22

Interacting bosonic fields

The usual formalism of interacting relativistic quantum field theory is purely perturbative and leads to formal, typically divergent expansions. It is natural to ask whether behind these expansions there exists a non-perturbative theory acting on a Hilbert space and satisfying some natural axioms (such as the Wightman or Haag–Kastler axioms). It is not difficult to give a whole list of models of increasing difficulty, well defined perturbatively, whose non-perturbative construction seems conceivable. There were times when it was hoped that by constructing them one by one we would eventually reach models in dimension 4 relevant for particle physics. The branch of mathematical physics devoted to constructing these models is called *constructive quantum field theory*.

The simplest class of non-trivial models of constructive quantum field theory is the bosonic theory in $1+1$ dimensions with an interaction given by an arbitrary bounded from below polynomial. It is called the $P(\varphi)_2$ *model*, where P is a polynomial, φ denotes the neutral bosonic field and $2 = 1 + 1$ stands for the space-time dimension. To our knowledge, it has no direct relevance for realistic physical systems, so its main motivation was as an intermediate step in the program of constructive quantum field theory.

The work on the $P(\varphi)_2$ model was successful and led to the development of a number of interesting and deep mathematical tools. The constructive program continued, with the construction of more difficult models, such as the Yukawa$_2$ and $\lambda\varphi_3^4$, as well as a deep analysis of Yang–Mills$_4$. Unfortunately, it seems that no models of direct physical relevance have so far been constructed within this program.

In this chapter we would like to describe some elements of the construction of the $P(\varphi)_2$ model. We will restrict ourselves to space-cutoff models and the net of local algebras associated with this model. We will not discuss the construction of the translation invariant model, which can be found in the literature. We believe that even such a limited treatment of this theory is a good illustration of many concepts of quantum field theory.

22.1 Free bosonic fields

22.1.1 Klein–Gordon equation

The simplest non-interacting relativistic model of quantum field theory describes neutral scalar bosons. It has already been discussed in Sect. 19.2. Let us recall the basic elements of its theory.

https://doi.org/10.1017/9781009290876.023 Published online by Cambridge University Press

The dual phase space can be taken to be the space of real space-compact solutions of the Klein–Gordon equation

$$\left(\partial_t^2 - \Delta_{\mathrm{x}} + m^2\right)\zeta(t,\mathrm{x}) = 0, \quad (t,\mathrm{x}) \in \mathbb{R}^{1,d}. \tag{22.1}$$

The model is called *massive*, resp. *massless* if $m > 0$, resp. $m = 0$. Introducing the operator $\epsilon = (-\Delta_x + m^2)^{\frac{1}{2}}$, we can rewrite (22.1) as

$$(\partial_t^2 + \epsilon^2)\zeta = 0.$$

We are in the framework of Subsect. 18.3.2 (and hence also of Subsect. 9.3.1, with $c = (2\epsilon)^{-1}$). We parametrize the dual phase space by the time-zero initial conditions

$$\vartheta(\mathrm{x}) := \dot{\zeta}(0,\mathrm{x}), \quad \varsigma(\mathrm{x}) = \zeta(0,\mathrm{x}). \tag{22.2}$$

It is natural to enlarge the dual phase space so that, in terms of the initial conditions, it is

$$\epsilon^{\frac{1}{2}} L^2(\mathbb{R}^d,\mathbb{R}) \oplus \epsilon^{-\frac{1}{2}} L^2(\mathbb{R}^d,\mathbb{R}). \tag{22.3}$$

(This is a special case of the space $\mathcal{Y}_{\mathrm{dyn}}$ defined in Subsect. 18.3.2.) The vector $\varsigma \in \epsilon^{-\frac{1}{2}} L^2(\mathbb{R}^d,\mathbb{R})$ describes the "position" and $\vartheta \in \epsilon^{\frac{1}{2}} L^2(\mathbb{R}^d,\mathbb{R})$ describes the "momentum".

The dual phase space can be treated as a Kähler space with the symplectic form

$$(\vartheta_1,\varsigma_1)\omega(\vartheta_2,\varsigma_2) = \int_{\mathbb{R}^d} (\vartheta_1(\mathrm{x})\varsigma_2(\mathrm{x}) - \vartheta_2(\mathrm{x})\varsigma_1(\mathrm{x}))\,\mathrm{d}\mathrm{x},$$

the conjugation τ and the Kähler anti-involution j:

$$\tau = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}, \quad \mathrm{j} = \begin{bmatrix} 0 & -\epsilon \\ \epsilon^{-1} & 0 \end{bmatrix}. \tag{22.4}$$

The dynamics is generated by the classical Hamiltonian

$$h_0(\varsigma,\vartheta) = \frac{1}{2}\int_{\mathbb{R}^d} \left(\vartheta^2(\mathrm{x}) + |\nabla_{\mathrm{x}}\varsigma(\mathrm{x})|^2 + m^2\varsigma^2(\mathrm{x})\right)\mathrm{d}\mathrm{x}.$$

One can show that the linear Klein–Gordon equation with initial conditions (22.2) possesses a unique solution, which for the Cauchy data (ϑ,ς) will be denoted $\mathrm{e}^{ta}(\vartheta,\varsigma)$. These solutions satisfy the following basic requirements of a (classical) relativistic field theory:

Theorem 22.1 (1) Locality. *If $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R}^{1,d})$, $\chi \equiv 1$ on an open set $\mathcal{O} \subset \mathbb{R}^{1,d}$ and ζ is a solution in $\mathbb{R}^{1,d}$, then $\chi\zeta$ is a solution in $\mathcal{O}$.*

(2) Causality. *If the Cauchy data (ϑ,ς) are supported in a set $K \subset \mathbb{R}^d$, then $\mathrm{e}^{ta}(\vartheta,\varsigma)$ is supported in $J(\{0\}\times K) = \big\{(t,\mathrm{x}) \in \mathbb{R}^{1,d} :\ \mathrm{dist}(\mathrm{x},K) \leq |t|\big\}$.*

(3) Covariance. *The Poincaré group acting as in Subsect. 19.2.10 preserves the space of solutions with Cauchy data in (22.3).*

22.1.2 Quantization of linear Klein–Gordon equation

Let us first briefly recall the results of Sect. 18.3 about the quantization of the linear Klein–Gordon equation. Consider the complexified dual phase space

$$\epsilon^{-\frac{1}{2}} L^2(\mathbb{R}^d) \oplus \epsilon^{\frac{1}{2}} L^2(\mathbb{R}^d). \tag{22.5}$$

The positive frequency subspace of (22.5) is taken as the one-particle space, as usual denoted by $\mathcal{Z}$. As in (8.32), and then (18.31), we parametrize it by the time-zero momenta, identifying it with $(2\epsilon)^{\frac{1}{2}} L^2(\mathbb{R}^d)$. The time reversal becomes the usual complex conjugation. One can introduce position and momentum operators for the Fock CCR representation on $\Gamma_{\mathrm{s}}(\mathcal{Z})$ as in Subsect. 8.2.7. Let us recall their definition in the present context:

Definition 22.2 *The* time-zero position and momentum operators*, often called the* φ and π fields*, are defined as*

$$\begin{aligned}
&\varphi(\vartheta) := a^*(\vartheta) + a(\vartheta), \ \ \vartheta \in \epsilon^{\frac{1}{2}} L^2(\mathbb{R}^d, \mathbb{R}),\\
&\pi(\varsigma) := \tfrac{\mathrm{i}}{2}\big(a^*(\epsilon\varsigma) - a(\epsilon\varsigma)\big), \ \ \ \varsigma \in \epsilon^{-\frac{1}{2}} L^2(\mathbb{R}^d, \mathbb{R}).
\end{aligned}$$

Definition 22.3 *The* time-zero φ and π fields at $\mathrm{x} \in \mathbb{R}^d$ *are defined as*

$$\begin{aligned}
\varphi(\mathrm{x}) &:= \varphi(\delta_{\mathrm{x}}),\\
\pi(\mathrm{x}) &:= \pi(\delta_{\mathrm{x}}),
\end{aligned}$$

where δ_{x} *is the Dirac mass at point* x*.*

Remark 22.4 *Comparing Defs. 22.2 and 22.3, we see that the notation* $\varphi(\cdot)$ *and* $\pi(\cdot)$ *is somewhat ambiguous. We use this convention, however, and make no attempt to improve on it.*

Note that δ_{x} does not belong to the one-particle space. We treat the fields $\varphi(\mathrm{x})$ and $\pi(\mathrm{x})$ as "operator-valued distributions" that become well-defined closed operators only after smearing with appropriate test functions:

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi(\mathrm{x})\vartheta(\mathrm{x})\mathrm{d}\mathrm{x} &= \varphi(\vartheta),\\
\int_{\mathbb{R}^d} \pi(\mathrm{x})\varsigma(\mathrm{x})\mathrm{d}\mathrm{x} &= \pi(\varsigma);
\end{aligned}$$

see Remark 3.54. Formally, $\varphi(\mathrm{x})$ and $\pi(\mathrm{x})$ satisfy the following form of the CCR:

$$\begin{aligned}
[\varphi(\mathrm{x}), \varphi(\mathrm{y})] &= [\pi(\mathrm{x}), \pi(\mathrm{y})] = 0,\\
[\varphi(\mathrm{y}), \pi(\mathrm{x})] &= \mathrm{i}\delta(\mathrm{x} - \mathrm{y})\mathbb{1}.
\end{aligned} \tag{22.6}$$

Definition 22.5 Free fields *are defined as*

$$\begin{aligned}
\varphi_0(t, \vartheta) &:= \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\epsilon)}\varphi(\vartheta)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\epsilon)}, \ \ \ \vartheta \in \epsilon^{\frac{1}{2}} L^2(\mathbb{R}^d, \mathbb{R}),\\
\varphi_0(t, \mathrm{x}) &:= \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\epsilon)}\varphi(\mathrm{x})\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\epsilon)}, \ \ \ \mathrm{x} \in \mathbb{R}^d.
\end{aligned}$$

(The subscript 0 indicates that we are dealing with free or, in other words, non-interacting fields.) The equations satisfied by the free fields can be expressed as an operator identity on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$

$$\partial_t^2 \varphi_0(t, \vartheta) + \varphi_0(t, \epsilon^2 \vartheta) = 0, \tag{22.7}$$

or, equivalently, as a distributional identity

$$\left(\partial_t^2 - \Delta_{\mathrm{x}} + m^2\right) \varphi_0(t, \mathrm{x}) = 0.$$

Definition 22.6 *For* $(\vartheta, \varsigma) \in \epsilon^{-\frac{1}{2}} L^2(\mathbb{R}^d, \mathbb{R}) \oplus \epsilon^{\frac{1}{2}} L^2(\mathbb{R}^d, \mathbb{R})$, *we also introduce the corresponding* phase space fields *and* Weyl operators

$$\phi(\vartheta, \varsigma) := \varphi(\vartheta) + \pi(\varsigma), \qquad W(\vartheta, \varsigma) := \mathrm{e}^{\mathrm{i}\phi(\vartheta, \varsigma)}. \tag{22.8}$$

If ζ *is the solution with the Cauchy data* (ϑ, ς), *we will also write* $\phi(\zeta)$ *and* $W(\zeta)$ *instead of (22.8).*

22.1.3 Free dynamics and free local algebras

For concreteness, until the end of the chapter we assume that $d = 1$.

For $A \in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$, we define

$$\begin{aligned}
\alpha_0^t(A) &:= \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\epsilon)} A \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\epsilon)}, \\
\alpha_0^{\mathrm{x}}(A) &:= \mathrm{e}^{\mathrm{i}\mathrm{x}\mathrm{d}\Gamma(D)} A \mathrm{e}^{-\mathrm{i}\mathrm{x}\mathrm{d}\Gamma(D)}, \\
\alpha_0^{x}(A) &:= \alpha_0^t \circ \alpha_0^{\mathrm{x}}(A), \qquad x = (t, \mathrm{x}) \in \mathbb{R}^{1,1},
\end{aligned}$$

where $D = D_{\mathrm{x}}$ is the momentum operator. Clearly, $\mathbb{R}^{1,1} \ni x \mapsto \alpha_0^x$ is a strongly continuous group of $*$-automorphisms of $B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$.

In the following definition, all the algebras are concrete and are contained in $B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$.

Definition 22.7 (1) *For a bounded open interval* $I \subset \mathbb{R}$, *the corresponding* time-zero local algebra *is defined as*

$$\mathfrak{R}(I) := \left\{W(\vartheta, \varsigma) \ : \ \vartheta, \varsigma \in C_{\mathrm{c}}^{\infty}(I, \mathbb{R})\right\}''.$$

(2) *The following algebra plays the role of the algebra of all observables:*

$$\mathfrak{O} := \Big(\bigcup_{I \subset \mathbb{R}} \mathfrak{R}(I)\Big)^{\mathrm{cpl}}.$$

(3) *For a bounded open set* $\mathcal{O} \subset \mathbb{R}^{1,1}$, *the corresponding* free local algebra *is defined as*

$$\mathfrak{M}_0(\mathcal{O}) := \left\{\alpha_0^t(A) \ : \ A \in \mathfrak{R}(I), \ \{t\} \times I \subset \mathcal{O}\right\}''.$$

As described in Subsect. 19.2.7, one can also quantize the free dynamics by abstract CCR algebras. Recall that if $\mathcal{O} \subset \mathbb{R}^{1,1}$ is an open set, then $\mathfrak{A}(\mathcal{O})$ is then the Weyl CCR C^*-algebra generated by elements $W(Gf)$ satisfying the Weyl commutation relations, with $f \in C_c^\infty(\mathcal{O})$, where $Gf(x) = \int G(x-y)f(y)\mathrm{d}y$ and G is the Pauli–Jordan function. These algebras possess a distinguished representation called the Fock representation,

$$\pi_{\mathrm{F}} : \mathfrak{A}(\mathcal{O}) \to B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big).$$

Proposition 22.8 *The following hold:*

(1) $\mathfrak{M}_0(\mathcal{O}) = \pi_{\mathrm{F}}(\mathfrak{A}(\mathcal{O}))''$.
(2) $\alpha_0^x\left(\mathfrak{M}_0(\mathcal{O})\right) = \mathfrak{M}_0(\mathcal{O}+x), \quad x \in \mathbb{R}^{1,1}$.
(3) $\mathfrak{O} = \Big(\bigcup\limits_{\mathcal{O}\subset\mathbb{R}^{1,1}} \mathfrak{M}_0(\mathcal{O})\Big)^{\mathrm{cpl}}$.

Proof Recall first that the Klein–Gordon equation satisfies the *causality property*, i.e.

$$\operatorname{supp}\big(\vartheta(t),\varsigma(t)\big) \subset J\big(\{0\}\times\operatorname{supp}(\vartheta,\varsigma)\big), \quad t \in \mathbb{R}. \tag{22.9}$$

To prove (1), we first note that

$$\mathfrak{M}_0(\mathcal{O}) = \left\{W\big(\mathrm{e}^{at}(\vartheta,\varsigma)\big) \ : \ \{t\}\times\big(\operatorname{supp}(\vartheta,\varsigma)\big) \subset \mathcal{O}\right\}''.$$

We then use Thm. 19.15 and the fact that the Green's function $G(t,s)$ is $\frac{\sin((t-s)\epsilon)}{\epsilon}$. Statement (2) is obvious. Statement (3) follows from the fact that

$$\mathfrak{R}(I) \subset \mathfrak{M}\big(]-\epsilon_0,\epsilon_0[\times I\big) \subset \mathfrak{R}\big(I+]-\epsilon_0,\epsilon_0[\big), \quad \epsilon_0 > 0. \tag{22.10}$$

The first inclusion in (22.10) is obvious; the second follows from causality. □

22.1.4 Q-space representation

Let τ be the canonical conjugation on $\mathcal{Z} = (2\epsilon)^{\frac{1}{2}}L^2(\mathbb{R})$ defined as $\tau\Psi(\mathrm{x}) = \overline{\Psi(\mathrm{x})}$. Recall from Subsect. 18.3.2 that τ corresponds to time reversal.

Let $T^{\mathrm{rw}} : \Gamma_{\mathrm{s}}(\mathcal{Z}) \to L^2(Q,\mathrm{d}\mu)$ be the real-wave (or Q-space) representation associated with τ, as in Subsect. 9.3.5. In the sequel we will freely identify objects on $\Gamma_{\mathrm{s}}(\mathcal{Z})$ and on $L^2(Q,\mathrm{d}\mu)$, using T^{rw}. We will also use the same symbol to denote a measurable function V on Q and the operator of multiplication by V acting on $L^2(Q,\mathrm{d}\mu)$. We are in the framework of Subsect. 9.3.1 with $c = (2\epsilon)^{-1}$.

Operators $\varphi(\vartheta)$, $\vartheta \in \epsilon^{\frac{1}{2}}L^2(\mathbb{R})$ commute with one another. In particular, polynomials in the variable φ, that is, functions of the form

$$V(\varphi) = \sum_{j=0}^{n} \int V_j(\mathrm{x}_1,\dots,\mathrm{x}_j)\varphi(\mathrm{x}_1)\cdots\varphi(\mathrm{x}_j)\mathrm{d}\mathrm{x}_1\cdots\mathrm{x}_j,$$

can be interpreted as functions on Q, and as (usually unbounded) operators on $L^2(Q,\mathrm{d}\mu) \simeq \Gamma_{\mathrm{s}}(\mathcal{Z})$.

We can also consider the Wick quantization of the polynomial V. We will use the following alternative notation:

$$\mathrm{Op}^{a^*,a}(V) = :V(\varphi):. \tag{22.11}$$

The notation on the r.h.s. of (22.11) is explained in Prop. 9.53. Clearly, $:V(\varphi):$ is a polynomial in the variable φ.

22.2 $P(\varphi)$ interaction

22.2.1 Nonlinear Klein–Gordon equation

Now let P be a real polynomial and $g : \mathbb{R} \to \mathbb{R}$ a real function. Let us consider the perturbed classical Hamiltonian

$$h(\vartheta, \varsigma) = h_0(\vartheta, \varsigma) + \int_{\mathbb{R}} g(\mathrm{x}) P\big(\varsigma(\mathrm{x})\big)\mathrm{dx}. \tag{22.12}$$

For stability reasons, we require that g be positive and that the polynomial P, and hence the Hamiltonian h, be bounded from below.

Formally, the associated field equation is the following *non-linear Klein–Gordon equation*:

$$\begin{cases} \partial_t^2 \varphi(t,\mathrm{x}) - \Delta_x \varphi(t,\mathrm{x}) + m^2 \varphi(t,\mathrm{x}) + g(x) P'(\varphi(t,\mathrm{x})) = 0, \\ \varsigma(\mathrm{x}) = \zeta(0,\mathrm{x}), \qquad \vartheta(\mathrm{x}) := \dot{\zeta}(0,\mathrm{x}). \end{cases} \tag{22.13}$$

22.2.2 Formal quantization of non-linear Klein–Gordon equation

Let us try to quantize the classical Hamiltonian (22.12). Let us assume that we can give a meaning to the formal expression

$$H = \mathrm{d}\Gamma(\epsilon) + \int_{\mathbb{R}} g(\mathrm{x}) P\big(\varphi(\mathrm{x})\big)\mathrm{dx}. \tag{22.14}$$

Set

$$\varphi(t,\mathrm{x}) := \mathrm{e}^{\mathrm{i}tH} \varphi(\mathrm{x}) \mathrm{e}^{-\mathrm{i}tH}.$$

Then formally we have

$$\partial_t^2 \varphi(t,\mathrm{x}) - \Delta_x \varphi(t,\mathrm{x}) + m^2 \varphi(t,\mathrm{x}) + g(x) P'\big(\varphi(t,\mathrm{x})\big) = 0,$$

which can be rephrased as saying that we have quantized the non-linear Klein–Gordon equation (22.13).

There are two deep difficulties with the formal expression (22.14):

(1) First, $\varphi(\mathrm{x})$ does not make sense as a self-adjoint operator, so expressions like $\varphi(\mathrm{x})^p$ do not make sense (even after integration against test functions). This problem is called the *ultraviolet divergence*, and is caused by the requirement that the associated field theory should be *local*. For classical field equations it corresponds to the well-known difficulty with multiplying distributions.

https://doi.org/10.1017/9781009290876.023 Published online by Cambridge University Press

(2) If $g(x) \equiv 1$, one encounters the second problem: the integral over $\mathbb{R}$ in (22.14) may not converge. This is called the *infinite-volume divergence* and is caused by the requirement that the field theory should be *translation invariant*.

One can try to tackle these problems as follows. First one modifies the Hamiltonian, introducing ultraviolet and space cutoffs. This leads to the (still formal) expression

$$H_\kappa(g) = \mathrm{d}\Gamma(\epsilon) + \int_{\mathbb{R}} g(\mathrm{x}) P\big(\varphi_\kappa(\mathrm{x})\big)\mathrm{dx}.$$

Here, $g \in C_{\mathrm{c}}^\infty(\mathbb{R})$ is a *space cutoff* and $\varphi_\kappa(\mathrm{x})$ is the *ultraviolet cutoff field*

$$\varphi_\kappa(\mathrm{x}) := \int_{\mathbb{R}} \varphi(\mathrm{y})\rho_\kappa(\mathrm{y}-\mathrm{x})\mathrm{dy}, \tag{22.15}$$

where $\rho \in C_{\mathrm{c}}^\infty(\mathbb{R})$ is a *cutoff function* with $\int_{\mathbb{R}} \rho(\mathrm{y})\mathrm{dy} = 1$, $\rho_\kappa(\mathrm{y}) = \kappa\rho(\kappa\mathrm{y})$ and $\kappa \gg 1$ is an *ultraviolet cutoff parameter*.

Now, $\rho_\kappa \in \mathcal{Z} = (2\epsilon)^{\frac{1}{2}} L^2(\mathbb{R})$, except if $m = 0$. The case $m = 0$ is exceptional, because then

$$\|\epsilon^{-\frac{1}{2}}\rho_\kappa\|^2 = (2\pi)^{-1} \int_{\mathbb{R}} |k|^{-1}|\hat{\rho}|^2(\kappa^{-1}k)\mathrm{d}k < \infty$$

iff $\hat{\rho}(0) = \int_{\mathbb{R}} \rho(\mathrm{y})\mathrm{dy} = 0$. In the rest of this chapter we will always assume that $m > 0$.

Note that the interaction term

$$\int_{\mathbb{R}} g(\mathrm{x}) P\big(\varphi_\kappa(\mathrm{x})\big)\mathrm{dx}$$

now makes sense as a self-adjoint operator on the Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$.

Next one tries to remove the ultraviolet cutoff, letting $\kappa \to \infty$ and trying to prove the existence of a (non-trivial) limit

$$H_\infty(g) = \lim_{\kappa\to\infty} \big(H_\kappa(g) - R_\kappa(g)\big) \tag{22.16}$$

in some appropriate sense, where $R_\kappa(g)$ are the so-called *counterterms* related to the well-known need to renormalize various physical constants of the model.

In dimension $1+1$ one can use the counterterms

$$R_\kappa(g) := \int_{\mathbb{R}} g(\mathrm{x})\Big(P\big(\varphi_\kappa(\mathrm{x})\big) - :P\big(\varphi_\kappa(\mathrm{x})\big):\Big)\mathrm{dx}$$

obtained by the *Wick ordering* of the interaction term. It is then possible to give a meaning to the expression

$$H_\infty(g) := \mathrm{d}\Gamma(\epsilon) + \int_{\mathbb{R}} g(\mathrm{x}) :P\big(\varphi(\mathrm{x})\big): \mathrm{dx}$$

as a bounded below self-adjoint operator on $\Gamma_{\mathrm{s}}(\mathcal{Z})$, as we will see in Subsect. 22.2.5.

Then one tries to take the infinite-volume limit, which means putting $g = 1$. This requires a change of the representation of the CCR – it cannot be done on the original Fock space. This is related to a general argument called *Haag's theorem*.

In higher space dimensions it is no longer possible to give meaning to $H_\infty(g)$ as a self-adjoint operator on the Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$. In dimension $1+2$ one can quantize the classical non-linear Klein–Gordon equation if the degree of P is not greater than 4. However, even with a spatial cut-off the resulting Hamiltonian acts on a Hilbert space supporting a representation of the CCR not equivalent to the Fock representation. This is related to the so-called *wave function renormalization*.

In dimensions $1+3$ or higher it is believed that interacting scalar bosonic quantum fields do not exist.

22.2.3 $P(\varphi)_2$ interaction as a Wick polynomial

Until the end of this chapter we assume that $d = 1$ and $m > 0$.

Recall that

$$\begin{aligned}\varphi(\mathrm{x}) \;&=\; \varphi(\delta_{\mathrm{x}}) = a^*(\delta_{\mathrm{x}}) + a(\delta_{\mathrm{x}}),\\ \varphi_\kappa(\mathrm{x}) = \varphi\big(\rho_\kappa(\cdot - \mathrm{x})\big) &= a^*\big(\rho_\kappa(\cdot - \mathrm{x})\big) + a\big(\rho_\kappa(\cdot - \mathrm{x})\big)\\ &= a^*\big(\chi(\kappa^{-1}D_{\mathrm{x}})\delta_{\mathrm{x}}\big) + a\big(\chi(\kappa^{-1}D_{\mathrm{x}})\delta_{\mathrm{x}}\big),\end{aligned}$$

where $\chi = \hat{\rho} \in \mathcal{S}(\mathbb{R})$ satisfies $\chi(0) = 1$.

Let us fix a real bounded below polynomial

$$P(\lambda) = \sum_{p=0}^{2n} a_p \lambda^p. \tag{22.17}$$

Clearly, $\deg P = 2n$ has to be even and $a_{2n} > 0$. We also fix a space cutoff function $g \in L^2(\mathbb{R})$.

As explained in Subsect. 22.2.2, instead of the operator $P\big(\varphi_\kappa(\mathrm{x})\big)$, we prefer to use its Wick-ordered version

$$:P\big(\varphi_\kappa(\mathrm{x})\big):= \sum_{p=0}^{2n} a_p{:}\varphi_\kappa(\mathrm{x})^p{:}.$$

We refer to Prop. 9.53, where this notation is explained. In particular, we recall from (9.60) that

$${:}\varphi_\kappa(\mathrm{x})^p{:} = \sum_{r=0}^{p} \binom{p}{r} a^*\big(\rho_\kappa(\cdot - \mathrm{x})\big)^r a\big(\rho_\kappa(\cdot - \mathrm{x})\big)^{p-r}. \tag{22.18}$$

Definition 22.9 *The operator*

$$V_\kappa := \int_{\mathbb{R}} g(\mathrm{x}) :P\big(\varphi_\kappa(\mathrm{x})\big): \mathrm{dx}$$

is called an ultraviolet cutoff Wick-ordered interaction.

https://doi.org/10.1017/9781009290876.023 Published online by Cambridge University Press

In this subsection we investigate V_κ as a Wick polynomial.

Definition 22.10 *Let us set*

$$a(\mathrm{k}) := (2\pi)^{-\frac{1}{2}} a(\mathrm{e}^{\mathrm{ik}\cdot\mathrm{x}}), \qquad a^*(\mathrm{k}) := (2\pi)^{-\frac{1}{2}} a^*(\mathrm{e}^{\mathrm{ik}\cdot\mathrm{x}}).$$

Using the notation of Sect. 9.4, we obtain that

$$\begin{aligned} M_{p,\kappa} &:= \int_{\mathbb{R}} g(\mathrm{x}) :\varphi_\kappa(\mathrm{x})^p : \mathrm{dx} \\ &= \sum_{r=0}^{p} \binom{p}{r} \int_{\mathbb{R}^p} w_{p,\kappa}(\mathrm{k}_1,\dots,\mathrm{k}_r,\mathrm{k}_{r+1},\dots,\mathrm{k}_p) \\ &\quad \times a^*(\mathrm{k}_1)\cdots a^*(\mathrm{k}_r) a(-\mathrm{k}_{r+1})\cdots a(-\mathrm{k}_p)\mathrm{dk}_1\cdots\mathrm{dk}_p, \end{aligned}$$

for

$$w_{p,\kappa}(\mathrm{k}_1,\dots,\mathrm{k}_p) = (2\pi)^{1-2p}\hat{g}\Big(\sum_{i=1}^{p}\mathrm{k}_i\Big)\prod_{j=1}^{p}\chi(\kappa^{-1}\mathrm{k}_j). \tag{22.19}$$

We denote by $w_{p,\infty}$ the function on $\mathbb{R}^p$ obtained by setting $\kappa=\infty$ in (22.19), i.e.

$$w_{p,\infty}(\mathrm{k}_1,\dots,\mathrm{k}_p) = (2\pi)^{1-2p}\hat{g}\Big(\sum_{i=1}^{p}\mathrm{k}_i\Big), \tag{22.20}$$

which allows us to define $M_{p,\infty}$.

Lemma 22.11 *The kernels $w_{p,\kappa}$ are in $\otimes^p \epsilon^{\frac{1}{2}} L^2(\mathbb{R})$ for $0<\kappa\leq\infty$ and, for any $\delta>0$,*

$$\|w_{p,\kappa} - w_{p,\infty}\|_{\otimes^p \epsilon^{\frac{1}{2}} L^2(\mathbb{R})} \leq C_\delta \|g\|_{L^2(\mathbb{R})}\kappa^{-\delta}.$$

Remark 22.12 *Lemma 22.11 still holds if $g\in L^{1+\delta}(\mathbb{R})$ for some $\delta>0$.*

Proof It clearly suffices to prove the corresponding statements with $w_{p,\kappa}$ replaced by $w_{p,\kappa}\prod_{i=1}^{p}\epsilon(\mathrm{k}_i)^{-\frac{1}{2}}$, and $\otimes^p\epsilon^{\frac{1}{2}}L^2(\mathbb{R})$ replaced by $L^2(\mathbb{R}^p)$. We use the bound

$$\prod_{j=1}^{p} a_j \leq \sum_{i=1}^{p}\Big(\prod_{j\neq i} a_j\Big)^{p/(p-1)}, \tag{22.21}$$

which follows from the inequality

$$\Big(\prod_{i=1}^{p}\lambda_i\Big)^{1/p} \leq \sum_{j=1}^{p}\lambda_j$$

applied to $\lambda_i = \prod_{j\neq i} a_j^{p/(p-1)}$. Applying (22.21) to $a_i=\omega(\mathrm{k}_i)^{-\frac{1}{2}}$, we obtain that $w_{p,\infty}$, and hence $w_{p,\kappa}$ for $\kappa<\infty$, belong to $L^2(\mathbb{R}^p)$. The bound on $\|w_{p,\kappa}-w_{p,\infty}\|$ is a direct computation, using (22.21). □

From Prop. 9.50 we see that

$$V_\kappa = \sum_{p=0}^{2n} a_p M_{p,\kappa}$$

is well defined as a Hermitian operator on $\mathrm{Dom}\, N^n$ for $\kappa \leq \infty$. We will use the notation

$$V := V_\infty = \int_{\mathbb{R}} g(\mathrm{x}) : P\big(\varphi(\mathrm{x})\big) : \mathrm{dx}.$$

Lemma 22.13 (1) *V_κ and V with domain* $\mathrm{Dom}\, N^n$ *are densely defined Hermitian operators.*

(2) *There exists $\delta > 0$ such that*

$$\|(V - V_\kappa)(N+1)^{-n}\| \leq C\|g\|_{L^2(\mathbb{R})}\kappa^{-\delta}.$$

Proof It suffices to apply the N_τ estimates of Prop. 9.50. □

22.2.4 $P(\varphi)_2$ *interaction as a multiplication operator*

In this subsection we study the operators V_κ as multiplication operators in the Q-space representation.

Proposition 22.14 *Assume that $g \in L^2(\mathbb{R})$. Then the following are true:*

(1) *V_κ and V are multiplication operators by functions in* $\bigcap\limits_{1\leq p<\infty} L^p(Q, \mathrm{d}\mu)$.

(2) *For any $\delta > 0$, there exists a constant $C_\delta > 0$ such that*

$$\|V_\kappa - V\|_{L^p(Q,\mathrm{d}\mu)} \leq C_\delta (p-1)^n \kappa^{-\delta}, \quad p > 2.$$

(3) *Assume in addition that $g \in L^1(\mathbb{R})$ and $g \geq 0$. Then there exist constants $C > 0$, κ_0 such that, for $\kappa \geq \kappa_0$,*

$$V_\kappa \geq -C(\log \kappa)^n.$$

Proof Note that $\epsilon = (D_{\mathrm{x}}^2 + m^2)^{\frac{1}{2}}$ is a real operator and that $\epsilon^{-\frac{1}{2}}\rho_\kappa(\cdot - \mathrm{x})$ is real. It follows that $\varphi_\kappa(\mathrm{x})$ is a multiplication operator in the Q-space representation. Hence, by Prop. 9.53, the same is true of the operators $\int_{\mathbb{R}} g(\mathrm{x}){:}\varphi_\kappa(\mathrm{x})^p{:}\mathrm{dx}$ for $p \in \mathbb{N}$.

For $2 \leq p < \infty$, let us now consider the map $a = (p-1)^{-\frac{1}{2}}\mathbb{1}$ on $(2\epsilon)^{\frac{1}{2}}L^2(\mathbb{R})$. By Thm. 9.30, we know that $\Gamma(a) = (p-1)^{-N/2}$ is a contraction from $L^2(Q)$ into $L^p(Q)$. It follows that

$$\|\Psi\|_{L^p(Q)} \leq (p-1)^{n/2}\|\Psi\|_{L^2(Q)}, \quad \Psi \in \bigoplus_{p=0}^{n} \Gamma_{\mathrm{s}}^p\big((2\epsilon)^{\frac{1}{2}}L^2(\mathbb{R})\big). \tag{22.22}$$

From Lemma 22.13 we know that $(V - V_\kappa)\Omega \to 0$. So V_κ is Cauchy in $L^2(Q)$. Hence, by (22.22), it is Cauchy also in $\bigcap\limits_{1\leq p<\infty} L^p(Q)$. It follows that V_κ converges

https://doi.org/10.1017/9781009290876.023 Published online by Cambridge University Press

to a function W in $\bigcap\limits_{1\leq p<\infty} L^p(Q)$. Now set

$$\mathcal{S} = \mathrm{Span}\big\{\mathrm{e}^{\mathrm{i}\varphi(\vartheta)} \ : \ \vartheta \in C_{\mathrm{c}}^\infty(\mathbb{R},\mathbb{R})\big\}.$$

Clearly, $\mathcal{S}$ is dense in $L^2(Q)$ and $\mathcal{S} \subset L^\infty(Q) \cap \mathrm{Dom}\, N^p$ for all $p \in \mathbb{N}$. Using that $V_\kappa \to V$ on $\mathrm{Dom}\, N^n$, we see that $V\Psi = W\Psi$ for all $\Psi \in \mathcal{S}$. Hence, $V = W$. This completes the proof of (1).

To prove (2), we use (22.22), the fact that V_κ and V belong to $\overset{2n}{\underset{p=0}{\oplus}} \Gamma_{\mathrm{s}}^p\big(L^2(\mathbb{R})\big)$, and Lemma 22.13.

It remains to prove (3). It follows from (9.26) that, for any $f \in L^2(\mathbb{R},\mathbb{R})$, one has

$$:\varphi(f)^p := \sum_{m=0}^{[p/2]} \frac{p!}{m!(p-2m)!}\varphi(f)^{p-2m}\Big(-\frac{1}{4}(f|\epsilon^{-1}f)\Big)^m. \tag{22.23}$$

Applying (22.23) to $f = \epsilon^{-\frac{1}{2}}\chi(\kappa^{-1}D_{\mathrm{x}})\delta_{\mathrm{x}}$, we obtain that

$$:\varphi_\kappa(\mathrm{x})^p := \sum_{m=0}^{[p/2]} \frac{p!}{m!(p-2m)!} c(\kappa)^{2m}\varphi_\kappa(\mathrm{x})^{p-2m}, \tag{22.24}$$

for

$$\begin{aligned} c(\kappa) &= \big(\delta_{\mathrm{x}}|\tfrac{1}{2}\epsilon^{-1}\chi^2(\kappa^{-1}D_{\mathrm{x}})\delta_{\mathrm{x}}\big)^{\frac{1}{2}} \\ &= (4\pi)^{-\frac{1}{2}}\big(\textstyle\int_{\mathbb{R}}(\mathrm{k}^2+m^2)^{-\frac{1}{2}}\chi^2(\kappa^{-1}\mathrm{k})\mathrm{dk}\big)^{\frac{1}{2}} \simeq C(\log\kappa)^{\frac{1}{2}}. \end{aligned} \tag{22.25}$$

We will apply the bound

$$a^m b^{p-m} \leq \delta b^p + C_\delta a^p, \quad a, b \geq 0, \qquad 0 \leq m \leq p, \quad \delta > 0, \tag{22.26}$$

to the terms in the r.h.s. of (22.24), setting $b = \varphi_\kappa(\mathrm{x})$, $a = c(\kappa)$. For $p = 2n$, we obtain, picking δ small enough,

$$:\varphi_\kappa(\mathrm{x})^{2n} :\geq \frac{1}{2}\varphi_\kappa(\mathrm{x})^{2n} - C(\log\kappa)^n.$$

For $p < 2n$, we take $\delta = 1$ and obtain

$$|:\varphi_\kappa(\mathrm{x})^p:| \leq C\big(|\varphi_\kappa(\mathrm{x})^p| + (\log\kappa)^{p/2}\big).$$

Both inequalities should be understood as inequalities between functions on the Q-space.

Since $a_{2n} > 0$, we obtain finally that

$$:P\big(\varphi_\kappa(\mathrm{x})\big) :\geq -C(\log\kappa)^n, \quad \text{for } \kappa \geq \kappa_0. \tag{22.27}$$

Integrating (22.27), using that $g \geq 0$ and $g \in L^1(\mathbb{R})$, we obtain (3). □

Although the operators V_κ are bounded from below, this is not the case for the operator V. Nevertheless, the measure of the set $\big\{q \in Q \ : V(q) < 0\big\}$ is very small, as shown in the next proposition.

Proposition 22.15 *Assume that $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $g \geq 0$. Then $\mathrm{e}^{-TV} \in L^1(Q, \mathrm{d}\mu)$ for all $T \geq 0$.*

Proof Let f be a positive measurable function on Q and $t \geq 0$. Set

$$m_f(t) := \mu\big(\{f(q) > t\}\big).$$

Clearly, for any $p \geq 1$,

$$m_f(t) \leq \|f\|^p_{L^p(Q)} t^{-p}. \tag{22.28}$$

Moreover, if $F : \mathbb{R}^+ \to \mathbb{R}^+$ is C^1 with $F' \geq 0$, one has

$$\int_Q F(f)\mathrm{d}\mu = \int_0^{+\infty} m_f(t)F'(t)\mathrm{d}t. \tag{22.29}$$

Let C be the constant in Prop. 22.14. We claim that there exist $c_1, c_2, \delta > 0$ such that

$$\mu\Big(\big\{q \in Q \ : \ V(q) \leq -2C(\log\kappa)^n\big\}\Big) \leq c_1 \mathrm{e}^{-c_2\kappa^\delta}. \tag{22.30}$$

Applying (22.29) to $F(\lambda) = \mathrm{e}^{T\lambda}$ and $f = -V\mathbb{1}_{\{V\leq 0\}}$, and using (22.30), we obtain that $\mathrm{e}^{-TV} \in L^1(Q)$.

It remains to prove (22.30). Since $V_\kappa \geq -C(\log\kappa)^n$, it follows that

$$\big\{V(q) \leq -2C(\log\kappa)^n\big\} \subset \big\{|V - V_\kappa|(q) \geq C(\log\kappa)^n\big\}.$$

Hence,

$$\mu\Big(\big\{V(q) \leq -2C(\log\kappa)^n\big\}\Big) \leq m_{|V-V_\kappa|}\big(C(\log\kappa)^n\big) \leq (p-1)^{np}\kappa^{-\delta p}(\log\kappa)^{-np},$$

by (22.28). Choosing $p = \kappa^{\delta/n} + 1$ yields (22.30). $\square$

22.2.5 Space-cutoff $P(\varphi)_2$ Hamiltonian

Theorem 22.16 *Assume that $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $g \geq 0$. Then,*

(1) $\mathrm{d}\Gamma(\epsilon) + V$ *is essentially self-adjoint on* $\mathrm{Dom}\,\mathrm{d}\Gamma(\epsilon) \cap \mathrm{Dom}\,V$*;*
(2) *The operator $H = \big(\mathrm{d}\Gamma(\epsilon) + V\big)^{\mathrm{cl}}$ is bounded from below.*

Definition 22.17 *The operator H is called a* space-cutoff $P(\varphi)_2$ Hamiltonian.

Proof We use the formalism of Subsect. 21.2.4. As the real Hilbert space we choose $\mathcal{X} = L^2(\mathbb{R}, \mathbb{R})$, so that $L^2(\mathbb{R}, \mathbb{R}) \otimes \mathcal{X} = L^2(\mathbb{R}^2, \mathbb{R})$. We choose the covariance

$$C = (D_t^2 + \epsilon^2)^{-1} = (D_t^2 + D_{\mathrm{x}}^2 + m^2)^{-1}.$$

We consider the associated Gaussian path space introduced in Def. 21.24. By Thm. 21.26, $\Gamma_{\mathrm{s}}(\mathcal{Z})$ is the physical Hilbert space and $H_0 = \mathrm{d}\Gamma(\epsilon)$ is the free

https://doi.org/10.1017/9781009290876.023 Published online by Cambridge University Press

Hamiltonian associated with this path space. By Props. 22.14 and 22.15, the multiplicative perturbation V satisfies the hypotheses of Prop. 21.35. By Prop. 9.29, we also see that $\mathrm{e}^{-t\mathrm{d}\Gamma(\epsilon)} = \Gamma(\mathrm{e}^{-t\epsilon})$ is a contraction on $L^p(Q)$ for all $1 \leq p \leq \infty$. Nelson's hyper-contractivity theorem (Thm. 9.30) implies that it maps $L^2(Q)$ into $L^p(Q)$ if $\mathrm{e}^{-tm} \leq (p-1)^{-\frac{1}{2}}$. Therefore, all the hypotheses of Thm. 21.38 are satisfied. This completes the proof of the theorem. □

22.2.6 Interacting dynamics and local algebras

Definition 22.18 *For $l > 0$, we set*

$$\begin{aligned} V_l &:= \int_{[-l,l]} :P\big(\varphi(\mathrm{x})\big): \mathrm{dx}, \\ H_l &:= \big(\mathrm{d}\Gamma(\epsilon) + V_l\big)^{\mathrm{cl}}, \\ \alpha_l^t(A) &:= \mathrm{e}^{\mathrm{i}tH_l} A \mathrm{e}^{-\mathrm{i}tH_l}, \quad A \in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big), \end{aligned}$$

which exist by Thm. 22.16.

Theorem 22.19 (Existence of interacting dynamics) *The following hold:*

(1) *For all bounded open intervals I and $A \in \mathfrak{R}(I)$, there exists the limit*

$$\alpha^t(A) := \lim_{l \to +\infty} \alpha_l^t(A).$$

(2) *α^t uniquely extends to the algebra $\mathfrak{O}$.*

(3) *Set*

$$\alpha^x := \alpha^t \circ \alpha_0^{\mathrm{x}}, \qquad x = (t, \mathrm{x}).$$

Then $\mathbb{R}^{1,1} \ni x \mapsto \alpha^x$ is a group of $*$-automorphisms of $\mathfrak{O}$.

Definition 22.20 *For a bounded open set $\mathcal{O} \subset \mathbb{R}^{1,1}$, we set*

$$\mathfrak{M}(\mathcal{O}) := \big\{\alpha^t(A) \ : \ A \in \mathfrak{R}(I), \ \{t\} \times I \subset \mathcal{O}\big\}'',$$

called the interacting local W^*-algebras.

Theorem 22.21 (Properties of interacting local algebras) *The following hold:*

(1) *One has*

$$\alpha^x\big(\mathfrak{M}(\mathcal{O})\big) = \mathfrak{M}(\mathcal{O} + x), \quad x \in \mathbb{R}^{1,1}.$$

(2) *The local interacting algebras are regular, i.e.*

$$\mathfrak{M}(\mathcal{O}) = \bigcap_{\mathcal{O}^{\mathrm{cl}} \subset \mathcal{O}_1} \mathfrak{M}(\mathcal{O}_1) = \bigvee_{\mathcal{O}_1^{\mathrm{cl}} \subset \mathcal{O}} \mathfrak{M}(\mathcal{O}_1).$$

(3) *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are causally separated, then*

$$\mathfrak{M}(\mathcal{O}_1) \subset \mathfrak{M}(\mathcal{O}_2)'.$$

(4) *If $\mathcal{O}_1$ is causally dependent on $\mathcal{O}_2$, then*

$$\mathfrak{M}(\mathcal{O}_1) \subset \mathfrak{M}(\mathcal{O}_2).$$

(5)

$$\mathfrak{O} = \Big(\bigcup_{\mathcal{O} \subset \mathbb{R}^{1,1}} \mathfrak{M}(\mathcal{O}) \Big)^{\mathrm{cpl}}.$$

Proof of Thm. 22.19. Applying Trotter's product formula (Thm. 2.75), we obtain

$$\mathrm{e}^{\mathrm{i}tH_l} = \mathrm{s} - \lim_{n\to\infty} \big(\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\epsilon)/n} \mathrm{e}^{\mathrm{i}tV_l/n}\big)^n.$$

For $A \in \mathfrak{R}(I)$, this implies

$$\alpha_l^t(A) = \mathrm{s} - \lim_{n\to\infty} \big(\alpha_0^{t/n} \circ \gamma_l^{t/n}\big)^n (A), \tag{22.31}$$

for

$$\gamma_l^t(A) := \mathrm{e}^{\mathrm{i}tV_l} A \mathrm{e}^{-\mathrm{i}tV_l}.$$

For $l' > l$, we have

$$V_{l'} - V_l = \int_{[-l',l']\backslash[-l,l]} :P\big(\varphi(\mathrm{x})\big): \mathrm{dx},$$

hence $V_{l'} - V_l$ is affiliated to $\mathfrak{R}\big(]-l',l'[\backslash[-l,l]\big)$. This implies that $\gamma_{l'}^t = \gamma_l^t$ on $\mathfrak{R}(I)$ if $l, l' > |I|$. Moreover, by the causality property, we know that

$$\alpha_0^t : \mathfrak{R}(I) \to \mathfrak{R}\big(I + [-|t|, |t|]\big). \tag{22.32}$$

Using (22.31) and (22.32), this implies that if $l, l' > |I| + |T|$ and $|t| \le T$, then $\alpha_l^t = \alpha_{l'}^t$ on $\mathfrak{R}(I)$. This shows the existence of α^t on $\mathfrak{R}(I)$. Since $t \mapsto \alpha_t^l$ is a group of $*$-automorphisms, so is $t \mapsto \alpha^t$. This completes the proof of (1). By density, α^t uniquely extends to $\mathfrak{O}$, which proves (2).

To prove (3), we note that $\alpha_0^{\mathrm{x}} \alpha_l^t \alpha_0^{-\mathrm{x}} = \alpha_{l+\mathrm{x}}^t$, which implies (3), by letting $l \to \infty$. $\square$

Proof of Thm. 22.21. (1) follows by the definition of $\mathfrak{M}(\mathcal{O})$. (2) follows from the analogous property of the time-zero local algebras $\mathfrak{R}(I)$:

$$\mathfrak{R}(I) = \bigcap_{J \supset I^{\mathrm{cl}}} \mathfrak{R}(J) = \bigvee_{J^{\mathrm{cl}} \subset I} \mathfrak{R}(J), \tag{22.33}$$

which is immediate.

To prove (3) and (4), instead of α^t we can use the space-cutoff dynamics α_l^t for sufficiently large l to define $\mathfrak{M}(\mathcal{O}_i)$. We note that it follows from (22.31) and the causality property that

$$\alpha_l^t\big(\mathfrak{R}(I)\big) \subset \mathfrak{R}\big(I +]-T, T[\big), \qquad |t| < T. \tag{22.34}$$

https://doi.org/10.1017/9781009290876.023 Published online by Cambridge University Press

Then (3) and (4) follow easily from (22.34). To prove (5) we again use (22.34) to get that

$$\mathfrak{R}(I) \subset \mathfrak{M}\big(]-\epsilon_0, \epsilon_0[\times I\big) \subset \mathfrak{R}(J),$$

for some $J \supset I$. This clearly implies (5). $\square$

For completeness, let us note the following theorem, which says that the interacting local algebras can also be defined only in terms of the φ fields and the interacting dynamics.

Theorem 22.22 $\mathfrak{M}(\mathcal{O}) = \Big\{\alpha^t\big(W(\vartheta, 0)\big) \,:\, \vartheta \in C_{\mathrm{c}}^\infty(I, \mathbb{R}),\ \{t\} \times \operatorname{supp}\vartheta \subset \mathcal{O}\Big\}''$.

The above theorem follows easily from the following proposition.

Proposition 22.23 *For $\delta > 0$, set $\mathfrak{B}_\delta(I) = \big\{\alpha^t(\mathrm{e}^{\mathrm{i}\varphi(f)}),\ \operatorname{supp} f \subset I,\ |t| < \delta\big\}''$. Then*

$$\mathfrak{R}(I) = \bigcap_{\delta > 0} \mathfrak{B}_\delta(I).$$

Proof By (22.34), we know that $\mathfrak{B}_\delta(I) \subset \mathfrak{R}(I + [-\delta, \delta])$. Hence, by (22.33), $\bigcap\limits_{\delta>0} \mathfrak{B}_\delta(I) \subset \mathfrak{R}(I)$.

To prove the converse inclusion, by (22.33) it suffices again to show that, for all $J^{\mathrm{cl}} \subset I$ and small enough $\delta > 0$, one has

$$\mathfrak{R}(J) \subset \mathfrak{B}_\delta(I). \tag{22.35}$$

To prove (22.35), let us fix I and J with $J^{\mathrm{cpl}} \subset I$, and set $\delta_0 = \frac{1}{2}\mathrm{dist}(J, \mathbb{R}\backslash I)$. We will first prove that if $\delta < \delta_0$, $\operatorname{supp}(\vartheta, \varsigma) \subset J$, then

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\epsilon)} W(\vartheta, \varsigma) \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\epsilon)} \in \mathfrak{B}_\delta(I), \quad |t| < \delta. \tag{22.36}$$

Set

$$\begin{aligned} V_I &:= \int_I : P\big(\varphi(\mathrm{x})\big) : \mathrm{dx}, \qquad H_I = \big(\mathrm{d}\Gamma(\epsilon) + V_I\big)^{\mathrm{cl}}, \\ V_I^{(r)} &:= V_I \mathbb{1}_{\{|V_I| \leq r\}}, \qquad H_I^{(r)} := \big(\mathrm{d}\Gamma(\epsilon) + V_I - V_I^{(r)}\big)^{\mathrm{cl}}, \end{aligned}$$

where $r \in \mathbb{N}$. The Hamiltonians $H_I^{(r)}$ are well defined by the methods of Sect. 22.2.5, and one has

$$H_I^{(r)} = H_I - V_I^{(r)}, \tag{22.37}$$

since $V_I^{(r)}$ is bounded.

It is easy to see that $V_I - V_I^{(r)}$ tends to 0 in $\bigcap\limits_{1 \leq p < \infty} L^p(Q)$ and for $t > 0$, $r \in \mathbb{N}$, $\mathrm{e}^{-t(V_I - V_I^{(r)})}$ is uniformly bounded in $L^1(Q)$. Using the methods of Sect. 21.3, we prove that

$$\mathrm{e}^{-t\mathrm{d}\Gamma(\epsilon)} = \mathrm{s} - \lim_{r\to+\infty} \mathrm{e}^{-tH_I^{(r)}}, \quad t > 0,$$

hence also

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\epsilon)} = \mathrm{s} - \lim_{r\to+\infty} \mathrm{e}^{\mathrm{i}tH_I^{(r)}}, \quad t \in \mathbb{R}. \tag{22.38}$$

By (22.37) and Trotter's product formula, we have

$$\mathrm{e}^{\mathrm{i}tH_I^{(r)}} = \mathrm{s} - \lim_{p\to+\infty} \big(\mathrm{e}^{\mathrm{i}tH_I/p}\mathrm{e}^{-\mathrm{i}tV_I^{(r)}/p}\big)^p.$$

Noting that V_I and hence $V_I^{(r)}$ are affiliated to $\mathfrak{R}(I)$, we see that $\mathrm{e}^{H_I^{(r)}}A\mathrm{e}^{-\mathrm{i}tH_I^{(r)}} \in \mathfrak{B}_\delta(I)$ if $A \in \mathfrak{R}(J)$ and $|t| < \delta$. Since $\mathfrak{B}_\delta(I)$ is weakly closed, we get (22.36), using (22.38).

It follows from (22.36) that $W\big(\frac{1}{t}\big(\mathrm{e}^{\mathrm{i}t\epsilon}h - h\big)\big) \in \mathfrak{B}_\delta(I)$ for $h = \vartheta + \mathrm{i}\epsilon\varsigma$, $\mathrm{supp}(\varsigma, \vartheta) \subset J$ and $|t| < \delta$. Using the strong continuity in Thm. 9.5, we obtain that $W(\mathrm{i}\epsilon f) = \mathrm{e}^{\mathrm{i}\pi(f)} \in \mathfrak{B}_\delta(I)$ if $\mathrm{supp}\, f \subset J$ and $f \in \mathrm{Dom}\,\epsilon$. Hence, $\mathrm{e}^{\mathrm{i}\varphi(f)}, \mathrm{e}^{\mathrm{i}\pi(g)} \in \mathfrak{B}_\delta(I)$ for $\mathrm{supp}\, f, \mathrm{supp}\, g \subset J$. This implies (22.35) and ends the proof of the proposition. □

22.3 Scattering theory for space-cutoff $P(\varphi)_2$ Hamiltonians

In this section we describe, without proof, some properties of the $P(\varphi)_2$ model. In particular, we discuss its *scattering theory*. This theory provides an interesting example of the application of the concept of CCR representations, which arise naturally as the so-called *asymptotic fields*.

In the formulation of the scattering theory we will use the symplectic space $(\mathcal{Y}, \omega)$ associated with the Klein–Gordon equation described at the beginning of Subsect. 22.1.1. Recall that it is equipped with the free dynamics $\mathbb{R} \ni t \mapsto \mathrm{e}^{ta}$, and the free Hamiltonian H_0 implements this dynamics:

$$\mathrm{e}^{\mathrm{i}tH_0} W(\zeta)\mathrm{e}^{-\mathrm{i}tH_0} = W(\mathrm{e}^{ta}\zeta), \quad \zeta \in \mathcal{Y}.$$

22.3.1 Domain of the space-cutoff $P(\varphi)_2$ Hamiltonian

Let us start with some questions about the Hamiltonian H constructed in Thm. 22.16.

The domain of H is not explicitly known, except if $\deg P = 4$, when it is known that $\mathrm{Dom}\, H = \mathrm{Dom}\,\mathrm{d}\Gamma(\epsilon) \cap \mathrm{Dom}\, V$. However, noting that, for all $\delta > 0$, the Hamiltonian $\delta\mathrm{d}\Gamma(\epsilon) + V$ is also bounded below, one obtains the following bounds:

$$H_0 \leq C(H_0 + V + b\mathbb{1}), \text{ for some } C,\ b > 0. \tag{22.39}$$

These estimates are called *first-order estimates*. The following *higher-order estimates* are in practice a substitute for the lack of knowledge of the domain of H.

https://doi.org/10.1017/9781009290876.023 Published online by Cambridge University Press

They are an important technical ingredient of the proof of most results described in this section.

Proposition 22.24 *Assume the hypotheses of Thm. 22.16. Then there exists $b > 0$ such that, for all $r \in \mathbb{N}$,*

$$\begin{aligned}
\|N^r (H + b\mathbb{1})^{-r}\| &< \infty, \\
\|H_0 N^r (H + b\mathbb{1})^{-n-r}\| &< \infty, \\
\|N^r (H + b\mathbb{1})^{-1} (N + \mathbb{1})^{1-r}\| &< \infty.
\end{aligned} \tag{22.40}$$

22.3.2 Spectrum of space-cutoff $P(\varphi)_2$ Hamiltonians

The following theorem about the *essential spectrum* of space-cutoff $P(\varphi)_2$ Hamiltonians was proven in Dereziński–Gérard (2000).

Theorem 22.25 *Assume the hypotheses of Thm. 22.16. Then*

$$\mathrm{spec}_{\,\mathrm{ess}}(H) = [\inf \mathrm{spec}\,(H) + m, +\infty[.$$

Corollary 22.26 *Therefore, H possesses a non-degenerate ground state (that is,* $\inf \mathrm{spec}\, H$ *is a simple eigenvalue).*

Proof Noting that $m > 0$, the existence of a ground state follows immediately from Thm. 22.25. Using the representation of e^{-tH} of Prop. 21.34, we see that e^{-tH} is positivity improving in the Q-space representation. By the Perron–Frobenius theorem (Thm. 5.25), it follows that the ground state is non-degenerate. □

22.3.3 Asymptotic fields

Scattering theory of space-cutoff $P(\varphi)_2$ Hamiltonians is quite different from the usual scattering theory studied e.g. in the context of Schrödinger operators. It resembles the so-called *Haag–Ruelle scattering theory* developed in the setting of the axiomatic quantum field theory. Its main ingredients are the so-called *asymptotic fields.*

Theorem 22.27 *Assume the hypotheses of Thm. 22.16. Suppose in addition that*

$$|\mathrm{x}|^s g(\mathrm{x}) \in L^2(\mathbb{R}) \text{ for some } s > 1.$$

Then the following hold:

(1) *For all $\zeta \in \mathcal{Y}$, the strong limits*

$$W^{\pm}(\zeta) := \mathrm{s} - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} W(\mathrm{e}^{-ta}\zeta)\mathrm{e}^{-\mathrm{i}tH}$$

exist. They are called the asymptotic Weyl operators*.*

https://doi.org/10.1017/9781009290876.023 Published online by Cambridge University Press

(2) *The maps*

$$\mathcal{Y} \ni \zeta \mapsto W^{\pm}(\zeta) \in U\big(\Gamma_{\rm s}(\mathcal{Z})\big)$$

are two CCR representations over the symplectic space $(\mathcal{Y}, \omega)$.

(3) *The representations* $W^{\pm}$ *are regular, so that we can define the asymptotic fields* $\phi^{\pm}(\zeta)$ *by the identity*

$$W^{\pm}(\zeta) = \mathrm{e}^{\mathrm{i}\phi^{\pm}(\zeta)}.$$

(4) *One has*

$$\mathrm{e}^{\mathrm{i}tH} W^{\pm}(\zeta) \mathrm{e}^{-\mathrm{i}tH} = W^{\pm}(\mathrm{e}^{ta}\zeta),$$

i.e. the unitary group $\mathrm{e}^{\mathrm{i}tH}$ *implements the free dynamics* e^{ta} *in the CCR representations* $W^{\pm}$.

(5) *Let us equip the symplectic space* $(\mathcal{Y}, \omega)$ *with its canonical Kähler anti-involution* j *defined in (22.4). Let* $\mathcal{K}^{\pm}$ *be the corresponding space of vacua of* $W^{\pm}$ *(see Def. 11.41). Let* $\mathcal{H}_{\rm pp}(H)$ *be the point spectrum subspace for* H. *Then*

$$\mathcal{H}_{\rm pp}(H) \subset \mathcal{K}^{\pm}.$$

Proof The theorem is relatively easy to prove and can be found in Dereziński–Gérard (2000). The main step of the first statement is the so-called Cook argument: we prove that the time derivative of $t \mapsto \mathrm{e}^{\mathrm{i}tH} W(\mathrm{e}^{-ta}\zeta)\mathrm{e}^{-\mathrm{i}tH}$ applied to a vector from a dense set is integrable. □

Let us note that Thm. 22.27 can be generalized to cover a much larger class of Hamiltonians. In particular, as proven in Dereziński–Gérard (1999), it holds under rather weak assumptions for the operators called sometimes *Pauli–Fierz Hamiltonians*. Operators of this form are well motivated from the physical point of view – they often appear in non-relativistic quantum physics.

It is natural to ask what type of CCR representations are defined in Thm. 22.27. Statement (5) suggests that a distinguished role is played by the Fock representation. In fact, one can prove that for space-cutoff $P(\varphi)_2$ Hamiltonians no other sectors exist.

Theorem 22.28 *Suppose that the assumptions of Thm. 22.27 hold. Then the following are true:*

(1) *The CCR representations* $W^{\pm}$ *are of Fock type for the anti-involution* j;
(2) $\mathcal{K}^{\pm} = \mathcal{H}_{\rm pp}(H)$.

Proof To prove (1) we use the number quadratic forms $n^{\pm}$ associated with the CCR representations $W^{\pm}$, defined in Subsect. 11.4.5. Let $\mathcal{V} \subset (2\epsilon)^{\frac{1}{2}} L^2(\mathbb{R})$ be a finite-dimensional space and $\Psi \in \mathrm{Dom}\,|H|^{\frac{1}{2}}$. Then, using

$$a^{\pm\sharp}(h)\Psi = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} a^{\sharp}(\mathrm{e}^{\mathrm{i}t\epsilon}h)\mathrm{e}^{-\mathrm{i}tH}\Psi,\ \ \Psi \in \mathrm{Dom}\,|H|^{\frac{1}{2}}, \tag{22.41}$$

https://doi.org/10.1017/9781009290876.023 Published online by Cambridge University Press

we obtain that

$$\begin{aligned} n_{\mathcal{V}}^{\pm}(\Psi) &= \sum_{i=1}^{\dim \mathcal{V}} \|a^{\pm}(e_i)\Psi\|^2 \\ &= \lim_{t\to\pm\infty} \Big(\mathrm{e}^{-\mathrm{i}tH}\Psi | \sum_{i=1}^{\dim \mathcal{V}} a^*(\mathrm{e}^{\mathrm{i}t\epsilon}e_i)a(\mathrm{e}^{\mathrm{i}t\epsilon}e_i)\mathrm{e}^{-\mathrm{i}tH}\Psi\Big) \\ &= \lim_{t\to\pm\infty} \big(\mathrm{e}^{-\mathrm{i}tH}\Psi | \mathrm{d}\Gamma(P_t)\mathrm{e}^{-\mathrm{i}tH}\Psi\big), \end{aligned}$$

where P_t is the orthogonal projection on the subspace $\mathrm{e}^{\mathrm{i}t\epsilon}\mathcal{V}$. We now note that

$$\mathrm{d}\Gamma(P_t) \le N \le C(H + b\mathbb{1}),$$

by the first-order estimates (22.39). Therefore,

$$n_{\mathcal{V}}^{\pm}(\Psi) \le C\big(\Psi, (H + b\mathbb{1})\Psi\big).$$

This implies that the number quadratic forms $n^{\pm} = \sup_{\mathcal{V}} n_{\mathcal{V}}^{\pm}$ are densely defined since $\mathrm{Dom}\,|H|^{\frac{1}{2}} \subset \mathrm{Dom}\, n^{\pm}$. By Thm. 11.52, this implies (1).

The proof of (2) is much more difficult and involves methods borrowed from N-body scattering theory; and see Dereziński–Gérard (2000) and Gérard–Panati (2008). □

The two statements of Thm. 22.28 taken together are sometimes called *asymptotic completeness*, since they give a complete understanding of the asymptotic CCR representations. This form of asymptotic completeness can be proven for a much larger class of Hamiltonians. In particular, in Dereziński–Gérard (1999) it has been proven, under rather weak assumptions, for a large class of massive Pauli–Fierz Hamiltonians. The crucial assumption used in the proofs of these statements is the existence of an energy gap in the spectrum of their 1-body Hamiltonians, which is usually called the positivity of their mass.

For space-cutoff $P(\varphi)_2$ Hamiltonians the condition $m > 0$ is needed to define the model itself. On the other hand, massless Pauli–Fierz Hamiltonians are easy to define. Thm. 22.27, with minor modifications, can be proven for a large class of such Hamiltonians. An outstanding question of scattering theory is what the conditions are for asymptotic completeness to hold in the case of massless Pauli–Fierz Hamiltonians.

The central concepts of the standard formulation of scattering theory, used in quantum mechanics, are the free Hamiltonian, and the wave and scattering operators. The reader may wonder why these concepts are missing from Thms. 22.27 and 22.28.

In reality, both wave operators and the scattering operator have a natural definition, which is essentially an application of the formalism of Sect. 11.4. The role of the free Hamiltonian is to some extent played by

$$K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\epsilon), \tag{22.42}$$

where $K := H\big|_{\mathcal{H}_{\mathrm{pp}}(H)}$.

Theorem 22.29 *Assume the hypotheses of Thm. 22.27. Then there exists a unique unitary operator* $S^{\pm}: \mathcal{H}_{\mathrm{pp}}(H) \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}) \to \Gamma_{\mathrm{s}}(\mathcal{Z})$ *such that*

$$S^{\pm}\Psi \otimes \Omega = \Psi, \quad \Psi \in \mathcal{H}_{\mathrm{pp}}(H),$$
$$S^{\pm}\mathbb{1} \otimes W(\zeta) = W^{\pm}(\zeta)S^{\pm}, \quad \zeta \in \mathcal{Y}.$$

It satisfies

$$S^{\pm}\big(K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\epsilon)\big) = HS^{\pm}.$$

Definition 22.30 *The operators* $S^{\pm}$ *are called* wave *or* Møller operators. $S := S^{+*}S^{-}$ *is called the* scattering operator.

Clearly, S is a unitary operator on $\mathcal{H}_{\mathrm{pp}}(H) \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$ commuting with (22.42).

22.4 Notes

The first general result on existence and uniqueness of solutions for non-linear Klein–Gordon equations is due to Ginibre–Velo (1985). More recent references can be found in the book by Tao (2006). The space-cutoff $P(\varphi)_2$ model was first constructed for $P(\varphi) = \varphi^4$ by Glimm–Jaffe (1968, 1970a), for general P by Segal (1970) and Simon–Høgh-Krohn (1972), using the theory of hyper-contractive semi-groups, and by Rosen (1970). The full translation invariant model was then constructed by Glimm–Jaffe (1970b) using local algebras, as in Subsect. 22.2.6. Later a construction by purely Euclidean arguments was given by Glimm–Jaffe–Spencer (1974), Guerra–Rosen–Simon (1973a,b, 1975) and Fröhlich–Simon (1977). The higher-order estimates for the $P(\varphi)_2$ model are due to Rosen (1971).

The construction of the asymptotic fields for a large class of models is due to Høgh-Krohn (1971). The spectral and scattering theory of space-cutoff $P(\varphi)_2$ models was studied by the authors in Dereziński–Gérard (2000) and by Gérard–Panati (2008), following an earlier similar work on Pauli–Fierz Hamiltonians by Dereziński–Gérard (2004).