# ON LONG-RANGE DEPENDENCE IN REGENERATIVE PROCESSES BASED ON A GENERAL ON/OFF SCHEME

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## Abstract

In this paper, we obtain a closed form for the covariance function of a general stationary regenerative process. It is used to derive exact asymptotics of the covariance function of stationary ON/OFF and workload processes, when ON and OFF periods are heavy-tailed and mutually dependent. The case of a G/G/1/0 queueing system with heavy-tailed arrival and/or service times is studied in detail.

*Keywords:* Regenerative process; ON/OFF process; G/G/1/0 queue; long-range dependence

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## 1. Introduction

Willinger *et al.* (1997) introduced the notion of ON/OFF processes to model data traffic in communication networks. In their model, an idealized source alternates between two regimes: an ON state, in which it produces data at a constant rate, and an OFF state, in which it produces no data. It is usually assumed that consecutive ON and OFF durations are independent, having generally different distributions. The corresponding ON/OFF times form an alternating renewal process and the ON/OFF process $A(t)$ is equal to 1 if $t$ falls into an ON period and is equal to 0 if $t$ is in an OFF period. The primary aim of Willinger *et al.* (1997) and subsequent works on ON/OFF processes was the explanation and modelling of observed long-range dependence and self-similarity in network traffic. By a *long-range dependence (LRD) property with rate* $\alpha \in (0, 1)$, we mean a regular decay of the autocovariance of a stationary process $X(t)$, $t \in \mathbb{R}$, i.e.

$$\operatorname{cov}(X(0), X(t)) = L(t)t^{-\alpha}, \qquad t > 0, \tag{1.1}$$

where $L$ is a function which is slowly varying at infinity.

In this paper, we obtain the LRD property of the stationary ON/OFF process $A(t)$ and the forward recurrence time $W(t)$ of the current busy period, in the case when the consecutive ON and OFF periods, $X^{\mathrm{on}}$ and $Y^{\mathrm{off}}$ respectively, are dependent. The random vectors $(X_j^{\mathrm{on}}, Y_j^{\mathrm{off}})$, $j \geq 1$, are assumed to be independent and identically distributed (i.i.d.). It turns out that in this case, similarly to in the case of independent periods $X^{\mathrm{on}}$ and $Y^{\mathrm{off}}$ (see Heath *et al.* (1998)), the decay rate of the covariances of $A(t)$ and $W(t)$ is determined by the decay rate of tail probabilities of ON and OFF periods (see Corollary 3.2, below). In Section 4 we specify the

above results for the G/G/1/0 queueing system (where customers arriving when the system is busy are rejected), in terms of interarrival and service length distributions.

The ON/OFF process (with independent or dependent periods $X^{\mathrm{on}}$ and $Y^{\mathrm{off}}$) is a particular case of a general class of regenerative processes or random tours; see e.g. Smith (1958). The probability distribution of a random tour starting at 0 is completely determined by the tour distribution $(\{\mathcal{Y}(t),\ t \geq 0\}, Z)$, where $Z$ is the length of tour (regeneration interval) and $\{\mathcal{Y}(t),\ t \geq 0\}$ is the tour process. In Lemma 2.1, below, we obtain a closed form of the covariance function of a general stationary random tour $\{X(t),\ t \geq 0\}$ in terms of the renewal function of regeneration points and the expectations $G^0(t) = \mathrm{E}[\mathcal{Y}(Z - t)\,\mathbf{1}_{\{t < Z\}}]$ (the 'backward tour mean'), $G^1(t) = \mathrm{E}[\mathcal{Y}(t)\,\mathbf{1}_{\{t < Z\}}]$ (the 'forward tour mean'), and $R(t) = \mathrm{E}[\mathcal{Y}_0(0)\mathcal{Y}_0(t)\,\mathbf{1}_{\{t < Z_0\}}]$, where $(\{\mathcal{Y}_0(t),\ t \geq 0\}, Z_0)$ is the initial tour and $\mathbf{1}_{\{\cdot\}}$ is the indicator function. The expression for $\mathrm{cov}(X(0), X(t))$ in Lemma 2.1, below, is similar to the equation for the covariance of the ON/OFF process and can be analyzed by renewal methods developed by Heath *et al.* (1998). For the processes $A(t)$ and $W(t)$ discussed above, the asymptotic behaviour of the functions $G^0(t)$, $G^1(t)$, and $R(t)$ as $t \to \infty$ is relatively easy to investigate; see Section 3. For regenerative processes, the asymptotic behaviour of $G^0(t)$, $G^1(t)$, and $R(t)$ may be rather difficult to investigate; Lemma 2.1, below, can be regarded as a preliminary step in establishing the LRD property of such processes.

The LRD property (1.1) is also related to the concept of *long-range count dependence (LRcD)*, introduced by Daley and Vesilo (1997). The LRcD property is defined in terms of asymptotic growth of the variance $\mathrm{var}(N(0, t])$ of the number of counts of a stationary point process in a (large) interval $(0, t]$. Daley and Vesilo (1997), (2000) obtained the LRcD property and the Hurst indices for several classes of point processes (not necessarily renewal) and queueing systems. Formally, a counting process is a particular case of the (integrated) regenerative process with tour $\mathcal{Y}(t) = \delta(t)$, where $\delta(\cdot)$ is the $\delta$-function. In Section 3 we obtain the asymptotics of the covariance function of counts of ON and OFF times in a finite moving interval $I + t$, from which the LRcD property in the renewal case easily follows.

## 2. A stationary regenerative process and its covariance function

Let $\{\mathcal{Y}(t),\ t \geq 0\}$ be a real-valued measurable stochastic process and let $Z > 0$ be a random variable defined on the same probability space. Let

$$(\{\mathcal{Y}_j(t),\ t \geq 0\}, Z_j), \qquad j = 1, 2, \ldots, \tag{2.1}$$

be i.i.d. copies of $(\{\mathcal{Y}(t),\ t \geq 0\}, Z)$. Let $(\{\mathcal{Y}_0(t),\ t \geq 0\}, Z_0)$ be independent of (2.1), generally having a different distribution. Define $T_0 := Z_0$, $T_n := \sum_{i=1}^{n} Z_i$, $n \geq 1$, and consider a stationary regenerative process with regeneration points $T_0, T_1, \ldots$ defined by the following equality:

$$X(t) := \mathcal{Y}_0(t)\,\mathbf{1}_{\{t < T_0\}} + \sum_{j=1}^{\infty} \mathcal{Y}_j(t - T_{j-1})\,\mathbf{1}_{\{T_{j-1} \leq t < T_j\}}, \qquad t \geq 0. \tag{2.2}$$

Assume that $\mu = \mathrm{E}[Z] < \infty$. Then, from Franken *et al.* (1981, Theorem 1.5.4), for any $p \geq 0$ we have

$$\mathrm{E}[|X(t)|^p] = \mu^{-1} \int_0^{\infty} \mathrm{E}[|\mathcal{Y}(s)|^p\,\mathbf{1}_{\{Z > s\}}]\,\mathrm{d}s = \mu^{-1} \int_0^{\infty} \mathrm{E}[|\mathcal{Y}(Z - u)|^p\,\mathbf{1}_{\{Z > u\}}]\,\mathrm{d}u. \tag{2.3}$$

Let $F(x) := P[Z \le x]$. Let $F^{\star n}$ denote the $n$th Stieltjes convolution of $F$ with itself (where $F^{\star 0}(x) := \mathbf{1}_{\{x \ge 0\}}$) and let

$$U(t) := E[N[0, t] \mid T_0 = 0] = \sum_{n=0}^{\infty} F^{\star n}(t)$$

be the renewal function, where $N[0, t] := \sum_{j=0}^{\infty} \mathbf{1}_{\{0 \le T_j \le t\}}$ is the number of renewal points in $[0, t]$ in a pure renewal process having a point at 0. Next, for any $t > 0$, define

$$z(t) := \int_0^t G^0(x) G^1(t - x) \, dx, \tag{2.4}$$

$$R(t) := E[X(0) X(t) \mathbf{1}_{\{t < T_0\}}] = E[\mathcal{Y}_0(0) \mathcal{Y}_0(t) \mathbf{1}_{\{t < T_0\}}], \tag{2.5}$$

where

$$G^0(t) := E[\mathcal{Y}(Z - t) \mathbf{1}_{\{t < Z\}}], \qquad G^1(t) := E[\mathcal{Y}(t) \mathbf{1}_{\{t < Z\}}]. \tag{2.6}$$

In the sequel, we call $G^0(t)$ a *backward tour mean* and $G^1(t)$ a *forward tour mean*.

In the following lemma, we obtain the mean and covariance of the stationary regenerative process in (2.2).

**Lemma 2.1.** *Assume that*

$$E[X^2(0)] = \mu^{-1} \int_0^{\infty} E[\mathcal{Y}^2(s) \mathbf{1}_{\{Z > s\}}] \, ds < \infty. \tag{2.7}$$

*Then, for any $t \ge 0$, we have*

$$E[X(t)] = \mu^{-1} \int_0^{\infty} G^0(s) \, ds = \mu^{-1} \int_0^{\infty} G^1(s) \, ds, \tag{2.8}$$

$$\text{cov}(X(0), X(t)) = R(t) + \mu^{-1} \int_0^t z(t - s) U(ds) - \left(\frac{m}{\mu^2}\right), \tag{2.9}$$

*where*

$$m := \int_0^{\infty} z(s) \, ds = (\mu \, E[X(0)])^2. \tag{2.10}$$

*Proof.* Relations (2.7) and (2.3) imply that $\int_0^{\infty} |G^i(s)|^p \, ds < \infty$, $p = 1, 2$, $i = 0, 1$; in particular,

$$\int_0^{\infty} (G^1(s))^2 \, ds = \int_0^{\infty} (E[\mathcal{Y}(s) \mathbf{1}_{\{s < Z\}}])^2 \, ds \le \int_0^{\infty} E[\mathcal{Y}^2(s) \mathbf{1}_{\{s < Z\}}] \, ds = \mu \, E[X^2(0)].$$

Therefore, the functions $z(t)$ and $R(t)$ in (2.4) and (2.5) respectively, are well-defined and bounded, i.e.

$$|z(t)| \le \mu \, E[X^2(0)], \quad |R(t)| \le E[X^2(0)], \qquad \text{for all } t \ge 0.$$

Relations (2.8) and (2.10) follow from (2.3) and the definitions in (2.4) and (2.6). Consider (2.9). Using (2.4), (2.5), (2.6), and the independence of disjoint tours, we obtain

$$
\begin{aligned}
\mathrm{E}[X(0)X(t)] &= \mathrm{E}[X(0)X(t)\,\mathbf{1}_{\{t<T_0\}}] + \sum_{i=0}^{\infty}\mathrm{E}[X(0)X(t)\,\mathbf{1}_{\{T_i\le t<T_{i+1}\}}] \\
&= R(t) + \sum_{n=0}^{\infty}\int_{\{0<s_0\le s_1<t\}}\mathrm{E}[\mathcal{Y}_0(0)\,\mathbf{1}_{\{T_0\in\,\mathrm{d}s_0\}}\,F^{n\star}(\mathrm{d}s_1-s_0)\mathcal{Y}(t-s_1)\,\mathbf{1}_{\{s_1<t\}}] \\
&= R(t) + \mu^{-1}\int_{\{0<s_0\le s_1<t\}}G^0(s_0)\,\mathrm{d}s_0 U(\mathrm{d}s_1-s_0)G^1(t-s_1) \\
&= R(t) + \mu^{-1}\int_0^t z(t-s)U(\mathrm{d}s),
\end{aligned}
$$

which proves (2.9).

## 3. The LRD property of stationary ON/OFF processes and forward recurrence time processes

Let $(X^{\mathrm{on}}, Y^{\mathrm{off}})$, $(X_1, Y_1)$, $(X_2, Y_2)$, ... be an i.i.d. sequence of random vectors taking values in $(0, \infty)^2$ and with arbitrarily dependent components, and let

$$
Z := X^{\mathrm{on}} + Y^{\mathrm{off}}, \qquad Z_i := X_i + Y_i, \quad i \ge 1. \tag{3.1}
$$

We shall assume that $\mu = \mathrm{E}[Z] = \mu_{\mathrm{on}} + \mu_{\mathrm{off}} < \infty$, where $\mu_{\mathrm{on}} := \mathrm{E}[X^{\mathrm{on}}]$ and $\mu_{\mathrm{off}} := \mathrm{E}[Y^{\mathrm{off}}]$. Consider a stationary version of the alternating point process

$$
0 \le X_0^{\mathrm{on}} < T_0 < T_0 + X_1 < T_1 < T_1 + X_2 < \cdots, \tag{3.2}
$$

where

$$
T_0 := X_0^{\mathrm{on}} + Y_0^{\mathrm{off}}, \qquad T_n := T_0 + \sum_{i=1}^n (X_i + Y_i), \quad n \ge 1.
$$

Intervals $[0, X_0^{\mathrm{on}})$, $[T_i, T_i + X_{i+1})$, $i = 0, 1, \ldots$, will be called *ON intervals* and intervals $[0 = X_0^{\mathrm{on}}, T_0)$, $[T_i + X_{i+1}, T_{i+1})$, $i = 0, 1, \ldots$, will be called *OFF intervals*. The initial time 0 falls into an ON or OFF interval depending on whether $X_0^{\mathrm{on}} > 0$ or $X_0^{\mathrm{on}} = 0$ respectively.

With the alternating point process in (3.2) we associate two stationary processes, the ON/OFF process $\{A(t),\ t \ge 0\}$ and the forward recurrence time of the busy period $\{W(t),\ t \ge 0\}$ as follows:

$$
\begin{aligned}
A(t) &:= \mathbf{1}_{\{X_0^{\mathrm{on}}>t\}} + \sum_{n=0}^{\infty}\mathbf{1}_{\{T_n\le t<T_n+X_{n+1}\}}, \\
W(t) &:= (X_0^{\mathrm{on}} - t)_+ + \sum_{n=0}^{\infty}(T_n + X_{n+1} - t)_+\,\mathbf{1}_{\{T_n\le t<T_{n+1}\}},
\end{aligned} \tag{3.3}
$$

where $a_+ = \max(a, 0)$. In the case of the G/G/1/0 queue, the process $W(t)$ represents the current workload in the system; see Section 4. In the sequel, we call $W(t)$ in (3.3) a *workload process*. It is clear from the definition that the processes $A(t)$ and $W(t)$ are regenerative in the

sense of (2.2), with the tour duration $Z$ given in (3.1), and the corresponding tour processes given by

$$\mathcal{Y}^A(t) := \mathbf{1}_{\{t < X^{\text{on}}\}}, \qquad \mathcal{Y}^W(t) := (X^{\text{on}} - t)_+.$$

By stationarity, we have $P[X_0^{\text{on}} \in dx] = \mu^{-1} P[X^{\text{on}} > x] \, dx, \ x > 0$, therefore

$$E[A(t)] = \mu^{-1} \int_0^\infty P[X^{\text{on}} > s] \, ds = \frac{\mu_{\text{on}}}{\mu},$$

$$E[W(t)] = \mu^{-1} \int_0^\infty E[(X^{\text{on}} - s)_+] \, ds = \frac{E[(X^{\text{on}})^2]}{2\mu}.$$

Moreover,

$$G_A^0(t) = P[Y^{\text{off}} < t \le X^{\text{on}} + Y^{\text{off}}] = \overline{F_{\text{off}}}(t) - F(t),$$
$$G_A^1(t) = P[X^{\text{on}} > t] = \overline{F_{\text{on}}}(t),$$
$$R_A(t) = P[X_0^{\text{on}} > t],$$
$$G_W^0(t) = E[(t - Y^{\text{off}})_+ \mathbf{1}_{\{X^{\text{on}} + Y^{\text{off}} > t\}}], \tag{3.4}$$
$$G_W^1(t) = E[(X^{\text{on}} - t)_+], \tag{3.5}$$
$$R_W(t) = \mu^{-1} \int_t^\infty x(x - t) \overline{F_{\text{on}}}(x) \, dx. \tag{3.6}$$

By definition and from (2.3), we have

$$\int_0^\infty G_W^0(t) \, dt = \int_0^\infty G_W^1(t) \, dt = \mu \, E[W(0)].$$

Next, for any $t > 0$, define

$$z_A(t) := \int_0^t \overline{F_{\text{off}}}(x) \overline{F_{\text{on}}}(t - x) \, dx,$$

$$z_W(t) := \int_0^t G_W^0(x) G_W^1(t - x) \, dx. \tag{3.7}$$

**Corollary 3.1.** (i) *For any $t \ge 0$, we have*

$$\text{cov}(A(0), A(t)) = \frac{\mu_{\text{off}} \mu_{\text{on}}}{\mu^2} - \frac{1}{\mu} \int_0^t z_A(t - s) U(ds). \tag{3.8}$$

(ii) *Assume in addition that $E[(X^{\text{on}})^3] < \infty$. Then, for any $t \ge 0$, we have*

$$\text{cov}(W(0), W(t)) = R_W(t) + \frac{1}{\mu} \int_0^t z_W(t - s) U(ds) - (E[W(0)])^2.$$

Note that (3.8), the equation for the covariance of $A(t)$, is the same as in the case of independent ON and OFF periods; see Heath *et al.* (1998, Equation (2.7)). Using the results of Heath *et al.* (1998), we therefore obtain the following corollary.

**Corollary 3.2.** *Let*

$$\overline{F_{\mathrm{on}}}(t) = t^{-\alpha_{\mathrm{on}}} L_{\mathrm{on}}(t), \tag{3.9}$$

$$\overline{F_{\mathrm{off}}}(t) = o(\overline{F_{\mathrm{on}}}(t)), \qquad t \to \infty, \tag{3.10}$$

*where $1 < \alpha_{\mathrm{on}} < 2$ and where $L_{\mathrm{on}}$ is a function, slowly varying at infinity. Assume, moreover, that there exists an $n \geq 1$ such that $F^{\star n}$ is nonsingular, where we recall that $F$ is the distribution function of $Z = X^{\mathrm{on}} + Y^{\mathrm{off}}$. Then we obtain*

$$\mathrm{cov}(A(0), A(t)) \sim \frac{\mu_{\mathrm{off}}^2}{(\alpha_{\mathrm{on}} - 1)\mu^3} t^{-(\alpha_{\mathrm{on}}-1)} L_{\mathrm{on}}(t), \qquad t \to \infty. \tag{3.11}$$

*Proof.* Note that (3.9) and (3.10) imply

$$\overline{F}(t) \sim L_{\mathrm{on}}(t) t^{-\alpha_{\mathrm{on}}}, \tag{3.12}$$

$$z_A(t) \sim \mu_{\mathrm{off}} L_{\mathrm{on}}(t) t^{-\alpha_{\mathrm{on}}}, \tag{3.13}$$

where (3.12) is proved in Lemma 3.1, below. Relation (3.13) follows from (3.12), (3.9), (3.10), and Heath *et al.* (1998, Lemma 4.2). Finally, (3.11) follows from (3.8), (3.13), and the argument in Heath *et al.* (1998, proof of Theorem 4.3).

Consider now the LRcD property of the introduced ON/OFF process. Let $I \subset (0, \infty)$ be a (fixed) interval of a finite Lebesgue measure $|I|$. Introduce the counting processes

$$N_{\mathrm{on}}(I) = \sum_{n=0}^{\infty} \mathbf{1}_{\{T_n \in I\}},$$

$$N_{\mathrm{off}}(I) = \mathbf{1}_{\{X_0^{\mathrm{on}} \in I\}} + \sum_{n=0}^{\infty} \mathbf{1}_{\{T_n + X_{n+1} \in I\}}.$$

Note that the processes $N_{\mathrm{on}}(I + t)$ and $N_{\mathrm{off}}(I + t)$, $t \geq 0$, are both stationary and

$$\mathrm{E}[N_{\mathrm{on}}(I)] = \mathrm{E}[N_{\mathrm{off}}(I)]$$

$$= \frac{|I|}{\mu},$$

$$\mathrm{cov}(N_{\mathrm{on}}(I), N_{\mathrm{on}}(I + t)) = \int_I \int_{I+t} \mathrm{P}[N_{\mathrm{on}}(\mathrm{d}x) = 1, N_{\mathrm{on}}(\mathrm{d}y) = 1] - \left(\frac{|I|}{\mu}\right)^2$$

$$= \mu^{-1} \int_I (U(I + t - x) - \mu^{-1}|I|) \, \mathrm{d}x, \tag{3.14}$$

$$\mathrm{cov}(N_{\mathrm{off}}(I), N_{\mathrm{off}}(I + t)) = \int_I \int_{I+t} \mathrm{P}[N_{\mathrm{off}}(\mathrm{d}x) = 1, N_{\mathrm{off}}(\mathrm{d}y) = 1] - \left(\frac{|I|}{\mu}\right)^2$$

$$= \mu^{-1} \int_I ((F_{\mathrm{off}} \star U \star F_{\mathrm{on}})(I + t - x) - \mu^{-1}|I|) \, \mathrm{d}x, \tag{3.15}$$

provided that $I \cap (I + t) = \varnothing$.

**Corollary 3.3.** *Assume the same conditions as in Corollary 3.2. Then, as $t \to \infty$, we obtain*

$$\left.\begin{array}{l} \text{cov}(N_{\text{on}}(I), N_{\text{on}}(I+t)) \\ \text{cov}(N_{\text{off}}(I), N_{\text{off}}(I+t)) \end{array}\right\} \sim \frac{|I|^2}{(\alpha_{\text{on}}-1)\mu^3} t^{-(\alpha_{\text{on}}-1)} L_{\text{on}}(t), \tag{3.16}$$

$$\left.\begin{array}{l} \text{var}(N_{\text{on}}[0,t]) \\ \text{var}(N_{\text{off}}[0,t]) \end{array}\right\} \sim \frac{2}{(\alpha_{\text{on}}-1)(2-\alpha_{\text{on}})(3-\alpha_{\text{on}})\mu^3} t^{3-\alpha_{\text{on}}} L_{\text{on}}(t). \tag{3.17}$$

*Proof.* Let $\overline{F_2}(x) := \mu^{-1} \int_x^\infty \overline{F}(y)\,dy$. From Heath *et al.* (1998, Equation (3.12)) and Lemma 3.1, below,

$$U(I+t) - \mu^{-1}|I| \sim \mu^{-1} \int_{I+t} \overline{F_2}(x)\,dx \sim \frac{|I|}{(\alpha_{\text{on}}-1)\mu^2} t^{-(\alpha_{\text{on}}-1)} L_{\text{on}}(t), \tag{3.18}$$

implying the first asymptotic relation in (3.16) by (3.14) and by the dominated convergence theorem. The second relation in (3.16) follows analogously from (3.15) and

$$(F_{\text{off}} \star U \star F_{\text{on}})(I+t) - \mu^{-1}|I| \sim \frac{|I|}{(\alpha_{\text{on}}-1)\mu^2} t^{-(\alpha_{\text{on}}-1)} L_{\text{on}}(t). \tag{3.19}$$

Relation (3.19) follows from Heath *et al.* (1998, Theorem 3.1(ii)), the fact that $\int_0^\infty \overline{F}(y)\,dy = \int_0^\infty \overline{F_{\text{on}} \star F_{\text{off}}}(y)\,dy = \mu$, and (3.18). Relation (3.17) is immediate from (3.16) and $\alpha_{\text{on}} \in (1, 2)$.

**Remark 3.1.** Note that the covariance $\text{cov}(A(0), A(t)) = \text{cov}(1 - A(0), 1 - A(t))$ as well as the covariances in (3.14) and (3.15) are symmetric with respect to the change of $F_{\text{on}}$ by $F_{\text{off}}$ and vice versa. Accordingly, Corollaries 3.2 and 3.3 remain valid, with obvious changes, if conditions on ON and OFF intervals are exchanged.

**Lemma 3.1.** *Let $X, Y \geq 0$ be arbitrarily dependent random variables such that*

(i) $P[X > t] = L(t)t^{-\gamma}$,

(ii) $P[Y > t] = o(P[X > t])$ *as $t \to \infty$,*

*where $\gamma > 0$ and $L$ is slowly varying at infinity. Let $Z := X + Y$, then*

$$P[Z > t] \sim L(t)t^{-\gamma}.$$

*Proof.* Let $0 < \varepsilon < 1$. Then we have

$$P[X > t] \leq P[X + Y > t]$$
$$\leq P[Y > \varepsilon t] + P[Y \leq \varepsilon t, X + Y > t]$$
$$\leq P[Y > \varepsilon t] + P[X > (1 - \varepsilon)t].$$

Dividing the above inequality by $P[X > t]$, we obtain

$$1 \leq \liminf_{t\to\infty} \frac{P[Z > t]}{L(t)t^{-\gamma}} \leq \limsup_{t\to\infty} \frac{P[Z > t]}{L(t)t^{-\gamma}} \leq (1 - \varepsilon)^{-\gamma}, \tag{3.20}$$

where we used the following fact:

$$\limsup_{t\to\infty} \frac{P[Y > \varepsilon t]}{L(t)t^{-\gamma}} \leq \limsup_{t\to\infty} \frac{P[Y > \varepsilon t]}{P[X > \varepsilon t]} \limsup_{t\to\infty} \frac{L(\varepsilon t)(\varepsilon t)^{-\gamma}}{L(t)t^{-\gamma}} = 0$$

(by assumptions (i) and (ii)), while $\limsup_{t \to \infty} L((1-\varepsilon)t)((1-\varepsilon)t)^{-\gamma}/L(t)t^{-\gamma} = (1-\varepsilon)^{-\gamma}$. As $\varepsilon > 0$ is arbitrary in (3.20), this completes the proof.

Next, we turn to the asymptotics of $\text{cov}(W(0), W(t))$. According to Corollary 3.1, this can be written as

$$\text{cov}(W(0), W(t))) = R_W(t) - \mu^{-1}h(t), \tag{3.21}$$

where the function $R_W(t)$ is defined in (3.6) and $h(t) := (m/\mu) - \int_0^t z_W(t-x)U(\mathrm{d}x)$, with $z_W(t)$ given in (3.7). The next corollary encompasses the following two cases:

(i)  $h(t) = o(R_W(t))$,

(ii)  $R_W(t) = o(h(t))$.

In case (i), the asymptotics of $\text{cov}(W(0), W(t))$ coincides with that of $R_W(t)$, and in case (ii), it coincides with the asymptotics of $-\mu^{-1}h(t)$.

**Corollary 3.4.** *Assume that the following conditions hold:* $\mathrm{E}[(X^{\mathrm{on}})^3] < \infty$, $F^{\star n}$ *is nonsingular for some* $n \geq 1$, *and*

$$\overline{F_{\mathrm{off}}}(t) = t^{-\alpha_{\mathrm{off}}} L_{\mathrm{off}}(t), \tag{3.22}$$

$$G_W^0(t) = o(\overline{F_{\mathrm{off}}}(t)), \qquad t \to \infty, \tag{3.23}$$

*where* $1 < \alpha_{\mathrm{off}} < 2$ *and* $L_{\mathrm{off}}$ *is slowly varying at infinity.*

(i) *Let*

$$\overline{F_{\mathrm{on}}}(t) = t^{-\alpha_{\mathrm{on}}} L_{\mathrm{on}}(t), \tag{3.24}$$

*where* $3 < \alpha_{\mathrm{on}} < \alpha_{\mathrm{off}} + 2$ *and* $L_{\mathrm{on}}$ *is slowly varying at infinity. Then*

$$\text{cov}(W(0), W(t)) \sim R_W(t) \sim \frac{1}{\mu(\alpha_{\mathrm{on}} - 3)(\alpha_{\mathrm{on}} - 2)} t^{-(\alpha_{\mathrm{on}}-3)} L_{\mathrm{on}}(t). \tag{3.25}$$

(ii) *Let*

$$\overline{F_{\mathrm{on}}}(t) = o(t^{-2}\overline{F_{\mathrm{off}}}(t)). \tag{3.26}$$

*Then*

$$\text{cov}(W(0), W(t)) \sim -\mu^{-1}h(t) \sim \frac{m}{(\alpha_{\mathrm{off}} - 1)\mu^3} t^{-(\alpha_{\mathrm{off}}-1)} L_{\mathrm{off}}(t). \tag{3.27}$$

*Proof.* (i) Note that (3.24) implies

$$R_W(t) = \frac{1}{\mu} \int_t^\infty x(x-t)x^{-\alpha_{\mathrm{on}}} L_{\mathrm{on}}(x)\,\mathrm{d}x \sim \frac{1}{\mu(\alpha_{\mathrm{on}} - 3)(\alpha_{\mathrm{on}} - 2)} t^{-(\alpha_{\mathrm{on}}-3)} L_{\mathrm{on}}(t).$$

Hence, by (3.21), (3.25) reduces to

$$h(t) = o(t^{-(\alpha_{\mathrm{on}}-3)} L_{\mathrm{on}}(t)). \tag{3.28}$$

Let us prove that $\mathrm{E}[(X^{\mathrm{on}})^3] < \infty$ and that (3.22) and (3.23) (where $1 < \alpha_{\mathrm{off}} < 2$) imply

$$z_W(t) = o(\overline{F}(t)) = o(L_{\mathrm{off}}(t)t^{-\alpha_{\mathrm{off}}}), \qquad t \to \infty. \tag{3.29}$$

Indeed, by (3.5) and the Chebyshev inequality, we have $G_W^1(t) \leq t^{-2} \mathrm{E}[(X^{\mathrm{on}})^3]$. Using this fact and (3.23), we obtain

$$
\begin{aligned}
z_W(t) &= \int_0^{t/2} G_W^0(x) G_W^1(t-x)\, \mathrm{d}x + \int_{t/2}^t G_W^0(x) G_W^1(t-x)\, \mathrm{d}x \\
&\leq C\left(\frac{t}{2}\right)^{-2} \int_0^\infty G_W^0(x)\, \mathrm{d}x + o(L_{\mathrm{off}}(t) t^{-\alpha_{\mathrm{off}}}) \sup_{t/2 \leq x \leq t} \frac{L_{\mathrm{off}}(x) x^{-\alpha_{\mathrm{off}}}}{L_{\mathrm{off}}(t) t^{-\alpha_{\mathrm{off}}}} \int_0^\infty G_W^1(x)\, \mathrm{d}x \\
&= O(t^{-2}) + o(L_{\mathrm{off}}(t) t^{-\alpha_{\mathrm{off}}}) \\
&= o(L_{\mathrm{off}}(t) t^{-\alpha_{\mathrm{off}}}),
\end{aligned}
$$

since $\alpha_{\mathrm{off}} < 2$. This proves (3.29). Using (3.29) and Heath *et al.* (1998, Theorem 3.1(iii)), we obtain

$$
h(t) \sim -\frac{m}{(\alpha_{\mathrm{off}} - 1)\mu^2} t^{-(\alpha_{\mathrm{off}} - 1)} L_{\mathrm{off}}(t), \qquad t \to \infty, \tag{3.30}
$$

which implies (3.28), since $\alpha_{\mathrm{on}} < \alpha_{\mathrm{off}} + 2$.

(ii) Equation (3.27) follows from (3.21), (3.30), and $R_W(t) = o(h(t))$; in other words, it follows from

$$
R_W(t) = \mu^{-1} \int_t^\infty x(x-t) \overline{F_{\mathrm{on}}}(x)\, \mathrm{d}x = o(t^{-(\alpha_{\mathrm{off}} - 1)} L_{\mathrm{off}}(t)).
$$

The last fact follows from (3.26), (3.22), and $\int_t^\infty x(x-t) x^{-2} \overline{F_{\mathrm{off}}}(x)\, \mathrm{d}x = O(t^{1-\alpha_{\mathrm{off}}} L_{\mathrm{off}}(t))$, which completes the proof.

**Corollary 3.5.** *Assume that the following conditions hold:* $\overline{F_{\mathrm{on}}}(t) = t^{-\alpha_{\mathrm{on}}} L_{\mathrm{on}}(t)$, $\overline{F_{\mathrm{off}}}(t) = t^{-\alpha_{\mathrm{off}}} L_{\mathrm{off}}(t)$, $F^{n\star}$ *is nonsingular for some* $n \geq 1$, *and (3.23), where* $3 < \alpha_{\mathrm{on}} < 4$, $1 < \alpha_{\mathrm{off}} < 2$, *and* $\alpha_{\mathrm{on}} \neq \alpha_{\mathrm{off}} + 2$. *Then,* $W(t)$ *in (3.3) has the LRD property with rate* $\min(\alpha_{\mathrm{on}} - 3, \alpha_{\mathrm{off}} - 1) \in (0, 1)$.

Note that all the conditions of Corollaries 3.2–3.5, with the exception of (3.23) and the obviously mild condition that $F^{\star n}$ is nonsingular for some $n \geq 0$, are formulated in terms of marginal distributions of ON and OFF periods. We show below that assumption (3.23) is also not very restrictive, in the sense that it holds in the case of independent ON and OFF periods.

**Proposition 3.1.** *Assume that* $X^{\mathrm{on}}$ *and* $Y^{\mathrm{off}}$ *are independent,* $\mathrm{E}[(X^{\mathrm{on}})^3] < \infty$, *and (3.22) is satisfied, with* $1 < \alpha_{\mathrm{off}} < 2$. *Then (3.23) holds.*

*Proof.* Decompose $G_W^0(t) = J_1(t) + J_2(t)$, where

$$
J_1(t) := \int_0^{t(1-\varepsilon)} (t-u) \overline{F_{\mathrm{on}}}(t-u) F_{\mathrm{off}}(\mathrm{d}u), \qquad J_2(t) := \int_{t(1-\varepsilon)}^t (t-u) \overline{F_{\mathrm{on}}}(t-u) F_{\mathrm{off}}(\mathrm{d}u).
$$

Using $t \overline{F_{\mathrm{on}}}(t) \leq C t^{-2}$, for all $t > 0$, we obtain, for each $\varepsilon > 0$,

$$
J_1(t) \leq C \int_0^{t(1-\varepsilon)} (t-u)^{-2} F_{\mathrm{off}}(\mathrm{d}u) \leq C(t\varepsilon)^{-2},
$$

where the constant $C$ does not depend on $t$ and $\varepsilon$. On the other hand, using (3.22), we have

$$
\begin{aligned}
J_2(t) &\leq C \int_{t(1-\varepsilon)}^{t} F_{\text{off}}(\mathrm{d}u) \\
&= C \frac{L_{\text{off}}(t)}{(t(1-\varepsilon))^{\alpha_{\text{off}}}} \left( \frac{L_{\text{off}}(t(1-\varepsilon))}{L_{\text{off}}(t)} - 1 \right) + C L_{\text{off}}(t) \left( \frac{1}{(t(1-\varepsilon))^{\alpha_{\text{off}}}} - \frac{1}{t^{\alpha_{\text{off}}}} \right) \\
&=: J_3(t) + J_4(t).
\end{aligned}
$$

Here, the term $J_4(t) = O(\varepsilon L_{\text{off}}(t) t^{-\alpha_{\text{off}}})$; in other words, the ratio $J_4(t)/(L_{\text{off}}(t) t^{-\alpha_{\text{off}}})$ can be made arbitrarily small uniformly in $t \geq 1$ by taking $\varepsilon > 0$ to be small enough. Clearly, $J_i(t) = o(L_{\text{off}}(t) t^{-\alpha_{\text{off}}})$, $i = 1, 3$, for any $\varepsilon > 0$ fixed. This proves (3.23).

## 4. Application to the G/G/1/0 queue

Consider a G/G/1/0 queueing system, with general interarrival distribution $F_\tau(x) = \mathrm{P}[\tau \leq x]$ and general service distribution $F_\sigma(x) = \mathrm{P}[\sigma \leq x]$, $x \in (0, \infty)$. We assume that $\mu_\sigma := \mathrm{E}[\sigma] < \infty$ and $\mu_\tau := \mathrm{E}[\tau] < \infty$. The interarrival and service times $\tau_i$, $\sigma_i$, $i = 1, 2, \ldots$, are all independent, and customers arriving when the system is busy are rejected. The number of customers in the system at time $t$, $A(t)$, is equal to 1 or 0. As noted in Section 1, $A(t)$ is a particular case of the ON/OFF process with generally dependent ON and OFF intervals whose joint distribution is given by

$$
\begin{aligned}
\mathrm{P}[X^{\text{on}} \in \mathrm{d}x, Y^{\text{off}} \in \mathrm{d}y] &:= \sum_{i=1}^{\infty} \mathrm{P}[\sigma_1 \in \mathrm{d}x, \tau_1 + \cdots + \tau_{i-1} \leq x, \tau_1 + \cdots + \tau_i \in \mathrm{d}y + x] \\
&= \mathrm{P}[\sigma \in \mathrm{d}x] \int_0^x \sum_{i=1}^{\infty} \mathrm{P}[\tau_1 + \cdots + \tau_{i-1} \in \mathrm{d}u, \tau_i \in \mathrm{d}y + x - u] \\
&= F_\sigma(\mathrm{d}x) \int_0^x H(\mathrm{d}u) F_\tau(x - u + \mathrm{d}y), \qquad (4.1)
\end{aligned}
$$

where

$$
H(t) := \sum_{n=0}^{\infty} F_\tau^{\star n}(t)
$$

is the renewal function of the arrival process. A stationary version of the corresponding G/G/1/0 queue and the processes $A(t)$, $W(t)$, $t \geq 0$, can be constructed as in Borovkov (1976, Chapter 7) or Baccelli and Brémaud (1994, Chapter 2.5).

We obtain the following results about the asymptotics of the covariances of the processes $A(t)$ and $W(t)$ in the G/G/1/0 queue with heavy tailed arrival and/or service times.

**Proposition 4.1.** *Assume that $F^{\star n}$ is nonsingular for some $n \geq 1$, where $F(x) = \mathrm{P}[X^{\text{on}} + Y^{\text{off}} \leq x]$.*

  (i) *Let $\overline{F_\tau}(x) = L_\tau(x) x^{-\alpha}$ and $\overline{F_\sigma}(x) = o(\overline{F_\tau}(x))$, where $L_\tau$ is slowly varying at infinity and $1 < \alpha < 2$. Then, as $t \to \infty$, we have*

$$
\mathrm{cov}(A(0), A(t)) \sim \frac{\mu_\sigma^2}{(\alpha - 1) \mu^2 \mu_\tau} t^{-(\alpha - 1)} L_\tau(t),
$$

  *where $\mu := \mu_\sigma + \mu_{\text{off}}$ and $\mu_{\text{off}}$ is given in (4.5).*

(ii) *Let* $\overline{F_\sigma}(x) = L_\sigma(x)x^{-\beta}$ *and* $\overline{F_\tau}(x) = o(\overline{F_\sigma}(x))$, *where* $L_\sigma$ *is slowly varying at infinity and* $1 < \beta < 2$. *Then, as* $t \to \infty$, *we have*

$$\mathrm{cov}(A(0), A(t)) \sim \frac{(\mu - \mu_\sigma)^2}{(\beta - 1)\mu^3} t^{-(\beta-1)} L_\sigma(t).$$

Proposition 4.1 is a direct consequence of Corollary 3.2 and Lemma 4.1, below. Similarly, Corollary 3.4 and Lemmas 4.1–4.2, below, yield the following result.

**Proposition 4.2.** *Assume that the following conditions hold:* $\mathrm{E}[\sigma^3] < \infty$, $F^{\star n}$ *is nonsingular for some* $n \geq 1$, *and* $\overline{F_\tau}(x) = x^{-\alpha} L_\tau(x)$, *where* $1 < \alpha < 2$ *and* $L_\tau$ *is slowly varying at infinity.*

(i) *Let* $\overline{F_\sigma}(x) = x^{-\beta} L_\sigma(x)$, *where* $3 < \beta < \alpha + 2$ *and* $L_\sigma$ *is slowly varying at infinity. Then*

$$\mathrm{cov}(W(0), W(t)) \sim \frac{1}{\mu(\beta - 3)(\beta - 2)} t^{-(\beta-3)} L_\sigma(t).$$

(ii) *Let* $\overline{F_\sigma}(x) = o(x^{-2}\overline{F_\tau}(x))$. *Then*

$$\mathrm{cov}(W(0), W(t)) \sim \frac{m}{(\alpha - 1)\mu^3} t^{-(\alpha-1)} L_\tau(t),$$

*where* $m := (\mathrm{E}[\sigma^2])^2/4$ *and* $\mu$ *is the same as in Proposition 4.1.*

The following lemma is a straightforward consequence of (4.1).

**Lemma 4.1.** *We have*

$$\overline{F_{\mathrm{on}}}(x) = \overline{F_\sigma}(x), \tag{4.2}$$

$$\mu_{\mathrm{on}} = \mu_\sigma, \tag{4.3}$$

$$\overline{F_{\mathrm{off}}}(x) = \int_0^\infty H(\mathrm{d}u) \int_u^\infty F_\sigma(\mathrm{d}w) \overline{F_\tau}(w - u + x), \tag{4.4}$$

$$\mu_{\mathrm{off}} = \mu_\tau \int_0^\infty \overline{F_\sigma}(u) H(\mathrm{d}u) - \mu_\sigma, \tag{4.5}$$

$$\overline{F_{\mathrm{off}}}(x) \leq q\overline{F_\tau}(x), \tag{4.6}$$

*where* $q := \int_0^\infty \overline{F_\sigma}(u) H(\mathrm{d}u) = (\mu_\sigma + \mu_{\mathrm{off}})/\mu_\tau$. *If, in addition,* $F_\tau$ *is long-tailed, i.e. for every* $z \geq 0$

$$\lim_{x \to \infty} \frac{\overline{F_\tau}(x + z)}{\overline{F_\tau}(x)} = 1, \tag{4.7}$$

*then*

$$\overline{F_{\mathrm{off}}}(x) \sim q\overline{F_\tau}(x), \qquad x \to \infty. \tag{4.8}$$

*Proof.* Note that the renewal equation $H(x) = 1 + \int_0^t F_\tau(x - u)H(\mathrm{d}u)$ implies that, for every $x \geq 0$,

$$\int_0^x \overline{F_\tau}(x - u)H(\mathrm{d}u) = 1, \qquad \int_0^x \left( \int_0^{x-u} \overline{F_\tau}(z)\,\mathrm{d}z \right) H(\mathrm{d}u) = x. \tag{4.9}$$

Relations (4.2), (4.3), and (4.4) are immediate by (4.1) and (4.9). Consider (4.5). Using (4.4) and (4.1), we have

$$
\mu_{\text{off}} = \int_0^\infty H(\mathrm{d}u) \int_u^\infty F_\sigma(\mathrm{d}w) \int_{w-u}^\infty \overline{F_\tau}(z)\,\mathrm{d}z
$$
$$
= \mu_\tau \int_0^\infty F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}u) - \int_0^\infty F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}u) \int_0^{w-u} \overline{F_\tau}(z)\,\mathrm{d}z.
$$

From the second equality in (4.9), (4.5) therefore follows.

Next, using (4.4), (4.5), (4.7), and the dominated convergence theorem, we obtain

$$
\overline{F_{\text{off}}}(x) = \overline{F_\tau}(x) \int_0^\infty \left\{ \int_0^\infty \frac{\overline{F_\tau}(x+z)}{\overline{F_\tau}(x)} F_\sigma(u+\mathrm{d}z) \right\} H(\mathrm{d}u) \tag{4.10}
$$
$$
\sim \overline{F_\tau}(x) \int_0^\infty \overline{F_\sigma}(u) H(\mathrm{d}u)
$$
$$
= q \overline{F_\tau}(x),
$$

which proves (4.8). Relation (4.6) is immediate from (4.10) and $\overline{F_\tau}(x+z) \le \overline{F_\tau}(x)$, $x, z \ge 0$, and does not require (4.7). Therefore, the proof is complete.

**Lemma 4.2.** *Assume that* $\mathrm{E}[\sigma^3] < \infty$ *and* $\overline{F_\tau}(x) = L(x)x^{-\alpha}$, *where* $1 < \alpha < 2$ *and* $L$ *is slowly varying at infinity. Let* $G_W^0(t)$ *be defined as in* (3.4), *from the joint distribution in* (4.1). *Then* $G_W^0(t) = o(\overline{F_\tau}(t))$.

*Proof.* From (4.1), we have

$$
G_W^0(t) = \int_0^t (t-u) \int_{t-u}^\infty \mathrm{P}[X^{\text{on}} \in \mathrm{d}x, Y^{\text{off}} \in \mathrm{d}u]
$$
$$
= \int_0^t (t-u) \int_{t-u}^\infty F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z) F_\tau(w-z+\mathrm{d}u)
$$
$$
= Q_1(t) + Q_2(t),
$$

where

$$
Q_1(t) := \int_0^t F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z) \int_0^w x F_\tau(w-z+t-\mathrm{d}x),
$$
$$
Q_2(t) := \int_t^\infty F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z) \int_0^t x F_\tau(w-z+t-\mathrm{d}x).
$$

The second term can be easily evaluated as follows using integration by parts, (4.9), and the Chebyshev inequality:

$$
Q_2(t) = \int_t^\infty F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z) \left( t\overline{F_\tau}(w-z) - \int_0^t \overline{F_\tau}(w-z+t-x)\,\mathrm{d}x \right)
$$
$$
\le t \int_t^\infty F_\sigma(\mathrm{d}w) \int_0^w \overline{F_\tau}(w-z) H(\mathrm{d}z)
$$
$$
= t \overline{F_\sigma}(t)
$$
$$
\le t^{-2} \mathrm{E}[\sigma^3]
$$
$$
= o(L(t)t^{-\alpha}). \tag{4.11}
$$

To evaluate $Q_1(t)$, we split it up, i.e.

$$Q_1(t) = \int_0^{t_*} F_\sigma(\mathrm{d}w) \cdots + \int_{t_*}^t F_\sigma(\mathrm{d}w) \cdots =: Q_{11}(t) + Q_{12}(t),$$

where $t_* = t^\gamma$ with $(1+\alpha)/3 < \gamma < 1$. Similarly, as above, using integration by parts and (4.9), we have

$$\begin{aligned}
Q_{12}(t) &= \int_{t_*}^t F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z)\left(w\overline{F_\tau}(t-z) - \int_0^w \overline{F_\tau}(w-z+t-x)\,\mathrm{d}x\right)\\
&\leq \int_{t_*}^t w F_\sigma(\mathrm{d}w) \int_0^w \overline{F_\tau}(t-z)H(\mathrm{d}z)\\
&\leq \int_{t_*}^t w F_\sigma(\mathrm{d}w) \int_0^w \overline{F_\tau}(w-z)H(\mathrm{d}z)\\
&= \int_{t_*}^t w F_\sigma(\mathrm{d}w)\\
&\leq t t_*^{-3}\, \mathrm{E}[\sigma^3]\\
&= o(L(t)t^{-\alpha}).
\end{aligned} \tag{4.12}$$

Consider the main term, $Q_{11}(t)$. Integrating by parts, we have

$$\begin{aligned}
Q_{11}(t) &= \int_0^{t_*} F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z) \int_0^w x F_\tau(w-z+t-\,\mathrm{d}x)\\
&= \int_0^{t_*} F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z)\left(w\overline{F_\tau}(t-z) - \int_0^w \overline{F_\tau}(w-z+t-x)\,\mathrm{d}x\right)\\
&= P_1(t) - P_2(t),
\end{aligned}$$

where

$$P_1(t) := \int_0^{t_*} w F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z)(\overline{F_\tau}(t-z) - \overline{F_\tau}(t)),$$

$$P_2(t) := \int_0^{t_*} F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z) \int_0^w (\overline{F_\tau}(w-z+t-x) - \overline{F_\tau}(t))\,\mathrm{d}x.$$

In view of (4.11) and (4.12), it remains to show that $P_i(t) = o(L(t)t^{-\alpha})$, $i = 1, 2$. We show that this relation holds for $P_2(t)$; the proof for $P_1(t)$ is analogous. To that end, write

$$P_2(t) = \overline{F_\tau}(t)(p'(t) + p''(t)),$$

where

$$p'(t) := \int_0^{t_*} F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z) \int_0^w \left(\frac{w-z+t-x}{t}\right)^{-\alpha}\left(\frac{L(w-z+t-x)}{L(t)} - 1\right)\mathrm{d}x,$$

$$p''(t) := \int_0^{t_*} F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z) \int_0^w \left(\left(1 + \frac{w-z-x}{t}\right)^{-\alpha} - 1\right)\mathrm{d}x.$$

Using the dominated convergence theorem, a well-known property of slowly varying functions, and $H(w) = O(w)$, we see that

$$
\begin{aligned}
p'(t) &= O\left(\delta_t \int_0^{t_*} F_\sigma(\mathrm{d}w) \int_0^w w H(\mathrm{d}z)\right) \\
&= O\left(\delta_t \int_0^\infty w H(w) F_\sigma(\mathrm{d}w)\right) \\
&= O(\delta_t) \\
&= o(1),
\end{aligned}
$$

where $\delta_t \to 0$ as $t \to 0$. Finally,

$$
\begin{aligned}
|p''(t)| &\le C t^{-1} \int_0^{t_*} F_\sigma(\mathrm{d}w) \int_0^w H(\mathrm{d}z) \int_0^w (w - z - x)\, \mathrm{d}x \\
&\le C t^{-1} \int_0^{t_*} w H(w) F_\sigma(\mathrm{d}w) \\
&= O(t^{-1}) \\
&= o(1).
\end{aligned}
$$

This completes the proof.

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