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BANACH ALGEBRAS WEAK* GENERATED BY THEIR IDEMPOTENTS

THOMAS VILS PEDERSEN

Laboratoire de Mathématiques Pures, Université Bordeaux 1, 351 Cours de la Libération, F-33405 Talence Cédex, France

Present address: Department of Mathematics and Physics, The Royal Veterinary and Agricultural University, Thorvaldsensvej 40, DK-1871 Frederiksberg C, Denmark (vils@dina.kvl.dk)

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Abstract For a closed set E contained in the closed unit interval, we show that the big Lipschitz algebra $\Lambda_\gamma(E)$ $(0 < \gamma < 1)$ is sequentially weak* generated by its idempotents if and only if it is weak* generated by its idempotents if and only if the little Lipschitz algebra $\lambda_\gamma(E)$ is generated by its idempotents, and we describe a class of perfect symmetric sets for which this holds. Moreover, we prove that $\Lambda_1(E)$ is sequentially weak* generated by its idempotents if and only if E is of measure zero. Finally, we show that the quotient algebras

$$\mathcal{A}_\beta/\overline{J_\beta(E)}^{\text{weak}^*}$$

of the Beurling algebras need not be weak* generated by their idempotents, when E is of measure zero and $\beta \geqslant \frac{1}{2}$.

Keywords: Banach algebras; idempotents; weak* topology; Lipschitz algebras

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1. Introduction

A commutative Banach algebra $\mathcal{B}$ is said to be generated by its idempotents if the algebra $\mathcal{S}$ of all linear combinations of idempotents in $\mathcal{B}$ is norm dense in $\mathcal{B}$. In [**8**], we showed that a certain condition on a closed set E contained in the closed unit interval $\mathbb{I}$ is equivalent to the 'little' Lipschitz algebra $\lambda_\gamma(E)$ $(0 < \gamma < 1)$ being generated by its idempotents. Since the linear combinations of idempotents in $\lambda_\gamma(E)$ are exactly the simple functions, the result can also be considered as a characterization of the closed sets $E \subseteq \mathbb{I}$ for which every function in $\lambda_\gamma(E)$ can be approximated by simple functions.

In this paper, we continue this investigation, but in the context of weak* topologies. Suppose that $\mathcal{B}$ is the dual space of a Banach space and thus is equipped with a weak* topology. We say that $\mathcal{B}$ is weak* generated by its idempotents if $\mathcal{S}$ is weak* dense in $\mathcal{B}$, and that $\mathcal{B}$ is sequentially weak* generated by its idempotents if $\mathcal{S}$ is sequentially weak* dense in $\mathcal{B}$. The dual spaces that we shall be concerned with are the 'big' Lipschitz algebras and the Beurling algebras.

2. Lipschitz algebras

For a closed set $E \subseteq \mathbb{I}$ and $0 < \gamma \leqslant 1$, let $\Lambda_\gamma(E)$ be the Lipschitz algebra of functions f on E for which

$$p_\gamma(f) = \sup\left\{\frac{|f(t) - f(s)|}{|t - s|^\gamma} : t, s \in E,\ t \neq s\right\} < \infty.$$

With the norm

$$\|f\|_{\Lambda_\gamma(E)} = \|f\|_\infty + p_\gamma(f) \quad (f \in \Lambda_\gamma(E))$$

(where $\|\cdot\|_\infty$ is the uniform norm on E), it is easily seen that $\Lambda_\gamma(E)$ is a Banach algebra with character space E. For $0 < \gamma \leqslant 1$, let $\lambda_\gamma(E)$ be the closed subalgebra of $\Lambda_\gamma(E)$ of functions f satisfying

$$|f(t) - f(s)| = o(|t - s|^\gamma)$$

uniformly as $t - s \to 0$.

In this paper, we shall make use of the fact that $\Lambda_\gamma(E)$ $(0 < \gamma \leqslant 1)$ is a dual space. For $t \in E$, let $\delta_t \in \Lambda_\gamma(E)^*$ be the point evaluation functional at t, and let

$$Y_\gamma(E) = \overline{\operatorname{span}\{\delta_t : t \in E\}}$$

(norm closure in $\Lambda_\gamma(E)^*$). Johnson [**2**, Section 4] proved that

$$Y_\gamma(E)^* = \Lambda_\gamma(E)$$

via the duality $\langle \varphi, f\rangle = \langle f, \varphi\rangle$ $(\varphi \in Y_\gamma(E),\ f \in \Lambda_\gamma(E))$. Since $\|\delta_t - \delta_s\|_{Y_\gamma(E)} \leqslant |t - s|^\gamma$ for $t, s \in E$, it follows that $Y_\gamma(E)$ is separable. On bounded subsets of $\Lambda_\gamma(E)$, the weak* topology is thus metrizable (see, for example, [**1**, Theorem V.5.1]) and agrees with the topology of pointwise convergence on E. When $0 < \gamma < 1$, we further have $Y_\gamma(E) = \lambda_\gamma(E)^*$ and thus $\Lambda_\gamma(E) = \lambda_\gamma(E)^{**}$ [**2**, Theorem 4.7]. For $\gamma = 1$, this can be shown to hold exactly when E is of measure zero (see the proof of Proposition 2.3 and [**13**, Theorem 3.3.3]).

We now turn our attention to the idempotents in $\Lambda_\gamma(E)$. For a closed set $E \subseteq \mathbb{I}$, let $\mathcal{S}(E)$ be the linear span of the idempotents in $\Lambda_\gamma(E)$. A function in $\Lambda_\gamma(E)$ belongs to $\mathcal{S}(E)$ if and only if it assumes only finitely many values on E, so in particular $\mathcal{S}(E) \subseteq \lambda_\gamma(E)$. For $x, y \in E$, let

$$\rho_{E,\gamma}(x, y) = \sup\{|f(x) - f(y)| : f \in \mathcal{S}(E) \text{ and } p_\gamma(f) \leqslant 1\}.$$

For a closed set $E \subseteq \mathbb{I}$, we proved in [**8**, Theorem 3.3] that $\lambda_\gamma(E)$ $(0 < \gamma < 1)$ is generated by its idempotents if and only if $\rho_{E,\gamma}(x, y) = |x - y|^\gamma$ for every $x, y \in E$.

In [**8**, Proposition 3.1], we observed that $\Lambda_\gamma(E)$ is not generated by its idempotents when $E \subseteq \mathbb{I}$ is an infinite, closed set. It therefore seems natural to study the problem in the weak* topology. As in [**8**, Proposition 1.1], it is easily seen that E is totally disconnected if $\Lambda_\gamma(E)$ is weak* generated by its idempotents.

Following Weaver [**11**], we say that a subalgebra $\mathcal{B}$ of $\Lambda_\gamma(E)$ separates points uniformly (in $\Lambda_\gamma(E)$) if there exists a constant M such that, for every $x, y \in E$, there exists $f \in \mathcal{B}$ with $|f(x) - f(y)| = |x - y|^\gamma$ and $p_\gamma(f) \leqslant M$. In particular, $\mathcal{S}(E)$ separates points uniformly if and only if there exists a constant $C > 0$ such that $\rho_{E,\gamma}(x, y) \geqslant C|x - y|^\gamma$ for every $x, y \in E$. Using the order structure of $\Lambda_\gamma(E)$, Weaver (see [**11**, Theorem B] and also [**13**, Corollary 4.1.9]) proved that if $\mathcal{B}$ is a weak* closed subalgebra of $\Lambda_\gamma(E)$ which separates points uniformly, then $\mathcal{B} = \Lambda_\gamma(E)$.

We do not know (but doubt) whether multiplication is weak* continuous in $\Lambda_\gamma(E)$, so in order to conclude that the weak* closure of $\mathcal{S}(E)$ is closed under multiplication, we shall need the following transfinite induction. For a subspace $\mathcal{B}$ of $\Lambda_\gamma(E)$, let $\mathcal{B}^{(1)}$ be the sequential weak* closure of $\mathcal{B}$. Inductively, for a non-limit ordinal α, let $\mathcal{B}^{(\alpha)} = (\mathcal{B}^{(\alpha-1)})^{(1)}$, and for a limit ordinal α, let $\mathcal{B}^{(\alpha)} = \bigcup_{\alpha' < \alpha} \mathcal{B}^{(\alpha')}$. A theorem of Banach (see, for example, [**1**, Theorem V.12.10]) asserts that there exists a countable ordinal α such that $\mathcal{B}^{(\alpha)}$ equals the weak* closure of $\mathcal{B}$. Now, suppose that $\mathcal{B}$ is an algebra. On bounded subsets of $\Lambda_\gamma(E)$, the weak* topology agrees with the topology of pointwise convergence on E, so it follows that $\mathcal{B}^{(1)}$ is an algebra. Also, an increasing union of algebras is again an algebra, so we deduce that the weak* closure of $\mathcal{B}$ is an algebra.

Applying this to $\mathcal{B} = \mathcal{S}(E)$, it follows from Weaver's result that if $\mathcal{S}(E)$ separates points uniformly, then $\Lambda_\gamma(E)$ is weak* generated by its idempotents. Moreover, we shall see that in this case $\Lambda_\gamma(E)$ is actually sequentially weak* generated by its idempotents. Weaver's result (that is, the implication (d) $\Rightarrow$ (b)) is the main part of the following result. We link it to generation of $\lambda_\gamma(E)$ by its idempotents and prove that the condition that $\mathcal{S}(E)$ separates points uniformly can be formally weakened. In Theorem 2.5, we shall see that this allows us to handle the case left open in [**8**] concerning certain perfect symmetric sets.

Theorem 2.1. *For $0 < \gamma < 1$ and a closed set $E \subseteq \mathbb{I}$, the following conditions are equivalent.*

(a) *$\lambda_\gamma(E)$ is generated by its idempotents.*

(b) *$\Lambda_\gamma(E)$ is weak* generated by its idempotents.*

(c) *$\Lambda_\gamma(E)$ is sequentially weak* generated by its idempotents.*

(d) *There exists a constant $C > 0$ such that $\rho_{E,\gamma}(x, y) \geqslant C|x - y|^\gamma$ for every $x, y \in E$ (that is, $\mathcal{S}(E)$ separates points uniformly).*

(e) *There exists a constant $C > 0$ such that $\rho_{E,\gamma}(x, y) \geqslant C|x - y|$ for every $x, y \in E$.*

Moreover, conditions (c) and (d) are also equivalent for $\gamma = 1$.

Proof. We prove the implications (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (c). Let $f \in \Lambda_\gamma(E)$ and extend f to a function $g \in \Lambda_\gamma([0, 2\pi])$ with $g(2\pi) = g(0)$. For $n \in \mathbb{N}$, let $\sigma_n(g)$ be the nth Fejér sum of g. It follows from [**4**, p. 64] or [**9**, Corollary 2.3 and pp. 150, 151] that $\sigma_n(g) \to g$ weak* in $\Lambda_\gamma([0, 2\pi])$ as $n \to \infty$. Moreover, $f_n = \sigma_n(g)|_E \in \lambda_\gamma(E)$ for $n \in \mathbb{N}$ and (f_n) is a bounded sequence in $\Lambda_\gamma(E)$ with $f_n \to f$ pointwise on E as $n \to \infty$, so we deduce that $f_n \to f$ weak* in $\Lambda_\gamma(E)$ as $n \to \infty$. Since $\mathcal{S}(E)$ is norm dense in $\lambda_\gamma(E)$, the conclusion follows.

(c) $\Rightarrow$ (d). For $x \in E$, let $f_x(t) = |x - t|^\gamma (t \in E)$. Then $\{f_x : x \in E\}$ is a bounded set in $\Lambda_\gamma(E)$, so it follows from a theorem of Banach (see, for example, [**1**, Theorem V.12.11]) that there exists a constant M such that, for every $x \in E$, there exists a sequence (f_{xn}) in $\mathcal{S}(E)$ satisfying $f_{xn} \to f_x$ weak* in $\Lambda_\gamma(E)$ as $n \to \infty$ and $p_\gamma(f_{xn}) \leqslant M$ for $n \in \mathbb{N}$. For $x, y \in E$, we have $f_{xn} \to f_x$ pointwise on E as $n \to \infty$, so $|f_{xn}(x) - f_{xn}(y)| \to |x - y|^\gamma$ as $n \to \infty$. Hence

$$\rho_{E,\gamma}(x, y) \geqslant M^{-1}|x - y|^\gamma,$$

which proves the implication.

(d) $\Rightarrow$ (e). This implication is obvious.

(e) $\Rightarrow$ (b). There exists a constant M such that, for every $x, y \in E$, there exists $g \in \mathcal{S}(E)$ with $|g(x) - g(y)| = |x - y|$ and $p_\gamma(g) \leqslant M$. Since the polynomials are weak* dense in Λ_γ, it suffices to show that τ belongs to the weak* closure of $\mathcal{S}(E)$. However, we shall prove that every function $f \in \Lambda_1(E)$ belongs to the weak* closure of $\mathcal{S}(E)$, since we will need this below for the case $\gamma = 1$. We have

$$\mathbb{I} \setminus E = \bigcup_{n=1}^{\infty} V_n,$$

where (V_n) are pairwise-disjoint, open intervals in $\mathbb{I}$, and for $N \in \mathbb{N}$, we have

$$\mathbb{I} \setminus \bigcup_{n=1}^{N} V_n = \bigcup_{n=0}^{N} [x_{Nn}, y_{Nn}],$$

where $y_{Nn} < x_{N,n+1}$ $(n = 0, \dots, N - 1)$. For $n = 0, \dots, N$, we choose $g_{Nn} \in \mathcal{S}(E)$ with $g_{Nn}(x_{Nn}) = 0$, $g_{Nn}(y_{Nn}) = y_{Nn} - x_{Nn}$ and $p_\gamma(g_{Nn}) \leqslant M$. We then define $f_N \in \mathcal{S}(E)$ by

$$f_N = \frac{f(y_{Nn}) - f(x_{Nn})}{y_{Nn} - x_{Nn}} g_{Nn} + f(x_{Nn}) \quad \text{on } [x_{Nn}, y_{Nn}] \ (n = 0, \dots, N),$$

and let $h_N = f_N - f$. Then $h_N(x_{Nn}) = h_N(y_{Nn}) = 0$ for $n = 0, \dots, N$, so it follows that

$$p_\gamma(h_N) \leqslant 2 \max_{n=0,\dots,N} p_\gamma(h_N|_{[x_{Nn}, y_{Nn}]}).$$

Since $p_\gamma(f_N|_{[x_{Nn}, y_{Nn}]}) \leqslant M p_1(f)$ for $n = 0, \dots, N$, we thus have $p_\gamma(h_N) \leqslant 2(M+1)p_1(f)$. Moreover, it is easily seen that $\max\{y_{Nn} - x_{Nn} : n = 0, \dots, N\} \to 0$ as $N \to \infty$, so

$$\|h_N\|_\infty \leqslant p_\gamma(h_N) \max_{n=0,\dots,N} (y_{Nn} - x_{Nn}) \to 0$$

as $N \to \infty$. Hence $f_N \to f$ weak* in $\Lambda_\gamma(E)$ as $N \to \infty$. Since $\Lambda_1(E)$ is weak* dense in $\Lambda_\gamma(E)$ (see the proof of (a) $\Rightarrow$ (c)), the implication follows.

(b) $\Rightarrow$ (a). Let $\varphi \in \lambda_\gamma(E)^* = Y_\gamma(E)$ and suppose that $\varphi \perp \mathcal{S}(E)$. Since $Y_\gamma(E)^* = \Lambda_\gamma(E)$, it follows from (b) that $\varphi = 0$, so we deduce that $\mathcal{S}(E)$ is norm dense in $\lambda_\gamma(E)$, as required.

For $\gamma = 1$, the proof of the implication (c) $\Rightarrow$ (d) remains valid, and the reverse implication follows from the proof of the implication (e) $\Rightarrow$ (b). $\square$

Remark 2.2. It follows from the comment after Theorem 1.4 in [**12**] that condition (d) is equivalent to the local condition that, for every $x \in \mathbb{I}$, there exist constants $C_x, \delta_x > 0$ such that $\rho_{E,\gamma}(x-\delta, x+\delta) \geqslant C_x\delta^\gamma$ for $0 \leqslant \delta \leqslant \delta_x$, and it is easily seen that condition (e) can be 'localized' in the same way.

In the case $\gamma = 1$, we can simplify the previous theorem.

Proposition 2.3. *Let $E \subseteq \mathbb{I}$ be a closed set. Then $\mathcal{S}(E)$ separates points uniformly in $\Lambda_1(E)$ if and only if E is of measure zero.*

Proof. If E is of measure zero, then it follows from Lemma 2.3 (iii) in [**8**] that $\rho_{E,1}(x,y) = |x-y|$ for every $x, y \in E$, so $\mathcal{S}(E)$ separates points uniformly in $\Lambda_1(E)$. Conversely, if E is of positive measure, then

$$\frac{m([t-\delta, t+\delta] \setminus E)}{\delta} \to 0 \quad \text{as } \delta \to 0$$

(where m is the Lebesgue measure on $\mathbb{I}$) for almost every $t \in E$ [**10**, p. 141]. Since $\rho_{E,1}(x,y) \leqslant m([x,y] \setminus E)$ for every $x, y \in E$ with $x \leqslant y$, it follows that $\mathcal{S}(E)$ does not separate points uniformly in $\Lambda_1(E)$. $\square$

Combining the proposition with Theorem 2.1 immediately gives us the following result.

Theorem 2.4. *Let $E \subseteq \mathbb{I}$ be a closed set. Then $\Lambda_1(E)$ is sequentially weak* generated by its idempotents if and only if E is of measure zero.*

We do not know whether there exists a closed set $E \subseteq \mathbb{I}$ of positive measure for which $\Lambda_1(E)$ is weak* generated by its idempotents. For the little Lipschitz algebras, it follows from Proposition 2.3 and [**13**, Theorem 4.4.2] that $\lambda_1(E)$ is generated by its idempotents when E is of measure zero. Moreover, it is easily seen that $\lambda_1(E)$ is generated by its idempotents when E is a finite union of closed intervals, but this still leaves open the question of whether there exists a totally disconnected set E of positive measure for which $\lambda_1(E)$ is generated by its idempotents.

2.1. Perfect symmetric sets

In [**8**], we showed that the class of closed sets $E \subseteq \mathbb{I}$ for which $\lambda_\gamma(E)$ $(0 < \gamma < 1)$ is generated by its idempotents strictly contains the class of closed sets of measure zero and is strictly contained in the class of closed, totally disconnected sets. This was done partly by considering the following perfect symmetric sets. (For full details, see [**5**, Chapter I] or [**8**].)

Let $\underline{\xi} = (\xi_n)$ be a sequence with $0 < \xi_n < \frac{1}{2}$ for $n \in \mathbb{N}$ and let

$$E_{\underline{\xi}} = \left\{ \sum_{n=1}^{\infty} \varepsilon_n \xi_1 \cdots \xi_{n-1}(1-\xi_n) : \varepsilon_n = 0 \text{ or } 1 \text{ for } n \in \mathbb{N} \right\}.$$

Then $E_{\underline{\xi}}$ is a perfect, closed set with empty interior and Lebesgue measure

$$m(E_{\underline{\xi}}) = \lim_{n\to\infty} 2^n \xi_1 \cdots \xi_n.$$

We have

$$E_{\underline{\xi}} = \bigcap_{n=1}^{\infty} E_n = \mathbb{I} \setminus \bigcup_{n=1}^{\infty} V_n,$$

where $E_n = \bigcup_{k=1}^{2^n} E_{nk}$ and $E_{n1}, \dots, E_{n2^n}$ are disjoint, closed intervals each of length $\xi_1 \cdots \xi_n$, and where $V_n = \bigcup_{k=1}^{2^{n-1}} V_{nk}$ and $V_{n1}, \dots, V_{n2^{n-1}}$ are disjoint, open intervals each of length $l_n = \xi_1 \cdots \xi_{n-1}(1-2\xi_n)$. We denote by $V_{n\cdot}$ (respectively, $E_{n\cdot}$) any one of the V_{nk} (respectively, E_{nk}).

When $\xi_n = \frac{1}{2}(1-2^{-an})$ for $n \in \mathbb{N}$ for some $a > 0$, we write $E(a)$ for $E_{\underline{\xi}}$. Observe that $m(E(a)) > 0$ and that

$$l_n = 2^{-(n-1)}(1-2^{-a}) \cdots (1-2^{-a(n-1)})2^{-an} \geqslant 2m(E(a))2^{-(a+1)n}$$

for $n \in \mathbb{N}$. For $0 < \gamma < 1$, we showed in [**8**, Examples 3.6 and 3.15] that if $\gamma(a+1) < 1$, then $\lambda_\gamma(E(a))$ is generated by its idempotents, whereas if $\gamma(a+1) > 1$, then $\lambda_\gamma(E(a))$ is not generated by its idempotents. It follows from Theorem 2.1 that these results also hold for (sequential) weak* generation of $\Lambda_\gamma(E(a))$ by its idempotents. For $\gamma(a+1) < 1$, the proof of [**8**, Example 3.15] consists of verifying (a stronger version of) condition (d) from Theorem 2.1. We shall now show that condition (e) holds for the case $\gamma(a+1) = 1$, and hence that [**8**, Example 3.15] extends to this case. We have not been able to verify directly the formally stronger condition (d) in this case.

Theorem 2.5. *Let $0 < \gamma \leqslant 1$. Then $\lambda_\gamma(E(a))$ is generated by its idempotents if and only if $\gamma(a+1) \leqslant 1$.*

For $\gamma = 1$, this result follows from the previous theorem, and as mentioned above, for $\gamma(a+1) \neq 1$, the result follows from [**8**]. To prove the remaining case, $\gamma < 1$ and $\gamma(a+1) = 1$, we need the following two lemmas.

Lemma 2.6. *Let $0 < \gamma < 1$ and let $E \subseteq \mathbb{I}$ be a closed set. Let $0 \leqslant x_0 < x_1 < \cdots < x_N \leqslant 1$ be points in E. Suppose that there exists a constant $C > 0$ such that*

$$\rho_{E,\gamma}(x_n, x_{n+1}) \geqslant C(x_{n+1} - x_n) \quad (n = 0, \dots, N-1).$$

Then

$$\rho_{E,\gamma}(x_0, x_N) \geqslant C(C+4)^{-1}(x_N - x_0).$$

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Proof. For $n = 0, \dots, N-1$, choose $f_n \in \mathcal{S}(E)$ with $f_n(x_n) = Cx_n$, $f_n(x_{n+1}) = Cx_{n+1}$ and $p_\gamma(f_n) \leqslant 2$, and define $f \in \mathcal{S}(E)$ by $f = f_n$ on $[x_n, x_{n+1}]$ $(n = 0, \dots, N-1)$. Let $s \in [x_m, x_{m+1}]$ and $t \in [x_n, x_{n+1}]$ with $m < n$. Then

$$\begin{aligned}|f(t) - f(s)| &\leqslant |f(t) - f(x_n)| + |f(x_n) - f(x_{m+1})| + |f(x_{m+1}) - f(s)| \\ &\leqslant 2((t - x_n)^\gamma + (x_{m+1} - s)^\gamma) + C(x_n - x_{m+1}) \\ &\leqslant (4 + C)(t - s)^\gamma.\end{aligned}$$

Hence $p_\gamma(f_n) \leqslant 4 + C$. Since $f_n(x_N) - f_n(x_0) = C(x_N - x_0)$, the conclusion follows. $\square$

The next lemma is similar to [**8**, Example 3.15]. For $n \in \mathbb{N}$ and $1 \leqslant k \leqslant 2^{n-1}$, write $V_{nk} = (a_{nk}, b_{nk})$ with $b_{nk} < a_{n,k+1}$ $(1 \leqslant k \leqslant 2^{n-1} - 1)$. Then

$$s(n,k) = \min\{b_{nk_2} - a_{nk_1} : 1 \leqslant k_1,\ k_2 \leqslant 2^{n-1} \text{ and } k_2 - k_1 = k\}$$

is the minimum distance spanned by k of the intervals $V_{n\cdot}$. For notational convenience, we let

$$\rho_{E,\gamma}(F) = \rho_{E,\gamma}(x,y)$$

for a closed interval $F = [x,y]$ with $x, y \in E$.

Lemma 2.7. *Let $0 < \gamma < 1$ and let a be such that $\gamma(a+1) = 1$. Then there exists a constant $C > 0$ such that*

$$\rho_{E(a),\gamma}(E_{p\cdot}) \geqslant Cm(E_{p\cdot})$$

for $p \in \mathbb{N}$.

Proof. Let $p \in \mathbb{N}$ and let $F = [0, \xi_1 \cdots \xi_p]$. Let $n \geqslant p+1$ and let f_n be a continuous function which is linear with increase $2^{-(n-p-1)}m(F)$ on each V_{nk} $(1 \leqslant k \leqslant 2^{n-p-1})$ and is constant on the contiguous intervals. Then $f_n \in \mathcal{S}(E)$ and $f_n(\xi_1 \cdots \xi_p) - f_n(0) = m(F)$. It follows from [**8**, Lemma 3.7] that

$$\begin{aligned}p_\gamma(f_n) &= \max\biggl\{\frac{(k+1)2^{-(n-p-1)}m(F)}{s(n,k)^\gamma} : 0 \leqslant k \leqslant 2^{n-p-1} - 1\biggr\} \\ &\leqslant 2\max\biggl\{\frac{(k+1)2^{-n}}{s(n,k)^\gamma} : 0 \leqslant k \leqslant 2^{n-p-1} - 1\biggr\}.\end{aligned}$$

We have

$$\frac{2^{-n}}{s(n,0)^\gamma} = \frac{2^{-n}}{l_n^\gamma} \leqslant (2m(E(a)))^{-\gamma}.$$

Let $1 \leqslant m \leqslant n - p - 1$. It follows from [**8**, Lemma 3.11] that

$$\begin{aligned}\frac{2^{m-n}}{s(n, 2^m - 1)^\gamma} &= \frac{2^{m-n}}{(\xi_1 \cdots \xi_{n-m-1}(1 - 2\xi_{n-m} \cdots \xi_n))^\gamma} \\ &\leqslant \frac{2^{m-n}}{(2^{n-m-2}m(E(a)))^\gamma} \leqslant \biggl(\frac{4}{m(E(a))}\biggr)^\gamma.\end{aligned}$$

Also, for $2^{m-1} \leqslant k \leqslant 2^m - 1$, we have

$$\frac{s(n,k)}{k} \geqslant \frac{s(n,2^m-1)}{2^m-1}$$

by [**8**, Lemma 3.13]. Hence

$$\begin{aligned}\frac{(k+1)2^{-n}}{s(n,k)^{\gamma}} &\leqslant 2^{m+1-n}\frac{s(n,k)^{1-\gamma}}{s(n,2^m-1)} \\ &\leqslant \frac{2^{m+1-n}}{s(n,2^m-1)^{\gamma}} \leqslant 2\biggl(\frac{4}{m(E(a))}\biggr)^{\gamma}.\end{aligned}$$

Consequently, $p_{\gamma}(f_n) \leqslant 2(4/m(E(a)))^{\gamma}$, which finishes the proof. □

We can now give the following proof.

Proof of Theorem 2.5. Let $F \subseteq \mathbb{I}$ be a closed interval. First, suppose that there exists V_{nk} such that $m(V_{nk} \cap F) \geqslant m(F)/3$. Then

$$\rho_{E(a),\gamma}(F) \geqslant \rho_{E(a),\gamma}(V_{nk} \cap F) = m(V_{nk} \cap F)^{\gamma} \geqslant m(F)^{\gamma}/3^{\gamma} \geqslant m(F)/3^{\gamma}.$$

Conversely, if $m(V_{nk} \cap F) < m(F)/3$ for every V_{nk}, then there exist $V_{n_1k_1}, V_{n_2k_2} \subseteq F$ with $b_{n_2k_2} - a_{n_1k_1} \geqslant m(F)/3$. With $N = \max\{n_1, n_2\}$ and $U = (a_{n_1k_1}, b_{n_2k_2})$, we have

$$U = \biggl(\bigcup_{E_{N\cdot} \subseteq U} E_{N\cdot}\biggr) \bigcup \biggl(\bigcup_{\substack{n \leqslant N,\\ V_{n\cdot} \subseteq U}} V_{n\cdot}\biggr),$$

as in [**8**, Example 3.15]. This is a finite union and $\rho_{E(a),\gamma}(V_{n\cdot}) = m(V_{n\cdot})^{\gamma} \geqslant m(V_{n\cdot})$ for $n \in \mathbb{N}$, so it follows from the two previous lemmas that

$$\rho_{E(a),\gamma}(F) \geqslant \rho_{E(a),\gamma}(U) \geqslant C(C+4)^{-1}m(U) \geqslant \tfrac{1}{3}C(C+4)^{-1}m(F).$$

Hence $\lambda_{\gamma}(E(a))$ is generated by its idempotents by Theorem 2.1. □

3. Beurling algebras

Let $\mathbb{T}$ be the unit circle and let

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t \quad (n \in \mathbb{Z})$$

be the Fourier coefficients of a function $f \in L^1(\mathbb{T})$. For $\beta \geqslant 0$, let $\mathcal{A}_{\beta}$ be the Beurling algebra of functions f on $\mathbb{T}$ for which

$$\|f\|_{\mathcal{A}_{\beta}} = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|(1+|n|)^{\beta} < \infty.$$

Then $\mathcal{A}_{\beta}$ is a Banach algebra of continuous functions on $\mathbb{T}$.

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For a closed set $E \subseteq \mathbb{T}$, consider the ideals

$$I_\beta(E) = \{f \in \mathcal{A}_\beta : f = 0 \text{ on } E\},$$
$$J_\beta(E) = \{f \in \mathcal{A}_\beta : f = 0 \text{ on a neighbourhood of } E\}.$$

It follows from [**6**, Corollary VIII.5.7] that $\overline{J_\beta(E)} \subseteq I \subseteq I_\beta(E)$ for every closed ideal I in $\mathcal{A}_\beta$ with E as hull (that is, $E = \{z \in \mathbb{T} : f(z) = 0 \text{ for every } f \in I\}$). For $\beta < \frac{1}{4}$, it is known [**3**, p. 65] that synthesis fails in $\mathcal{A}_\beta$, that is, there exists a closed set $E \subseteq \mathbb{T}$ for which $J_\beta(E)$ is not norm dense in $I_\beta(E)$. We do not know whether this extends to $\frac{1}{4} \leqslant \beta < 1$.

Observe that the quotient algebra $\mathcal{A}_\beta / I_\beta(E)$ can be identified with the restriction algebra

$$\mathcal{A}_\beta(E) = \{f \in C(E) : \text{there exists } g \in \mathcal{A}_\beta : f = g|_E\}.$$

Zouakia [**14**, Corollaire 5.13] proved that $\mathcal{A}_\beta / \overline{J_\beta(E)}$ (and thus $\mathcal{A}_\beta(E)$) is generated by its idempotents whenever E is of measure zero and $\beta < \frac{1}{2}$. In Corollary 2.8 and the comment on p. 1122 of [**8**], we proved that this result is optimal in the sense that, for $\beta \geqslant \frac{1}{2}$, there exists a closed set E of measure zero such that $\mathcal{A}_\beta / \overline{J_\beta(E)}$ is not generated by its idempotents.

The algebra $\mathcal{A}_\beta$ is a dual space, and, for $\beta > 0$ and a closed set $E \subseteq \mathbb{T}$, we shall see that $\overline{J_\beta(E)}^{\text{weak}^*}$ and $I_\beta(E)$ are weak* closed ideals in $\mathcal{A}_\beta$. It immediately follows from Zouakia's result that the Banach algebra $\mathcal{A}_\beta / \overline{J_\beta(E)}^{\text{weak}^*}$ is sequentially weak* generated by its idempotents whenever E is of measure zero and $0 < \beta < \frac{1}{2}$. We shall show that, for $\beta \geqslant \frac{1}{2}$, there exists a closed set E of measure zero such that $\mathcal{A}_\beta / \overline{J_\beta(E)}^{\text{weak}^*}$ is not weak* generated by its idempotents.

We write $\mathcal{PM}_\beta$ (pseudomeasures with weight $(1 + |n|)^\beta$) for the dual space of $\mathcal{A}_\beta$. For $T \in \mathcal{PM}_\beta$ and $n \in \mathbb{Z}$, let

$$\widehat{T}(n) = \langle \mathrm{e}^{-\mathrm{i}nt}, T \rangle.$$

This identifies $\mathcal{PM}_\beta$ with the set of all sequences $(\widehat{T}(n))$ for which

$$\|T\|_{\mathcal{PM}_\beta} = \sup_{n \in \mathbb{Z}} \frac{|\widehat{T}(n)|}{(1 + |n|)^\beta} < \infty,$$

and

$$\langle f, T \rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n) \widehat{T}(-n)$$

for $f \in \mathcal{A}_\beta$ and $T \in \mathcal{PM}_\beta$. Let $\mathcal{PF}_\beta$ (pseudofunctions with weight $(1 + |n|)^\beta$) be the closed subspace of $\mathcal{PM}_\beta$ of those $T \in \mathcal{PM}_\beta$ for which

$$\frac{\widehat{T}(n)}{(1 + |n|)^\beta} \to 0 \quad \text{as } |n| \to \infty.$$

Then

$$\langle T, f \rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n) \widehat{T}(-n) \quad (T \in \mathcal{PF}_\beta,\ f \in \mathcal{A}_\beta)$$

identifies $\mathcal{A}_\beta$ with the dual space of $\mathcal{PF}_\beta$ and thus induces a weak* topology on $\mathcal{A}_\beta$.

It is well known that $\mathcal{PM}_\beta$ is a Banach $\mathcal{A}_\beta$-module via the action

$$\langle f, gT\rangle = \langle fg, T\rangle \quad (f, g \in \mathcal{A}_\beta,\ T \in \mathcal{PM}_\beta).$$

We also have the following lemma.

Lemma 3.1. *Let $\beta \geqslant 0$. Then $\mathcal{PF}_\beta$ is a closed $\mathcal{A}_\beta$-submodule of $\mathcal{PM}_\beta$.*

Proof. For $m \in \mathbb{Z}$, define $\xi_m \in \mathcal{PF}_\beta$ by $\langle \xi_m, f\rangle = \hat{f}(m)(f \in \mathcal{A}_\beta)$. Then $\widehat{f\xi_m}(n) = \hat{f}(n+m)$ for $f \in \mathcal{A}_\beta$ and $n \in \mathbb{Z}$, so we deduce that $\mathcal{A}_\beta \xi_m \subseteq \mathcal{PF}_\beta$. Since $\operatorname{span}\{\xi_m : m \in \mathbb{Z}\}$ is dense in $\mathcal{PF}_\beta$, the result follows. □

Corollary 3.2. *For $\beta \geqslant 0$, multiplication is separately weak* continuous in $\mathcal{A}_\beta$.*

Proof. Let (f_i) be a net in $\mathcal{A}_\beta$ converging weak* to 0. For $g \in \mathcal{A}_\beta$ and $T \in \mathcal{PF}_\beta$, we have

$$\langle T, f_i g\rangle = \langle f_i g, T\rangle = \langle f_i, gT\rangle = \langle gT, f_i\rangle \to 0,$$

since $gT \in \mathcal{PF}_\beta$ by the previous lemma. Hence $f_i g \to 0$ weak*, which proves the result. □

Let $T \in \mathcal{PM}_\beta$. As usual, we define the support of T ($\operatorname{supp} T$) in the sense of distributions, that is, as the complement of the largest open set $U \subseteq \mathbb{T}$ for which $\langle f, T\rangle = 0$ for every $f \in C^\infty(\mathbb{T})$ with $\operatorname{supp} f \subseteq U$. For a closed set $E \subseteq \mathbb{T}$, let

$$\mathcal{PM}_\beta(E) = \{T \in \mathcal{PM}_\beta : \operatorname{supp} T \subseteq E\}.$$

Also, for a subset $X \subseteq \mathcal{A}_\beta$, let

$$X^\perp = \{T \in \mathcal{PM}_\beta : \langle f, T\rangle = 0 \text{ for every } f \in X\},$$
$$^\perp X = \{T \in \mathcal{PF}_\beta : \langle T, f\rangle = 0 \text{ for every } f \in X\}.$$

It follows from [**3**, p. 29] that

$$\mathcal{PM}_\beta(E) = (\overline{J_\beta(E)})^\perp,$$

so $\mathcal{PM}_\beta(E)$ is the dual space of the quotient algebra $\mathcal{A}_\beta/\overline{J_\beta(E)}$.

For a closed set $E \subseteq \mathbb{T}$, let

$$\mathcal{PF}_\beta(E) = \mathcal{PM}_\beta(E) \cap \mathcal{PF}_\beta.$$

Then

$$\mathcal{PF}_\beta(E) = (\overline{J_\beta(E)})^\perp \cap \mathcal{PF}_\beta = {}^\perp(\overline{J_\beta(E)})$$

and thus

$$(\mathcal{PF}_\beta(E))^\perp = \overline{J_\beta(E)}^{\text{weak}^*}.$$

For $\varepsilon > 0$ and $z \in \mathbb{T}$, let $f_\varepsilon(z) = (1-z)/(1+\varepsilon-z)$. It is easily seen that $f_\varepsilon \to 1$ weak* in $\mathcal{A}_0$ as $\varepsilon \to 0$. In particular, $I_0(\{1\})$ is not weak* closed in $\mathcal{A}_0$. In the weighted algebras, the situation is different.

Proposition 3.3. *Let $\beta > 0$ and let $E \subseteq \mathbb{T}$ be a closed set. Then*

$$\overline{J_\beta(E)}^{\mathrm{weak}^*} \quad \text{and} \quad I_\beta(E)$$

are weak closed ideals in $\mathcal{A}_\beta$.*

Proof. It follows from the previous corollary that $\overline{J_\beta(E)}^{\mathrm{weak}^*}$ is an ideal. For $z \in \mathbb{T}$, define $T_z \in \mathcal{PF}_\beta$ by $\hat{T}_z(n) = z^{-n} (n \in \mathbb{Z})$. Then

$$\langle T_z, f\rangle = f(z) \quad \text{for } f \in \mathcal{A}_\beta.$$

Hence weak* convergence in $\mathcal{A}_\beta$ implies pointwise convergence on $\mathbb{T}$, so we deduce that $I_\beta(E)$ is weak* closed. □

It follows that

$$\mathcal{A}_\beta(E) = ({}^\perp(I_\beta(E)))^*,$$
$$\mathcal{A}_\beta/\overline{J_\beta(E)}^{\mathrm{weak}^*} = (\mathcal{PF}_\beta(E))^*$$

are dual spaces as well as Banach algebras with E as their character space. We shall need the following version of [**8**, Proposition 2.7].

Proposition 3.4. *Let $\beta > 0$ and let $E \subseteq \mathbb{T}$ be a closed set. Suppose that there exists a non-zero measure μ with support contained in E such that $\hat{\mu}(n) = o(|n|^{\beta-1})$ as $|n| \to \infty$. Then*

$$\mathcal{A}_\beta/\overline{J_\beta(E)}^{\mathrm{weak}^*}$$

is not weak generated by its idempotents.*

Proof. In the proof of [**8**, Proposition 2.7], we showed that

$$0 \neq \mu' \in \mathcal{PM}_\beta(E) = (\mathcal{A}_\beta/\overline{J_\beta(E)})^*$$

(where μ' is defined in the sense of distributions), and that $\langle e, \mu'\rangle = 0$ for every idempotent e in $\mathcal{A}_\beta/\overline{J_\beta(E)}$. Our condition on μ implies that $\mu' \in \mathcal{PF}_\beta(E)$, so $\langle \mu', e\rangle = 0$ for every idempotent e in $\mathcal{A}_\beta/\overline{J_\beta(E)}^{\mathrm{weak}^*}$, which proves the result. □

Using the Ivasěv–Musatov Theorem (or Körner's improvement thereof), we can now prove our result.

Theorem 3.5. *For $\beta \geqslant \frac{1}{2}$, there exists a perfect, closed set $E \subseteq \mathbb{T}$ of measure zero such that*

$$\mathcal{A}_\beta/\overline{J_\beta(E)}^{\mathrm{weak}^*}$$

is not weak generated by its idempotents.*

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Proof. It follows from [**7**, Theorem 1.1 or Theorem 1.2] that there exists a perfect, closed set $E \subseteq \mathbb{T}$ of measure zero and a non-zero measure μ with support contained in E such that

$$\hat{\mu}(n) = O((|n| \log |n|)^{-1/2}) = o(|n|^{\beta-1})$$

as $|n| \to \infty$. The result thus follows from the previous proposition. $\square$

For $\beta \geqslant \frac{1}{2}$, we do not know whether there exists a closed set $E \subseteq \mathbb{T}$ of measure zero for which $\mathcal{A}_\beta(E)$ is not weak* generated by its idempotents. If weak* synthesis holds, that is, if

$$\overline{J_\beta(E)}^{\text{weak}^*} = I_\beta(E)$$

for every closed set $E \subseteq \mathbb{T}$, then this is of course the case, but we suspect that weak* synthesis fails.

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