Dv), D_j w) \quad \text{for any} \quad v, w \in V.$$

Here the nonlinear functions $a_j(x, t, \eta)$, $j = 1, \dots, n$, satisfy the following

ASSUMPTION (I). For $(x, t) \in G$, $\eta \in R^n - \{0\}$, $\xi \in R^n$ and $j$, $1 \leqq j \leqq n$,

1-1 $\qquad a_j(x, t, \eta) \in C^0(\Omega \times I \times R^n) \cap C^1(\Omega \times I \times (R^n - \{0\}))$,

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1-2
$$\sum_{i,j=1}^{n} (\partial/\partial \eta_i) a_j(x,t,\eta)\xi_i\xi_j \geqq \gamma|\eta|^{p-2}|\xi|^2,$$

1-3
$$|(\partial/\partial t) a_j(x,t,\eta)| \leqq \Lambda|\eta|^{p-1}.$$

Here $\gamma$, $\Lambda$ are some positive constants.

It is easy to see that the operator $\Delta_p$ satisfies the Assumption (I). However, it is assumed that $2 \leqq p$ in [N2]. In such a case we considered the existence and the regularity of solutions of (1.3) under the following assumptions in [N2]:

ASSUMPTION (II). The function $u_0(x)$ in (1.3) belongs to $D(\Phi)$ and there exists an element $z_0(x)$ in $L^2(\Omega)$ such that the inequality

$$(z_0, v-u_0) + \langle A(0)u_0, v-u_0\rangle + \Phi(v) - \Phi(u_0) \geqq (f(0), v-u_0) \tag{1.5}$$

holds for all $v \in D(\Phi)$.

ASSUMPTION (III). There exists $v_0$ in $D(\Phi)$ such that for any $t \in I$

$$\{\langle A(t)v, v-v_0\rangle + \Phi(v)\}/|v|_V \to \infty \text{ uniformly as } |v|_V \to \infty. \tag{1.6}$$

Concerning with the regularity of solutions of (1.3), we obtained the following results in [N2, p.276]: let the Assumptions (I), (II) and (III) be satisfied. If $f$, $f_t \in L^2(I, L^2(\Omega))$, it holds that $\|(|Du|^{(p-2)/2}Du)_t\|^2_{L^2(G)}$, $\|(a_j(\,.\,,\,.\,, Du))_t\|^{p'}_{L^{p'}(G)}$ and $\|(|Du|^{(p-2)}Du)_t\|^{p'}_{L^{p'}(G)} \leqq \gamma(u_0, z_0, f)$ for any solution $u$ of (1.3). Here $p'$ is the adjoint number of $p$, i.e., $p' = p/(p-1)$, and $\gamma(u_0, z_0, f) = C(1+|z_0|_2^2+|u_0|_2^2+|Du_0|_p^p+\|f\|^2_{L^2(I,L^2(\Omega))}+\|f_t\|^2_{L^2(I,L^2(\Omega))})$. In [N2] we have treated (1.3) when the operator $A$ has a nonlinear perturbed term of lower order.

In [N4] we considerd the nonlinear parabolic equation

$$\begin{cases} u \in L^\infty(I, W_0^{1,p}(\Omega)) \cap C(I, L^2(\Omega)),\ u_t \in L^2(I, L^2(\Omega)), \\ (u_t(t), v) + \langle A(t)u(t), v\rangle = \langle f(t), v\rangle \\ \qquad\qquad\qquad\qquad \text{for all } v \in W_0^{1,p}(\Omega) \ \text{ a.e. } \ t \in I, \\ u(x,0) = u_0. \end{cases} \tag{1.7}$$

under the following assumptions:

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ASSUMPTION (IV). $f \in L^{p'}(I, W^{1,p'}(\Omega))$ and $f_t \in L^2(I, L^2(\Omega))$.

ASSUMPTION (V). $u_0 \in W^{1,p}(\Omega)$ and $Du_0 \in L^2(\Omega)$. Further, there exists a function $z_0 \in L^2(\Omega)$ such that the equality

$$(z_0, v) + \langle A(0)u_0, v\rangle = \langle f(0), v\rangle \quad \text{for all} \ \ v \in W_0^{1,p}(\Omega)$$

holds

In [N4, p.51, p.62] we obtained the following results concerned to the regularity of any solution $u$ of (1.7): under the Assumptions (I), (IV) and (V) it holds that (1) if $2(n+1)/(n+3) < p \leqq 2$, then $D^2u \in L^{k(p_\infty)}(I', L_{\text{loc}}^{k(p_\infty)}(\Omega))$, $(Du)_t \in L^{k(p_\infty)}(\Omega \times I')$ and $D^2u \in L^{m(p_\infty)}(\Omega \times I')$, (2) if $2n/(n+2) < p \leqq 2(n+1)/(n+3)$, then $D^2u \in L^p(I, L_{\text{loc}}^p(\Omega))$, $(Du)_t \in L^p(\Omega \times I')$ and $D^2u \in L^{2p/(4-p)}(G)$. Here $D^2u = D_iD_ju$, $1 \leqq i, j \leqq n$, $I' = (a, b)$ with any $a, b$, $0 < a < b < T$, $k(p_\infty) = 4(n+1)(p-1)/((n+3)p-4)$ and $m(p_\infty) = 2(n+1)(p-1)/(2p+n-3)$. It is easy to see that $p < k(p_\infty)$ (resp. $p > k(p_\infty)$), if $2(n+1)/(n+3) < p \leqq 2$ (resp. $1 < p \leqq 2(n+1)/(n+3)$).

The other results on the regularity of solutions for nonlinear parabolic variational inequalities and nonlinear parabolic differential equations are referred to [N2] and [N4].

In the abstract framework of a Hilbert triple $\{V, H, V'\}$ G. Savaré has considered (1.3) when $A(t)$, $t \in I$, is a family of linear continuous coercive operators from $V$ to $V'$ and $\Phi$ is a proper convex lower semicontinuous functional on $V$. In [Sa] he obtained the estimate $\|u_1 - u_2\|_{i(I)}^2 \leqq C(|u_{1,0} - u_{2,0}|_H^2 + (\|f_1\|_{S(I)} + \|f_2\|_{S(I)})\|f_1 - f_2\|_{S(I)})$, where $u_1$ (resp. $u_2$) is any solution of (1.3) with $f = f_1$ (resp. $f_2$) and $u_0 = u_{1,0}$ (resp. $u_{2,0}$). Here $i(I) = L^2(I, V) \cap L^\infty(I, H)$ and $S(I) = L^2(I, V') + L^1(I, H) + B_{21}^{-1/2}(I, H)$ with the norm $\|v\|_{i(I)} = \|v\|_{L^2(I,V)} + \|v\|_{L^\infty(I,H)}$ and $\|v\|_{S(I)} = \inf(\|v_1\|_{L^2(I,V')} + \|v_2\|_{L^1(I,H)} + \|v_3\|_{B_{21}^{-1/2}(I,H)})$, where the infimum is taken for $v = v_1 + v_2 + v_3$ such that $v_1 \in L^2(I, V')$, $v_2 \in L^1(I, H)$ and $v_3 \in B_{21}^{-1/2}(I, H)$. The definition and the properties of the space $B_{21}^{-1/2}(I, H)$ are referred to [Sa].

At last we refer to the results in [C]. In [C] Y. Cheng considered the nonlinear elliptic equation with the Dirichlet boundary condition; $-\text{div}(|Du|^{p-2}Du) = f$ in $\Omega$ and $u = 0$ on $\Gamma$ in the weak sense that

$$\langle \Delta_p u, \phi\rangle = (f, \phi) \text{ for all } \phi \in C_0^\infty(\Omega). \tag{1.8}$$

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Y. Cheng obtained the followings for solutions of (1.8) in [C]: (1) when $2 \leqq p$, it holds that $|u_1 - u_2|_{1,p} \leqq C|f_1 - f_2|_{-1,p'}^{1/(p-1)}$, and (2) when $1 < p < 2$, it holds that $|u_1 - u_2|_{1,p} \leqq C(|f_1|_{-1,p'} + |f_2|_{-1,p'})^{(2-p)/(p-1)}|f_1 - f_2|_{-1,p'}$. Here $u_1$ (resp. $u_2$) is any solution of (1.8) for $f = f_1$ (resp. $f_2$).

This paper is constructed as follows: in Section 2 we prepare some lemmas and a proposition, which play important roles in the proof of our theorems. In Section 3 we give the proofs of our theorems.

## §2. Lemmas

Lemma 2.1. ([C, p.736, Theorem 3]) *There exists a positive constant $\gamma_0$ depending only on $p$ such that the following inequality holds:*

$$\sum_{j=1}^{n}(|\xi|^{p-2}\xi_j - |\eta|^{p-2}\eta_j)(\xi_j - \eta_j) \geqq \gamma_0(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \tag{2.1}$$

*for any $\xi$ and $\eta \in R^n$, where the right-hand side is defined to be 0, if $\xi = \eta = 0$.*

We can show easily the following lemma by Hölder's inequality:

Lemma 2.2. *Let $0 < r < 1$ and $r' = r/(r-1)$. If $F(x) \in L^r(Q)$, $F(x)H(x) \in L^1(Q)$ and $\int_Q |H(x)|^{r'} dx < \infty$, then it holds that*

$$\left(\int_Q |F(x)|^r dx\right)^{1/r} \leqq \left(\int_Q |F(x)H(x)| dx\right)\left(\int_Q |H(x)|^{r'} dx\right)^{-1/r'}. \tag{2.2}$$

*Here $Q$ is any bounded domain in $R^m$.*

By the same way as in [N1, p.77, Lemma 1.4] we can show the following

Lemma 2.3. *Let $v$ be a distribution in $Q$ and let $\{v_j\}_{j=1}^{\infty}$ be a sequence in a reflexive Banach space $X$, where $C_0^{\infty}(Q)$ is dense. Let norms $|v_j|_X$ be uniformly bounded. If $(v_j, \psi) \to (v, \psi)$ as $j \to \infty$ for any $\psi \in C_0^{\infty}(Q)$, then $v$ belongs to $X$ and the sequence $v_j$ converges weakly to $v$ in $X$ as $j \to \infty$.*

Next, we prepare a priori estimates for solutions of (1.1).

Proposition 2.1. *Under the assumptions in Theorem A there exists a unique solution $u$ of (1.1) with the following properties:*

(1) $u \in L^{\infty}(I, V) \cap C(I, L^2(\Omega)), \quad u_t \in L^{\infty}(I, L^2(\Omega))$

(2) $|u_t(t)|_2^2,\ |u(t)|_V^p \leqq \sigma(u_0, f)$ *uniformly in $t \in I$.*

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For solutions of (1.3) the assertions of the above proposition were proved in [N2, p.275, Theorem 1] under the Assumptions (I), (II) and (III) when $2 \leqq p$. In the cases that $1 < p \leqq 2$, if $n = 1, 2$, and $2n/(n+2) < p \leqq 2$, if $3 \leqq n$ we can show the same results as [N2] by the similar way to [N2] with slight modifications. Moreover, the above proposition was proved by J. Kacur in [K3, p.116, Theorem 5.2.3]. In [K3] the operator $\Delta_p$ was replaced by a genral nonlinear elliptic operator which was independent of $t$.

*Remark* 2.1. We extend $u(t)$ and $f(t)$ outside $I$ in such a way that

$$\begin{aligned} u(t) &= u(T), \qquad f(t) = f(T) \quad \text{for} \quad t > T; \\ u(t) &= u_0, \qquad f(t) = f(0) \quad \text{for} \quad t < 0. \end{aligned} \tag{2.3}$$

LEMMA 2.4. *Under the assumptions in Theorem A the inequality*

$$|(u(h) - u_0)/h|_2 \leqq C\|f_t\|_{L^2(I,L^2(\Omega))} \qquad (h \neq 0) \tag{2.4}$$

*holds for any solution $u$ of* (1.1). *Here the positive constant $C$ depends only on $T$.*

*Proof.* By Remark 2.1 it is easy to see that the estimate (2.4) is valid for $h < 0$. Therefore, we may assume that $0 < h$ from now on.

For any solution $u$ of (1.1) and a.e. $t \in I$ let us take $v = u_0$ in (1.1). After this we take $v = u(t)$ in (1.2). Adding these two inequalities, we get

$$\begin{aligned} (u_t(t) - (u_0)_t, u(t) - u_0) + \langle \Delta_p u(t) - \Delta_p u_0, u(t) - u_0 \rangle \\ \leqq (f(t) - f(0), u(t) - u_0) \quad \text{a.e. } t \in I. \end{aligned} \tag{2.5}$$

Using the fact that $0 \leqq \langle \Delta_p u(t) - \Delta_p u_0, u(t) - u_0 \rangle$, we get

$$\frac{1}{2}\frac{d}{dt}|u(t) - u_0|_2^2 \leqq (f(t) - f(0), u(t) - u_0) \quad \text{a.e. } t \in I. \tag{2.6}$$

Integrating the above over $(0, h)$ for any $h > 0$, we have

$$|u(h) - u_0|_2^2 \leqq 2\int_0^h (f(t) - f(0), u(t) - u_0)dt. \tag{2.7}$$

Dividing (2.7) by $h^2$ and using Hölder's inequality, we get

$$|(u(h) - u_0)/h|_2^2 \leqq \frac{2}{h^2}\int_0^h |(f(t) - f(0), u(t) - u_0)|dt \tag{2.8}$$

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$$\leqq 2\int_0^h |(f(t)-f(0))/h|_2 |(u(t)-u_0)/t|_2 dt$$
$$\leqq \int_0^h |(f(t)-f(0))/h|_2^2 dt + \int_0^h |(u(t)-u_0)/t|_2^2 dt,$$

because $0 < 1/h < 1/t$ for $0 < t < h$.

Here, we estimate the first term on the right-hand side of (2.8) as follows: at first,

$$\text{(2.9)} \quad |(f(t)-f(0))/h|_2^2 = |\frac{1}{h}\int_0^t f_t(s)ds|_2^2 = \int_\Omega \left(\frac{1}{h}\int_0^t f_t(s,x)ds\right)^2 dx$$
$$\leqq \int_\Omega \frac{t}{h^2}\int_0^t f_t^2(s,x)dsdx = \frac{t}{h^2}\int_0^t |f_t(s)|_2^2 ds.$$

Then, we have

$$\text{(2.10)} \quad \int_0^h |(f(t)-f(0))/h|_2^2 dt \leqq \int_0^h \left(\frac{t}{h^2}\int_0^t |f_t(s)|_2^2 ds\right) dt$$
$$\leqq \frac{1}{2}\int_0^h |f_t(s)|_2^2 ds \leqq \frac{1}{2}\|f_t\|^2_{L^2(I,L^2(\Omega))}.$$

Therefore, from (2.8) and (2.10) we get

$$\text{(2.11)} \quad |(u(h)-u_0)/h|_2^2 \leqq \frac{1}{2}\|f_t\|^2_{L^2(I,L^2(\Omega))} + \int_0^h |(u(t)-u_0)/t|_2^2 dt.$$

Let us use Gronwall's inequality to obtain the estimate (2.4). In this way we finish the proof of this lemma. □

## §3. Proofs of our theorems

At first we give the proof of Theorem A.

*Proof of Theorem A.* For any solution $u$ of (1.1) and a.e. $t \in I$, we take $v = u(t+h)$ in (1.1), where $|h| < \min(t, T-t)$. After this we take $v = u(t)$ in (1.1) for $t = t+h$. Adding these two inequalities, we have

$$\text{(3.1)} \quad (u_t(t+h)-u_t(t), u(t+h)-u(t))$$
$$+\langle \Delta_p u(t+h) - \Delta_p u(t), u(t+h)-u(t)\rangle$$
$$\leqq (f(t+h)-f(t), u(t+h)-u(t)) \qquad \text{a.e. } t \in I.$$

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Dividing (3.1) by $h^2$, we get

$$\begin{aligned}(3.2)\qquad &\frac{1}{2}\frac{d}{dt}|u(t+h)-u(t)|_2^2/h^2\\ &\qquad +\langle \Delta_p u(t+h)-\Delta_p u(t), u(t+h)-u(t)\rangle/h^2\\ &\leqq (\delta_h f(t), \delta_h u(t)) \qquad \text{a.e. } t\in I.\end{aligned}$$

where $\delta_h f(t) = (f(t+h)-f(t))/h$ and $\delta_h u(t) = (u(t+h)-u(t))/h$.

Let us integrate (3.2) over $I$ to get

$$\begin{aligned}(3.3)\qquad &\frac{1}{2}|u(T+h)-u(T)|_2^2/h^2 - \frac{1}{2}|u(h)-u_0|_2^2/h^2\\ &\qquad + \int_I \langle \Delta_p u(t+h)-\Delta_p u(t), u(t+h)-u(t)\rangle/h^2 dt\\ &\leqq \int_I (\delta_h f(t), \delta_h u(t))dt.\end{aligned}$$

By Lemma 2.1 it holds that

$$\begin{aligned}(3.4)\qquad &\gamma_0(|Du(t+h)|+|Du(t)|)^{p-2}|Du(t+h)-Du(t)|^2/h^2\\ &\leqq \sum_{j=1}^{n}\left(|Du(t+h)|^{p-2}D_j u(t+h) - |Du(t)|^{p-2}D_j u(t)\right)\cdot\\ &\qquad\qquad (D_j u(t+h)-D_j u(t))/h^2.\end{aligned}$$

Integrating (3.4) over $\Omega$, we get

$$\begin{aligned}(3.5)\qquad &\gamma_0\int_\Omega (|Du(t+h)|+|Du(t)|)^{p-2}|Du(t+h)-Du(t)|^2/h^2 dx\\ &\leqq \langle \Delta_p u(t+h)-\Delta_p u(t), u(t+h)-u(t)\rangle/h^2.\end{aligned}$$

Next let us set $F = |Du(t+h)-Du(t)|^2/h^2$, $H = (|Du(t+h)| + |Du(t)|)^{p-2}$, $r = p/2$ and $Q=\Omega$ in Lemma 2.2. Then, from (2.2), (3.5) and Proposition 2.1 we get

$$\begin{aligned}(3.6)\qquad &\left(\int_\Omega |(Du(t+h)-Du(t))/h|^p dx\right)^{2/p}\\ &\leqq \frac{1}{h^2}\left(\int_\Omega (|Du(t+h)|+|Du(t)|)^{p-2}|Du(t+h)-Du(t)|^2 dx\right)\cdot\\ &\qquad\qquad \left(\int_\Omega (|Du(t+h)|+|Du(t)|)^p dx\right)^{(2-p)/p}\\ &\leqq \sigma^{(2-p)/p}(u_0,f)(1/\gamma)^{-1}\langle \Delta_p u(t+h)-\Delta_p u(t), u(t+h)-u(t)\rangle/h^2.\end{aligned}$$

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Let us integrate (3.6) over $I$ to have

$$(3.7)\qquad \int_I \left( \int_\Omega |(Du(t+h) - Du(t))/h|^p dx \right)^{2/p} dt \leqq \sigma^{(2-p)/p}(u_0, f) \int_I \langle \Delta_p u(t+h) - \Delta_p u(t), u(t+h) - u(t) \rangle / h^2 dt.$$

Combining (3.7) with (3.3) multiplied by $\sigma^{(2-p)/p}(u_0, f)$, we obtain

$$(3.8)\qquad \begin{aligned} &\int_I \left( \int_\Omega |(Du(t+h) - Du(t))/h|^p dx \right)^{2/p} dt \\ &\leqq \sigma^{(2-p)/p}(u_0, f)(|(u(h) - u_0)/h|_2^2 + \int_I |(\delta_h f(t), \delta_h u(t))| dt) \\ &\leqq \sigma^{(2-p)/p}(u_0, f)(|(u(h) - u_0)/h|_2^2 + \int_I |\delta_h f(t)|_2 |\delta_h u(t)|_2 dt) \\ &\leqq \sigma^{(2-p)/p}(u_0, f)(|(u(h) - u_0)/h|_2^2 + \int_I |\delta_h u(t)|_2^2 dt + \int_I |\delta_h f(t))|_2^2 dt) \end{aligned}$$

By the similar calculations to (2.9) and (2.10) the last two terms in the blackts on the right-hand side of (3.8) are estimated as follows:

$$\begin{aligned} &\int_I |\delta_h f(t)|_2^2 dt \leqq \|f_t\|_{L^2(I, L^2(\Omega))}^2, \\ &\int_I |\delta_h u(t)|_2^2 dt \leqq \|u_t\|_{L^2(I, L^2(\Omega))}^2 \leqq \sigma(u_0, f). \end{aligned}$$

Here we have used Proposition 2.1 in the last inequality. In (3.8) let us use the above two estimates and Lemma 2.4. Then, we get

$$(3.9)\qquad \int_I \left( \int_\Omega |(Du(t+h) - Du(t))/h|^p dx \right)^{2/p} dt \leqq \sigma^{2/p}(u_0, f).$$

By virtue of Lemma 2.3 we finish the proof of our theorem, see [N3, p.184] in detail. □

*Proof of Theorem B.* For $i = 1$ and $i = 2$ let $u_i$ be any solution of the inequality

$$(3.10)_i\qquad \begin{cases} ((u_i)_t(s), u_i(s) - v) + \langle \Delta_p u_i(s), u_i(s) - v \rangle + \Phi(u_i(s)) - \Phi(v) \\ \qquad\qquad \leqq (f_i(s), u_i(s) - v) \quad \text{for all } v \in D(\Phi) \text{ a.e. } s \in I, \\ u_i(x, 0) = u_{i,0} \end{cases}$$

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In $(3.10)_1$ (resp. $(3.10)_2$) we take $v = u_2$ (resp. $u_1$). Then, let us add these two inequalities in order to get

$$(3.11)\qquad (((u_1)_t - (u_2)_t)(s), u_1(s) - u_2(s)) + \langle \Delta_p u_1(s) - \Delta_p u_2(s), u_1(s) - u_2(s)\rangle \leqq (f_1(s) - f_2(s), u_1(s) - u_2(s)) \text{ a.e. } s \in I.$$

Then, we have

$$(3.12)\qquad \frac{1}{2}\frac{d}{ds}|u_1(s) - u_2(s)|_2^2 + \langle \Delta_p u_1(s) - \Delta_p u_2(s), u_1(s) - u_2(s)\rangle \leqq (f_1(s) - f_2(s), u_1(s) - u_2(s)) \quad \text{a.e. } \ s \in I.$$

Let us apply Hölder's inequality to the right-hand side of (3.12) and integrate it over $(0, t)$ for any $t \in I$ to obtain the inequality

$$(3.13)\qquad |u_1(t) - u_2(t)|_2^2 + 2\int_0^t \langle \Delta_p u_1(s) - \Delta_p u_2(s), u_1(s) - u_2(s)\rangle ds \leqq |u_{1,0} - u_{2,0}|_2^2 + \int_0^t |f_1(s) - f_2(s)|_2^2 ds + \int_0^t |u_1(s) - u_2(s)|_2^2 ds.$$

After using the fact that $0 \leqq \langle \Delta_p u_1(s) - \Delta_p u_2(s), u_1(s) - u_2(s)\rangle$ for any $s$ let us apply Gronwall's inequality again to have

$$(3.14)\qquad |u_1(t) - u_2(t)|_2^2 \leqq C(|u_{1,0} - u_{2,0}|_2^2 + \int_0^T |f_1(t) - f_2(t)|_2^2 dt).$$

In this way we finish the proof for (1) in Theorem B.

Secondly, as in (3.6) we have the following: from Proposition 2.1

$$(3.15)\qquad \left(\int_\Omega |Du_1(t) - Du_2(t)|^p dx\right)^{2/p} \leqq (\sigma(u_{1,0}, f_1) + \sigma(u_{2,0}, f_2))^{(2-p)/p} \langle \Delta_p u_1(t) - \Delta_p u_1(t), u_1(t) - u_2(t)\rangle.$$

By virtue of (3.13)–(3.15) we have

$$(3.16)\qquad \int_I \left(\int_\Omega |Du_1(t) - Du_2(t)|^p dx\right)^{2/p} dt \leqq (\sigma(u_{1,0}, f_1) + \sigma(u_{2,0}, f_2))^{(2-p)/p} \cdot (|u_{1,0} - u_{2,0}|_2^2 + \|f_1 - f_2\|_{L^2(I, L^2(\Omega))}^2).$$

Thus, we finish the proof. ∎

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*Remark* 3.1. When $2 \leqq p$, it holds that in Theorem B the estimate (1) is valid and the estimate (2) is replaced by the following form:

$$\|u_1 - u_2\|^p_{L^p(I,V)} \leqq C(|u_{1,0} - u_{2,0}|^2_2 + \|f_1 - f_2\|^{p'}_{L^{p'}(I,V')})$$

with some positive constant $C$. The above assertion is deduced from (3.13), (3.14), Hölder's and Poincaré's inequalities and the inequality

$$\sum_{j=1}^{n}(|\xi|^{p-2}\xi_j - |\eta|^{p-2}\eta_j)(\xi_j - \eta_j) \geqq C_0|\xi - \eta|^p \quad \text{for any} \quad \xi, \eta \in R^n,$$

where $C_0$ depends only on $p$ and $n$, see [C].

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