ON THE PRODUCT OF TWO KUMMER SERIES

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1. Introduction. Let α, β, μ, ν, z be complex numbers such that 2μ and 2ν are not negative integers. Using the notation of **(4)** for generalized hypergeometric series, we set

$$(1) \qquad \phi(z) = {}_1F_1\left[\begin{matrix}\mu + \frac{1}{2} - \alpha; - z\\ 2\mu + 1\end{matrix}\right] {}_1F_1\left[\begin{matrix}\nu + \frac{1}{2} - \beta; z\\ 2\nu + 1\end{matrix}\right]$$

and define $a_n = a_n(\alpha, \beta, \mu, \nu)$ by

$$(2) \qquad \phi(z) = \sum_{n=0}^{\infty} a_n z^n.$$

It is evident that the function $\phi(z)$ does not change if the parameters are subjected to the transformation

$$S: (\alpha, \beta, \mu, \nu; z) \to (\beta, \alpha, \nu, \mu; - z);$$

this merely interchanges the two factors in (1). The function $\phi(z)$ also admits of a further, and less evident, transformation. Applying Kummer's transformation **(4**, 6.3 (7)**)** to the two series on the right of (1), we find that $\phi(z)$ can also be written as follows:

$$(3) \qquad \phi(z) = {}_1F_1\left[\begin{matrix}\mu + \frac{1}{2} + \alpha; z\\ 2\mu + 1\end{matrix}\right] {}_1F_1\left[\begin{matrix}\nu + \frac{1}{2} + \beta; - z\\ 2\nu + 1\end{matrix}\right].$$

Thus, $\phi(z)$ is invariant under the transformation

$$T: (\alpha, \beta, \mu, \nu; z) \to (- \alpha, - \beta, \mu, \nu; - z).$$

It follows from the above that the coefficients a_n satisfy

$$(4) \qquad a_n(\beta, \alpha, \nu, \mu) = a_n(- \alpha, - \beta, \mu, \nu) = (- 1)^n a_n(\alpha, \beta, \mu, \nu).$$

These relations of symmetry are not completely mirrored in the representation (1) of the generating function of the a_n. While it is true that the function $\phi(z)$ as a whole is invariant under both transformations S and T, the particular factorization (1) is invariant only under S but not under T.

The primary objective of this paper is the derivation of a generating function for the coefficients a_n which renders explicit both relations (4). Widening the scope of our problem somewhat, we shall in fact derive a *complete set of generating functions* of the form

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$$\psi(z) = \sum_{n=0}^{\infty} c_n a_n z^n, \tag{5}$$

where the c_n are quotients of products of factorials and the functions $\psi(z)$ are products of two (generalized) hypergeometric series. Any generating function of this type belongs to exactly one of four classes according to the invariance of the factorization of $\psi(z)$ under none, exactly one, or both of the transformations S and T. The set of generating functions to be given below is complete in the sense that each class is represented in it. Applying the transformations under which the factorizations are not invariant, we shall obtain $4 + 2 + 2 + 1$ different factorizations for generating functions of the form (5).

Our results do not answer completely the following question raised by a referee. Do there exist generating functions of the form (5) with factorizations which are invariant under ST but not under both S and T? It is easy to show that any factorization left invariant under ST and one of the transformations S and T is left invariant also under the other, but our method fails to show whether there exists a generating function with a factorization which, although invariant under ST, is changed by both S and T.

2. Representations of a_n in terms of terminating ${}_3F_2$. By Cauchy multiplication of the two series on the right of (2) we get the expression

$$a_n = \frac{(\nu + \frac{1}{2} - \beta)_n}{(2\nu + 1)_n\, n!}\, {}_3F_2\left[\begin{matrix} -2\nu - n,\ \mu + \frac{1}{2} - \alpha,\ -n; \\ 2\mu + 1,\ -\nu + \beta - n + \frac{1}{2} \end{matrix}\right] \tag{6}$$

We shall now utilize some results of a theory due to Whipple on transformations of functions ${}_3F_2$ with unit argument (2, chapter III). According to Whipple, any terminating ${}_3F_2$ can be represented as a product of factorials and a terminating ${}_3F_2$ in eighteen different ways. We divide the resulting eighteen representations of a_n into four classes, according to whether they are invariant under none, exactly one, or both of the transformations S and T. (Two representations which are obtained from each other by reversing the order of summation in the hypergeometric sum are hereby considered identical.) The representation (6) is typical for the class invariant under S but not T. The following are typical representatives of the other classes:

$$a_n = \frac{(\mu + \nu - \alpha - \beta + 1)_n}{(2\nu + 1)_n\, n!}\, {}_3F_2\left[\begin{matrix} 2\mu + 2\nu + n + 1,\ \mu + \frac{1}{2} - \alpha,\ -n; \\ 2\mu + 1,\ \mu + \nu - \alpha - \beta + 1 \end{matrix}\right]. \tag{7}$$

(not invariant);

$$a_n = \frac{(\mu + \nu - \alpha - \beta + 1)_n(\mu + \frac{1}{2} + \alpha)_n}{(2\mu + 1)_n\,(2\nu + 1)_n\, n!}\, {}_3F_2\left[\begin{matrix} -\mu - \nu - \alpha - \beta - n,\ \mu + \frac{1}{2} - \alpha,\ -n; \\ \mu + \nu - \alpha - \beta + 1,\ -\mu - \alpha - n + \frac{1}{2} \end{matrix}\right] \tag{8}$$

(invariant under T);

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$$(9)\quad a_n = \frac{(\mu + \frac{1}{2} + \alpha)_n(\nu + \frac{1}{2} - \beta)_n}{(2\mu + 1)_n(2\nu + 1)_n\, n!}\, {}_3F_2\left[\begin{matrix}\mu + \frac{1}{2} - \alpha, \nu + \frac{1}{2} + \beta, - n;\\ - \mu - \alpha - n + \frac{1}{2}, - \nu + \beta - n + \frac{1}{2}\end{matrix}\right]$$

(invariant under S and T).

Applying to these formulae the transformations under which they are not invariant, we get three new representations of the form (7) and one new representation of each of the forms (6) and (8). This, together with the reversed series, makes up Whipple's total of eighteen series.

3. The complete set of generating functions. We now assert that the following identities hold:

$$(10)\quad {}_1F_0\,[2\mu + 2\nu + 1; z]{}_3F_2\left[\begin{matrix}\mu + \nu + \frac{1}{2}, \mu + \nu + 1, \mu + \frac{1}{2} - \alpha; & -\dfrac{4z}{(1-z)^2}\\ 2\mu + 1, \mu + \nu - \alpha - \beta + 1\end{matrix}\right]$$
$$= \sum_{n=0}^{\infty} \frac{(2\nu + 1)_n(2\mu + 2\nu + 1)_n}{(\mu + \nu - \alpha - \beta + 1)_n}\, a_n z^n$$

(not invariant);

$$(11)\qquad {}_1F_1\left[\begin{matrix}\mu + \frac{1}{2} - \alpha; - z\\ 2\mu + 1\end{matrix}\right]{}_1F_1\left[\begin{matrix}\nu + \frac{1}{2} - \beta; z\\ 2\nu + 1\end{matrix}\right] = \sum_{n=0}^{\infty} a_n z^n$$

(invariant under S);

$$(12)\quad {}_1F_1\left[\begin{matrix}\mu + \frac{1}{2} - \alpha; - z\\ \mu + \nu - \alpha - \beta + 1\end{matrix}\right]{}_1F_1\left[\begin{matrix}\mu + \frac{1}{2} + \alpha; z\\ \mu + \nu + \alpha + \beta + 1\end{matrix}\right]$$
$$= \sum_{n=0}^{\infty} \frac{(2\mu + 1)_n(2\nu + 1)_n}{(\mu + \nu + \alpha + \beta + 1)_n(\mu + \nu - \alpha - \beta + 1)_n}\, a_n z^n$$

(invariant under T);

$$(13)\quad {}_2F_0\,[\mu + \tfrac{1}{2} - \alpha, \nu + \tfrac{1}{2} + \beta; - z]\ {}_2F_0\,[\mu + \tfrac{1}{2} + \alpha, \nu + \tfrac{1}{2} - \beta; z]$$
$$= \sum_{n=0}^{\infty} (2\mu + 1)_n(2\nu + 1)_n\, a_n z^n$$

(invariant under both S and T).

Here we have listed for completeness as (11) once again the generating function (2). Applying the transformations under which the factorizations are not invariant, we obtain three new generating functions of the form (10) and a new factorization for each of the functions (11) and (12).

The proof of (12) and (13) follows from (8) and (9) by the equations 4.3 (13) and 4.3 (15) of **(4)**. In order to prove (10), we denote the product on the left of (10) by $\psi(z)$ and observe that

$$\psi(z) = \sum_{p=0}^{\infty} c_p(- 4z)^p\, (1 - z)^{-2\mu-2\nu-1-2p},$$

where

$$c_p = \frac{(\mu + \nu + \frac{1}{2})_p(\mu + \nu + 1)_p(\mu + \frac{1}{2} - \alpha)_p}{(2\mu + 1)_p(\mu + \nu - \alpha - \beta + 1)_p\, p!}.$$

Using the binomial expansion and rearranging, we get

$$\begin{aligned}\psi(z) &= \sum_{p=0}^{\infty} c_p(-4z)^p \sum_{q=0}^{\infty} \frac{(2\mu + 2\nu + 2p + 1)_q}{q!} z^q \\ &= \sum_{n=0}^{\infty} z^n \sum_{p=0}^{n} (-4)^p c_p \frac{(2\mu + 2\nu + 2p + 1)_{n-p}}{(n-p)!} \\ &= \sum_{n=0}^{\infty} \frac{(2\mu + 2\nu + 1)_n}{n!} z^n \sum_{p=0}^{n} \frac{(-n)_p(\mu + \frac{1}{2} - \alpha)_p(2\mu + 2\nu + n + 1)_p}{(2\mu + 1)_p(\mu + \nu - \alpha - \beta + 1)_p\, p!}.\end{aligned}$$

The inner sum is readily expressed in terms of a_n by (7), and (10) follows.

It will be noted that the generating function (13), which possesses the highest degree of symmetry, diverges for every $z \neq 0$, unless both series on the left terminate. As a formal Cauchy product it retains a meaning in the case of divergence.

4. Identities of Cayley-Orr type. Evidently our results can be interpreted as identities between the coefficients in the expansion of certain products of hypergeometric series. Such identities were first studied by Cayley and Orr (see **2**, chapter X); more recently, the subject has been taken up again by Burchnall and Chaundy **(3)** and the author **(6)**. In fact, the implication (2) $\rightarrow$ (12) is a confluent form of equation (24) of **(3)**.

5. An application to the product of two Whittaker functions. In this section we shall use the notation of **(4)** for Whittaker functions and Jacobi polynomials. In **(5)** we have proved a result which can be stated thus: Let $\alpha, \beta, \mu, \nu, \rho, \tau$ be arbitrary complex numbers such that none of the numbers 2μ, 2ν, $2\mu + 2\nu$ is a negative integer, and let a_n be defined by (2). Then the following identity holds:

$$\text{(14)}\quad \left(\rho\frac{1-\tau}{2}\right)^{-\mu-\frac{1}{2}} M_{\alpha,\mu}\left(\rho\frac{1-\tau}{2}\right) \cdot \left(\rho\frac{1+\tau}{2}\right)^{-\nu-\frac{1}{2}} M_{\beta,\nu}\left(\rho\frac{1+\tau}{2}\right)$$
$$= \sum_{n=0}^{\infty} \frac{n!}{(2\mu + 2\nu + 1 + n)_n} a_n P_n^{(2\mu,2\nu)}(\tau)\rho^{-\mu-\nu-1} M_{\alpha+\beta,\mu+\nu+\frac{1}{2}+n}(\rho).$$

We now see that the coefficients a_n can be defined by any of the generating functions given in §3, in particular by the symmetric function (13). Also, making use of the results of Bailey **(1)** on cases where products of two hypergeometric functions can be expressed in terms of a single such function, we now could give a systematic account of those special cases of (14) where a_n can be expressed in terms of factorials only. Most of these cases were noted in **(5)**, using *ad hoc* methods. A further result can be obtained by applying equation (2.10) of **(1)**. We have, provided that 2μ is not an integer,

$$\text{(15)}\qquad a_n(\alpha, \alpha, \mu, -\mu) = \frac{\mu(\alpha - n/2 + \frac{1}{2})_n}{(\mu - n/2)_{n+1}\, n!}.$$

After some simplification we thus obtain from (14)

$$(16)\quad \left(\frac{1-\tau^2}{4}\right)^{-\frac{1}{2}} M_{\alpha,\mu}\left(\rho\frac{1-\tau}{2}\right) M_{\alpha,-\mu}\left(\rho\frac{1+\tau}{2}\right)$$
$$= \Gamma(2\mu+1) \sum_{n=0}^{\infty} \frac{\mu(\alpha - n/2 + \frac{1}{2})_n (2\mu+1)_n}{(2n)!(\mu - n/2)_{n+1}} P_n^{-2\mu}(\tau)\, M_{2\alpha, n+\frac{1}{2}}(\rho).$$

This expansion is a counterpart of the following result, which (in a different notation) can be found in **(5)**:

$$(17)\quad \left(\frac{1-\tau^2}{4}\right)^{-\frac{1}{2}} M_{\alpha,\mu}\left(\rho\frac{1-\tau}{2}\right) M_{\alpha,\mu}\left(\rho\frac{1+\tau}{2}\right)$$
$$= \Gamma(2\mu+1) \sum_{n=0}^{\infty} \frac{(\mu + \frac{1}{2} - \alpha)_n (\mu + \frac{1}{2} + \alpha)_n}{n!(2\mu+1)_n (4\mu + 1 + 2n)_{2n}} P_{2n+2\mu}^{-2\mu}(\tau) M_{2\alpha, 2\mu+\frac{1}{2}+2n}(\rho).$$

In both (16) and (17) P denotes the Legendre function of the first kind on the cut **(4**, 3.4(6)**)**. The limits of (16) and (17) as $\mu \to 0$ can be written in the form

$$(18)\quad e^{-\rho/2} L_{\alpha-\frac{1}{2}}\left(\rho\frac{1-\tau}{2}\right) L_{\alpha-\frac{1}{2}}\left(\rho\frac{1+\tau}{2}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2(4n)!}(-1)^n(\alpha - n + \tfrac{1}{2})_{2n} P_{2n}(\tau) M_{2\alpha, 2n+\frac{1}{2}}(\rho)$$

where L denotes the Laguerre function and P the Legendre polynomial. For $\alpha = \frac{1}{2}$ all terms on the right of (18) vanish except the first. The relation then becomes trivial, since $L_0 = 1$, $M_{1,\frac{1}{2}}(\rho) = e^{-\rho/2}$.

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