# A DECOMPOSITION OF MEASURES

NORMAN Y. LUTHER

Let $X$ be a set, $\mathscr{S}$ a $\sigma$-ring of subsets of $X$, and let $\mu$ be a measure on $\mathscr{S}$. Following (**1**), we define $\mu$ to be *semifinite* if

$$\mu(E) = \text{lub}\{\mu(F);\ F \in \mathscr{S},\ F \subset E, \text{ and } \mu(F) < \infty\} \quad \text{for every } E \in \mathscr{S}.$$

We show (Theorem 1) that every measure can be reduced to a semifinite measure for many practical purposes. In many cases, this reduction can be made even more significantly (Theorems 2 and 3). Finally, necessary and sufficient conditions that a semifinite measure be $\sigma$-finite are given as a corollary to Theorem 3.

We shall need the following concepts. A measure $\mu$ is *anti-semifinite* (or *degenerate*; cf. **3**, p. 127) if it takes on no finite, non-zero values, i.e., its range is contained in $\{0, \infty\}$. Clearly, the only measure which is both semifinite and anti-semifinite is the identically zero measure. If $\mu$ and $\nu$ are measures on $\mathscr{S}$, then following (**4**), we shall say that $\nu$ is *S-singular* with respect to $\mu$, denoted $\nu\, S\, \mu$, if for each $E \in \mathscr{S}$ there is a set $F \in \mathscr{S}$ such that $F \subset E$, $\nu(E) = \nu(F)$, and $\mu(F) = 0$. As is customary (**2**, p. 126), we use $\nu \perp \mu$ to signify that $\nu$ is *singular* with respect to $\mu$, i.e., that there is a set $A$ with $E \cap A \in \mathscr{S}$ for all $E \in \mathscr{S}$ (i.e., a *locally measurable* set $A$) such that $\nu(E \cap A) = 0 = \mu(E - A)$ for all $E \in \mathscr{S}$. Obviously, $\nu \perp \mu$ implies that $\nu\, S\, \mu$ and $\mu\, S\, \nu$. However, the converse fails, as we show below. Moreover, it is clear that although singularity is a symmetric relation, $S$-singularity does not possess this property. These facts suffice for our purposes. They, along with other facts regarding $S$-singularity, are recorded in (**4**).

THEOREM 1. *Let $\mu$ be a measure on a $\sigma$-ring $\mathscr{S}$.*
(i) *There exists a unique decomposition $\mu = \mu_1 + \mu_2$ such that:*
(a) *$\mu_1$ is a semifinite measure on $\mathscr{S}$.*
(b) *$\mu_2$ is an anti-semifinite measure on $\mathscr{S}$.*
(c) $\mu_1\, S\, \mu_2$.
(d) $\mu_2\, S\, \mu_1$.
(ii) *Let $\mu = \mu_1' + \mu_2'$, where $\mu_1'$ is a semifinite measure on $\mathscr{S}$ and $\mu_2'$ is an anti-semifinite measure on $\mathscr{S}$. Let $\mu = \mu_1 + \mu_2$ be the decomposition of* (i).
(a) *$\mu_1 \leqq \mu_1'$ and $\mu_2 \leqq \mu_2'$.*
(b) *If it is required that $\mu_1'\, S\, \mu_2'$, then $\mu_1'$ is unique; i.e., $\mu_1'$ is necessarily equal to $\mu_1$.*

---

Received November 28, 1966 and in revised form, July 26, 1967.



https://doi.org/10.4153/CJM-1968-092-0 Published online by Cambridge University Press

(c) *If it is required that $\mu_2' \, S \, \mu_1'$, then $\mu_2'$ is unique; i.e., $\mu_2'$ is necessarily equal to $\mu_2$.*

*Proof.* For a locally measurable set $F$, let $\mu_F$ denote the measure on $\mathscr{S}$ defined by

$$\mu_F(E) = \mu(F \cap E) \qquad \text{for all } E \in \mathscr{S}.$$

Let

$$\mathscr{M} = \{M \in \mathscr{S};\ M \text{ is } \mu\text{-}\sigma\text{-finite}\}$$

and define

$$\mu_1 = \text{lub}\{\mu_M;\ M \in \mathscr{M}\}.$$

Then $\mu_1$ is a measure on $\mathscr{S}$ by (**1**, Theorem 10.1) and one can readily verify that:

(1) $\mu_1(E) = \text{lub}\{\mu(M); M \subset E, M \in \mathscr{M}\}$ for all $E \in \mathscr{S}$;

(2) $\mu_1 = \mu$ on $\mathscr{M}$;

(3) For each $E \in \mathscr{S}$ there is an $M \in \mathscr{M}$ such that $M \subset E$ and

$$\mu_1(E) = \mu(M) = \mu_1(M)$$

(since $\mathscr{M}$ is closed under countable unions);

(4) $\mu_1$ is a semifinite measure on $\mathscr{S}$.

Next, we let

$$\mathscr{N} = \{N \in \mathscr{S}; \mu_1(N) = 0\}$$

and define

$$\mu_2 = \text{lub}\{\mu_N;\ N \in \mathscr{N}\}.$$

Then $\mu_2$ is a measure on $\mathscr{S}$ which satisfies:

(1)′ $\mu_2(E) = \text{lub}\{\mu(N); N \subset E, N \in \mathscr{N}\}$ for all $E \in \mathscr{S}$;

(2)′ $\mu_2 = \mu$ on $\mathscr{N}$;

(3)′ For each $E \in \mathscr{S}$ there is a set $N \in \mathscr{N}$ such that $N \subset E$ and

$$\mu_2(E) = \mu(N) = \mu_2(N);$$

(4)′ $\mu_2$ is an anti-semifinite measure on $\mathscr{S}$;

(5)′ $\mu_2 = 0$ on $\mathscr{M}$.

(1)′, (2)′, and (3)′ are obvious. (4)′ is shown as follows. Suppose there exists $E \in \mathscr{S}$ such that $\mu_2(E) = \delta$, where $\delta$ is finite and non-zero. There is a set $N \in \mathscr{N}$ such that $N \subset E$ and $\mu(N) = \mu_2(E) = \delta$; hence $N \in \mathscr{M}$ so that $\mu_1(N) = \mu(N) = \delta > 0$, contradicting $N \in \mathscr{N}$. The proof of (5)′ is similar: If $M \in \mathscr{M}$, there is a set $N \in \mathscr{N}$ such that $N \subset M$ (hence $N \in \mathscr{M}$) and $\mu_2(M) = \mu(N) = \mu_1(N) = 0$.

Next we show that $\mu = \mu_1 + \mu_2$. Let $E \in \mathscr{S}$. Since $\mu_1 = \mu$ on $\mathscr{M}$ and $\mu_2 = 0$ on $\mathscr{M}$, we can assume $E \notin \mathscr{M}$, hence $\mu(E) = \infty$. If $\mu_1(E) = \infty$, the result is obvious. Suppose $\mu_1(E) < \infty$. There exists a set $M \in \mathscr{M}$ such that $M \subset E$ and $\mu_1(M) = \mu(M) = \mu_1(E) < \infty$. Thus $\mu_1(E - M) = 0$ so that $E - M \in \mathscr{N}$. Hence $\mu_2(E) \geqq \mu_2(E - M) = \mu(E - M) = \mu(E) - \mu(M) = \infty$. Consequently, $\mu(E) = \infty = \mu_1(E) + \mu_2(E)$.

The last step in the existence part of this proof consists of showing that $\mu_1 S \mu_2$ and $\mu_2 S \mu_1$. Since the proofs are similar, we shall only prove that $\mu_1 S \mu_2$. Let $E \in \mathcal{S}$. There is a set $M \in \mathcal{M}$ such that $M \subset E$ and $\mu_1(E) = \mu_1(M)$. Moreover, $\mu_2(M) = 0$ by (5)′.

To demonstrate the uniqueness assertions, it obviously suffices to prove (ii)(b) and (ii)(c). In the process, we shall also prove (ii)(a). Suppose that $\mu = \mu_1 + \mu_2 = \mu_1' + \mu_2'$, where $\mu_1$ and $\mu_1'$ are semifinite measures on $\mathcal{S}$ and $\mu_2$ and $\mu_2'$ are anti-semifinite measures on $\mathcal{S}$. Clearly, it suffices to show: (A) if $\mu_1 S \mu_2$, then $\mu_1 \leqq \mu_1'$; and (B) if $\mu_2 S \mu_1$, then $\mu_2 \leqq \mu_2'$.

(A) Let $E \in \mathcal{S}$. Since $\mu_1 S \mu_2$, there is a set $F \in \mathcal{S}$ such that $F \subset E$, $\mu_1(F) = \mu_1(E)$, and $\mu_2(F) = 0$. It suffices to show that $\mu_1(F) \leqq \mu_1'(F)$ since then $\mu_1(E) = \mu_1(F) \leqq \mu_1'(F) \leqq \mu_1'(E)$. Suppose $\mu_1'(F) < \mu_1(F)$. There is a set $G \in \mathcal{S}$ such that $G \subset F$ and $\mu_1'(F) < \mu_1(G) < \infty$ since $\mu_1$ is semifinite. Then $\mu_1'(G) < \mu_1(G) < \infty$ and $\mu_2(G) = 0$ by monotonicity. Hence

$$\mu_2'(G) = \mu(G) - \mu_1'(G) = \mu_1(G) - \mu_1'(G),$$

contradicting the anti-semifiniteness of $\mu_2'$.

(B) Let $E \in \mathcal{S}$. Since $\mu_2 S \mu_1$, there is a set $F \in \mathcal{S}$ such that $F \subset E$, $\mu_2(F) = \mu_2(E)$, and $\mu_1(F) = 0$. Clearly, it suffices to show that $\mu_2(F) \leqq \mu_2'(F)$ in order to verify that $\mu_2(E) \leqq \mu_2'(E)$. Suppose $\mu_2'(F) < \mu_2(F)$. Then $\mu_2'(F) = 0$ and $\mu_2(F) = \infty$ by the anti-semifiniteness of $\mu_2$ and $\mu_2'$. Consequently, $\mu_1'(F) = \infty$; therefore, there is a set $G \in \mathcal{S}$ such that $G \subset F$ and $0 < \mu_1'(G) < \infty$ because $\mu_1'$ is semifinite. But $\mu_1(G) = 0$ and $\mu_2'(G) = 0$ by monotonicity. Hence $\mu_2(G) = \mu(G) = \mu_1'(G)$, contradicting the anti-semifiniteness of $\mu_2$. The proof is complete.

The following easily verified remarks pertain to the uniqueness of $\mu_1'$ and [or] $\mu_2'$ in the decomposition $\mu = \mu_1' + \mu_2'$, where $\mu_1'$ is a semifinite measure and $\mu_2'$ is an anti-semifinite measure. In particular, they indicate that Theorem 1 (ii) is quite optimal.

(I) $\mu_1' = \mu$ on $\mathcal{M} = \{M \in \mathcal{S}; M \text{ is } \mu\text{-}\sigma\text{-finite}\}$ and $\mu_2' = 0$ on $\mathcal{M}$. This is probably the strongest uniqueness statement that can be made in general.

(II) If one requires that $\mu_1' S \mu_2'$ $[\mu_2' S \mu_1']$, then $\mu_1'$ $[\mu_2']$ is unique, by Theorem 1 (ii), but $\mu_2'$ $[\mu_1']$ need not be unique.

(III) $\mu_1' S \mu_2'$ is necessary as well as sufficient for the uniqueness of $\mu_1'$. (To show this, one may copy, from the proof of Theorem 1, the argument that $\mu_1 S \mu_2$ since, necessarily, $\mu_2' = 0$ on $\mathcal{M}$ by Remark (I).) However, in general, $\mu_2' S \mu_1'$ is not necessary for the uniqueness of $\mu_2'$.

(IV) If $\mu$ is semifinite, then $\mu_1'$ is unique (in fact, $\mu_1' = \mu$) but $\mu_2'$ need not be unique.

(V) If $\mu$ is anti-semifinite, then $\mu_2'$ is unique (in fact, $\mu_2' = \mu$) but $\mu_1'$ need not be unique.

The following two simple examples may help to clarify the preceding remarks for the reader.

(a) Let $X$ be an uncountable set, and let $\mathscr{S}$ be its power set. Let $\mu_1(E) = \text{card}\,(E)$ and let

$$\mu_2(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ \infty & \text{if } E \text{ is uncountable.} \end{cases}$$

Let $\mu = \mu_1 + \mu_2$. Then $\mu$ is semifinite and $\mu_2$ is not unique. In fact, if $F$ is a given subset of $X$, and $\mu_2'(E) = \mu_2(E \cap F)$ for all $E \in \mathscr{S}$, then $\mu = \mu_1 + \mu_2'$.

(b) Let $X, \mathscr{S}$ be as in (a). Let

$$\mu(E) = \begin{cases} \infty & \text{if } E \neq \emptyset, \\ 0 & \text{if } E = \emptyset. \end{cases}$$

Then $\mu_2$ must be $\mu$, and $\mu_1$ can be any semifinite measure on $\mathscr{S}$. In this instance, $\mu_2 \, S \, \mu_1$ does not necessarily hold (e.g. if $\mu_1(E) = \text{card}(E)$).

That it is not possible to replace conditions (c) and (d) of Theorem 1 (i) by the condition $\mu_1 \perp \mu_2$ is illustrated by the following example. Let $(X, \mathscr{S})$ be the measurable space of (**2**, Exercise 31.9). Let $\mu_1(E)$ be the number of horizontal lines on which $E$ is full, let $\mu_2(E)$ be $\infty$ if $E$ is full on some vertical line, and 0 otherwise, and let $\mu = \mu_1 + \mu_2$. Then clearly, $\mu_1$ is semifinite (in fact, $\sigma$- finite), $\mu_2$ is anti-semifinite, $\mu_1 \, S \, \mu_2$, and $\mu_2 \, S \, \mu_1$. However, it is false that $\mu_1 \perp \mu_2$ since if there were a locally measurable set $A$ such that

$$\mu_1(E - A) = 0 = \mu_2(E \cap A)$$

for all $E \in \mathscr{S}$, then $A$ would be full on every horizontal line and countable on every vertical line, which is impossible.

Under what circumstances can one replace conditions (c) and (d) of Theorem 1 (i) by the condition $\mu_1 \perp \mu_2$? By the preceding example, it is not sufficient that $\mu_1$ be $\sigma$-finite. However, by (**4**, Theorem 3.3), it is sufficient that $\mu_1$ be strongly $\sigma$-finite. (See **4** for a discussion of strong $\sigma$-finiteness; finite measures on $\sigma$-rings and $\sigma$-finite measures on $\sigma$-algebras are common examples of strongly $\sigma$-finite measures.) As for conditions on $\mu$, it is obviously sufficient that $\mu$ be semifinite or anti-semifinite. Broader sufficient conditions are given in Proposition 1 and its application, Theorem 2, which follow.

PROPOSITION 1. *Let $\mu = \mu_1 + \mu_2$ be the decomposition of Theorem* 1 (i). *If $\mu$ is not semifinite* (*equivalently*, $\mu_2 \neq 0$), *then, by the maximal principal and the fact that $\mu_2 \, S \, \mu_1$, there exists a maximal disjoint subfamily $\mathscr{B}$ of*

$$\begin{aligned} \mathscr{G} &= \{G \in \mathscr{S}; \mu_1(G) = 0 \text{ and } \mu(G) = \infty\} \\ &= \{G \in \mathscr{S}; \mu_1(G) = 0 \text{ and } \mu_2(G) = \infty\}. \end{aligned}$$

*If some such $\mathscr{B}$ has the property that for each $E \in \mathscr{S}$ only countably many members of $\mathscr{B}$ intersect $E$, then $\mu_1 \perp \mu_2$; hence there is a unique decomposition $\mu = \mu_1 + \mu_2$ such that $\mu_1$ is a semifinite measure on $\mathscr{S}$, $\mu_2$ is an anti-semifinite measure on $\mathscr{S}$, and $\mu_1 \perp \mu_2$.*

*Proof*. Let $A = \bigcup\{B; B \in \mathscr{B}\}$. Then for every $E \in \mathscr{S}$ we have

$$A \cap E \in \mathscr{N} = \{N \in \mathscr{S}; \mu_1(N) = 0\}$$

and, consequently, $\mu_1(A \cap E) = 0$. Moreover, for each $E \in \mathscr{S}$ there is a set $N \in \mathscr{N}$ such that $N \subset E - A$ and $\mu_2(E - A) = \mu_2(N)$; but $\mu_2(N) = 0$ by the maximality of $\mathscr{B}$ so that $\mu_2(E - A) = 0$.

THEOREM 2. *Let $\mu$ be a measure on a $\sigma$-ring $\mathscr{S}$. Suppose there is a measure $\nu$ on $\mathscr{S}$ which is non-zero on every set in $\mathscr{S}$ of non-semifinite $\mu$-measure where either* (i) *$\nu$ is finite or* (ii) *$\mathscr{S}$ is a $\sigma$-algebra and $\nu$ is $\sigma$-finite. Then there is a unique decomposition $\mu = \mu_1 + \mu_2$ such that:*

(a) *$\mu_1$ is a semifinite measure on $\mathscr{S}$,*

(b) *$\mu_2$ is an anti-semifinite measure on $\mathscr{S}$,*

(c) $\mu_1 \perp \mu_2$.

*Proof*. Obviously, it suffices to prove the result under assumption (i). If $\mu$ is semifinite, we may take $\mu = \mu_1$ and $\mu_2 = 0$. Otherwise, letting $\mu = \mu_1 + \mu_2$ be the decomposition of Theorem 1 (i), we have $\mathscr{G} = \{G \in \mathscr{S}; \mu_1(G) = 0$ and $\mu(G) = \infty\}$ non-empty since $\mu_2 \neq 0$ and $\mu_2 \, S \, \mu_1$. Now $\mathscr{G}$ is closed under countable unions, thus $k = \text{lub}\{\nu(G); G \in \mathscr{G}\}$ is finite (and positive) and there is a set $G_1 \in \mathscr{G}$ such that $\nu(G_1) = k$. Indeed, the one-member family $\{G_1\}$ is a maximal disjoint subfamily of $\mathscr{G}$ since, if, $G_0 \in \mathscr{G}$ were disjoint from $G_1$, then $\nu(G_0) > 0$ by hypothesis, since every set in $\mathscr{G}$ is a set of non-semifinite $\mu$-measure; but then, $G_1 \cup G_0 \in \mathscr{G}$ and $\nu(G_1 \cup G_0) > k$, a contradiction. Proposition 1 applies and the proof is complete.

Along the same lines, we establish the following conditions under which both $\mu_1 \perp \mu_2$ and $\mu_1$ is $\sigma$-finite.

PROPOSITION 2. *If $\mu$ is not anti-semifinite, then by the maximal principle, there exists a maximal disjoint subfamily $\mathscr{C}$ of*

$$\mathscr{M}^+ = \{M \in \mathscr{S}; M \text{ is of positive and } \sigma\text{-finite } \mu\text{-measure}\}.$$

*If some such $\mathscr{C}$ has the property that for each $E \in \mathscr{S}$ only countably many members of $\mathscr{C}$ intersect $E$, then there is a unique decomposition $\mu = \mu_1 + \mu_2$ such that $\mu_1$ is a $\sigma$-finite measure on $\mathscr{S}$, $\mu_2$ is an anti-semifinite measure on $\mathscr{S}$, and $\mu_1 \perp \mu_2$.*

*Proof*. Let $A = \bigcup\{C; C \in \mathscr{C}\}$ and let $\mu = \mu_1 + \mu_2$ be the decomposition of Theorem 1 (i). Then for each $E \in \mathscr{S}$ we have $E \cap A \in \mathscr{M}$ and, consequently, $\mu_2(A \cap E) = 0$. Now, for each $E \in \mathscr{S}$ there is a set $M \in \mathscr{M}$ such that $M \subset E - A$ and $\mu_1(E - A) = \mu(M)$; but $\mu(M) = 0$ by the maximality of $\mathscr{C}$ so that $\mu_1(E - A) = 0$. Accordingly,

$$\mu_1(E) = \mu_1(E \cap A) = \mu(E \cap A) = \mu_A(E)$$

for all $E \in \mathscr{S}$, where $\mu_A$ is a $\sigma$-finite measure on $\mathscr{S}$.

As an application of Proposition 2, we establish a result which is essentially that of (**2**, Exercise 30.11). If $\nu$, $\eta$ are measures on a $\sigma$-ring $\mathscr{S}$, then $\eta$ is *absolutely continuous* with respect to $\nu$, denoted by $\eta \ll \nu$, if $\eta(E) = 0$ for every $E \in \mathscr{S}$ such that $\nu(E) = 0$.

THEOREM 3. *Let $\mu$ be a measure on a $\sigma$-ring $\mathscr{S}$. Suppose there is a measure $\nu$ on $\mathscr{S}$ such that $\mu \ll \nu$, where either* (i) *$\nu$ is finite or* (ii) *$\mathscr{S}$ is a $\sigma$-algebra and $\nu$ is $\sigma$-finite. Then there is a unique decomposition $\mu = \mu_1 + \mu_2$ such that:*

(a) *$\mu_1$ is a $\sigma$-finite measure on $\mathscr{S}$.*

(b) *$\mu_2$ is an anti-semifinite measure on $\mathscr{S}$.*

(c) $\mu_1 \perp \mu_2$.

The proof, in which Proposition 2 is applied, is similar to that of Theorem 2, and is left to the reader.

One should note that the hypotheses of Theorem 3 imply those of Theorem 2. Moreover, it is clear from the proofs that hypothesis (i) of Theorem 2 can be weakened to "$\nu$ is finite on every set of non-semifinite $\mu$-measure" and, likewise, (i) of Theorem 3 can be weakened to "$\nu$ is finite on every set of positive and $\sigma$-finite $\mu$-measure".

We cannot delete the hypothesis that "$\mathscr{S}$ is a $\sigma$-algebra" from (ii) of Theorems 2 or 3. This is illustrated by our earlier example since, if $\nu(E)$ is defined as the number of horizontal or vertical lines on which $E$ is full, then $\nu$ is $\sigma$-finite and $\mu \ll \nu$ in that example.

Since any measure possessing the decomposition of Theorem 3 with $\mu_2 \neq 0$ cannot be semifinite, we deduce the following.

COROLLARY. *Let $\mu$ be a semifinite measure on a $\sigma$-ring $\mathscr{S}$. Then $\mu$ is $\sigma$-finite if, and only if, there is a $\sigma$-finite measure $\nu$ on $\mathscr{S}$ such that $\mu \ll \nu$.*

*Proof.* If $\mu$ is $\sigma$-finite, let $\nu = \mu$. To prove the converse, let $E \in \mathscr{S}$. Since $\nu$ is $\sigma$-finite, there is a sequence of sets $E_i \in \mathscr{S}$ of finite $\nu$-measure such that $E \subset \bigcup E_i$. Then, for each $i$, $\nu_{E_i}$ is a finite measure on $\mathscr{S}$, $\mu_{E_i}$ is a semifinite measure on $\mathscr{S}$, and $\mu_{E_i} \ll \nu_{E_i}$; hence $\mu_{E_i}$ is a $\sigma$-finite measure on $\mathscr{S}$ by Theorem 3. Accordingly, for each $i$ there is a sequence of sets $F_{ij} \in \mathscr{S}$ such that $E_i \subset \bigcup_{j=1}^{\infty} F_{ij}$ and $\mu_{E_i}(F_{ij}) = \mu(E_i \cap F_{ij}) < \infty$ for all $j$. Since

$$E \subset \bigcup_{i,j} E_i \cap F_{ij},$$

$\mu$ is $\sigma$-finite.

In closing, we remark that most of the results of this paper fail for measures on rings or algebras. More specifically, suppose we extend the concepts of this paper to measures on a ring $\mathscr{R}$ in the obvious way, namely, by substituting $\mathscr{R}$ for $\mathscr{S}$. Then it is easily shown that every measure $\mu$ on $\mathscr{R}$ can be written as $\mu = \mu_1 + \mu_2$, where $\mu_1$ and $\mu_2$ are semifinite and anti-semifinite measures on $\mathscr{R}$, respectively. However, it may happen that (i) the decomposition is not unique and (ii) for every such decomposition, (a) neither $\mu_1 \, S \, \mu_2$ nor

$\mu_2 \, S \, \mu_1$, (b) $\mu_1$ is finite, and (c) $\mu \ll \mu_1$. (For such an example, let $X$ be the set of positive real numbers, $\mathscr{R}$ the class of all $E \subset X$ such that $E$ or its complement is finite, and let $\mu(E)$ be $\sum_{n \in E \cap J} 1/2^n$ or $\infty$ according as $E$ is finite or not, where $J$ denotes the set of positive integers.) In particular, Theorems 2 and 3 do not generalize to measures on rings or algebras.

## References

1. S. K. Berberian, *Measure and integration* (Macmillan, New York, 1965).
2. P. R. Halmos, *Measure theory* (Van Nostrand, New York, 1950).
3. E. Hewitt and K. Stromberg, *Real and abstract analysis* (Springer-Verlag, New York, 1965).
4. R. A. Johnson, *On the Lebesgue decomposition theorem*, Proc. Amer. Math. Soc. *18* (1967), 628–632.

*Washington State University,*
*Pullman, Washington*

https://doi.org/10.4153/CJM-1968-092-0 Published online by Cambridge University Press