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On Graded Categorical Groups and Equivariant Group Extensions

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Abstract. In this article we state and prove precise theorems on the homotopy classification of graded categorical groups and their homomorphisms. The results use equivariant group cohomology, and they are applied to show a treatment of the general equivariant group extension problem.

Introduction

If Γ is a group, then a Γ-graded categorical group is a groupoid $\mathbb{G}$ equipped with a grading functor $\mathrm{gr}\colon \mathbb{G} \to \Gamma$ and with a graded monoidal structure, by graded functors $\bigotimes\colon \mathbb{G} \times_\Gamma \mathbb{G} \to \mathbb{G}$ and $I\colon \Gamma \to \mathbb{G}$ and corresponding coherent 1-graded associativity and unit constraints, such that for each object X, there is an object X' with an arrow of grade 1 $X \otimes X' \to I$ (see Section 1 for the details). These graded categorical groups were originally introduced by Fröhlich and Wall in [9], where they presented a suitable abstract setting to study Brauer groups in equivariant situations. An illustrative example in that context is the *graded Picard categorical group*, $\mathrm{Pic}_\Gamma(R)$, defined by a ring R on which an action by ring automorphisms of a group Γ is given: the objects of $\mathrm{Pic}_\Gamma(R)$ are the invertible R-bimodules, and a morphism $P \to Q$ of grade $\sigma \in \Gamma$ is a pair (f, σ), where $f\colon P \xrightarrow{\sim} Q$ is an isomorphism of abelian groups with $f(rp) = {}^\sigma r f(p)$ and $f(pr) = f(p)^\sigma r$. The graded tensor functor is given by the tensor product of R-bimodules, the graded unit is defined by $I(\sigma) = (\sigma, \sigma)\colon R \to R$, and the associativity and unit 1-graded isomorphisms are the usual ones for the tensor product of bimodules (see the cited work [9] for more algebraic examples).

Furthermore, interesting graded categorical groups also arise from several problems in algebraic topology. To help motivate the reader, we consider the following example (see [6] for other instances): let Γ be a (discrete) group operating on a pointed space $(X, *)$ by pointed homeomorphisms. Then, the Γ-*graded categorical group of loops*, $\mathcal{P}_\Gamma(X, *)$, has the loops in X based on $*$, $\omega\colon [0, 1] \to X$, as its objects. A morphism $\omega \to \omega'$ of grade $\sigma \in \Gamma$ is a pair $([H], \sigma)$, where $[H]$ is the homotopy class real end loops of a homotopy $H\colon {}^\sigma\omega \to \omega'$, that is, of a map $H\colon [0, 1] \times [0, 1] \to X$ with $H(t, 0) = {}^\sigma\omega(t)$, $H(t, 1) = \omega'(t)$ and $H(0, s) = * = H(1, s)$. The composition is induced by the usual vertical composition of homotopies, according to the formula $([H'], \tau)([H], \sigma) = ([H' \circ {}^\tau H], \tau\sigma)$. The graded monoidal structure is induced by

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the H-group structure of the loop space $\Omega(X,*)$; thus, the graded tensor product is given on objects by concatenation of loops and on morphisms with the same grade by the horizontal composition of homotopies, and the 1-graded constraints are defined to be the homotopy classes of the respective standard homotopies proving the associativity and unit of the loop composition. This graded categorical group $\mathcal{P}_\Gamma(X,*)$ brings with it all information on the equivariant 2-type of $(X,*)$.

The main objective of this paper is to state and prove a precise classification theorem for graded categorical groups. The ungraded case, that is, the classification of categorical groups, was dealt with by Sinh in [17], where they were called Gr-categories (see also [1], [2], [5] or [12]). In this classification, two graded categorical groups over the same group of grades, say Γ, which are connected by a graded monoidal equivalence are considered the same. Hence, the problem arises of giving a complete invariant of this equivalence relation, which we solve by means of triples (G,A,k), consisting of a Γ-group G, a Γ-equivariant G-module A and a cohomology class $k \in H^3_\Gamma(G,A)$. Here, $H^n_\Gamma(G,A)$, $n \geq 0$, are the equivariant cohomology group studied in [7].

Our classification result point out the utility of graded categorical groups in homotopy theory: they arise as algebraic equivariant 2-types. Indeed, for any triple (G,A,k) as above, there is a pointed space $(X,*)$ on which the group Γ acts by pointed homeomorphisms, unique up to equivariant weak equivalence, such that $\pi_i(X,*) = 0$ if $i \neq 1,2$, $G = \pi_1(X,*)$ as Γ-group, $\pi_2(X,*) = A$ as Γ-equivariant G-module and k is the unique non-trivial "equivariant Postnikov invariant" of $(X,*)$ (*i.e.*, with the same invariants as the graded categorical group $\mathcal{P}_\Gamma(X,*)$). We should remark that by a *equivariant weak homotopy equivalence*, we mean a equivariant pointed map $f\colon (X,*) \to (Y,*)$ which is a weak homotopy equivalence. This should not be confused with the stronger notion of *weak equivariant-homotopy equivalence*, which is a equivariant pointed map that induces weak homotopy equivalences on the fixed point subspaces of all subgroups of Γ. The Postnikov invariant of a equivariant-homotopy 2-type is not an element of a cohomology group $H^3_\Gamma(G,A)$ as above, but rather an element of a Bredon-Moerdijk-Svensson 3-rd cohomology groups [3, 14], as it is showed in [4].

Moreover, with the aim of underlining the interest of our results on graded categorical groups for group theorists, we explain how from these results, we can deduce an appropriate treatment of the equivariant group extensions with a non-abelian kernel, including theory of obstructions. The conclusions we obtain are parallel to the known ones for group extensions by Schreier [16] and Eilenberg-MacLane [8], which appear now as the particular case in which Γ is trivial.

The article is organized in four sections. The first two are dedicated to stating a minimum of necessary concepts and terminology, by reviewing definitions and some basic facts concerning the homotopy category of Γ-graded categorical groups (Section 1) and the homotopy category of 3-cocycles of Γ-groups (Section 2). Section 3 is the main section, since it includes our theorems on homotopy classification of Γ-graded categorical groups and their homomorphisms. The last section is devoted to showing a cohomological solution to the problem of classifying all equivariant group extensions of any prescribed pair of Γ-groups, as an application of the results in Section 3.

1 Graded Categorical Groups

In this section we review the definition and establish some basic facts concerning graded categorical groups. Hereafter, Γ is a fixed group, which we regard as a category with exactly one object, say $*$, where the morphisms are the members of Γ and the composition law is the group operation: $* \xrightarrow{\sigma} * \xrightarrow{\tau} * = * \xrightarrow{\tau\sigma} *$.

A Γ-*grading* on a category $\mathbb{G}$, [9], is a functor $\mathrm{gr}\colon \mathbb{G} \to \Gamma$. For any morphism f in $\mathbb{G}$ with $\mathrm{gr}(f) = \sigma$, we refer to σ as the *grade* of f, and we say that f is a σ-*morphism*. The grading is *stable* if, for any object X of $\mathbb{G}$ and any $\sigma \in \Gamma$, there exists an isomorphism $X \xrightarrow{\sim} Y$ with domain X and grade σ; in other words, the grading is a cofibration in the sense of Grothendieck [10]. Suppose $(\mathbb{G}, \mathrm{gr})$ and $(\mathbb{H}, \mathrm{gr})$ are stably Γ-graded categories. A *graded functor* $F\colon (\mathbb{G}, \mathrm{gr}) \to (\mathbb{H}, \mathrm{gr})$ is a functor $F\colon \mathbb{G} \to \mathbb{H}$ preserving grades of morphisms. Observe from [10, Corollary 6.12] that every graded functor between stably Γ-graded categories is cocartesian. Suppose $F'\colon (\mathbb{G}, \mathrm{gr}) \to (\mathbb{H}, \mathrm{gr})$ is also a graded functor. A *graded natural equivalence* $\theta\colon F \to F'$ is a natural equivalence of functors such that all isomorphisms $\theta_X\colon FX \xrightarrow{\sim} F'X$ are of grade 1. If $(\mathbb{G}, \mathrm{gr})$ is a graded category, the category $\mathrm{Ker}\,\mathbb{G}$ is the subcategory consisting of all morphisms of grade 1; by [10, Proposition 6.10], a graded functor between stably graded categories $F\colon (\mathbb{G}, \mathrm{gr}) \to (\mathbb{H}, \mathrm{gr})$ is a graded equivalence if and only if the induced functor $F\colon \mathrm{Ker}\,\mathbb{G} \to \mathrm{Ker}\,\mathbb{H}$ is an equivalence of categories.

For a Γ-graded category $(\mathbb{G}, \mathrm{gr})$, we write $\mathbb{G} \times_\Gamma \mathbb{G}$ for the subcategory of the product category $\mathbb{G} \times \mathbb{G}$ whose arrows are those pairs of arrows of $\mathbb{G}$ with the same grade; this has an obvious grading, which is stable if and only if gr is.

A Γ-*graded monoidal category*, [9], (see [15, Chapter I, Section 4.5] for the general notion of fibred monoidal category) $\mathbb{G} = (\mathbb{G}, gr, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$, is a stably Γ-graded category $(\mathbb{G}, \mathrm{gr})$ together with graded functors

$$\bigotimes\colon \mathbb{G} \times_\Gamma \mathbb{G} \to \mathbb{G}, \quad I\colon \Gamma \to \mathbb{G}, \tag{1}$$

and graded natural equivalences

$$\begin{gathered} \mathbf{a}\colon (-\otimes-)\otimes- \xrightarrow{\sim} -\otimes(-\otimes-) \\ \mathbf{l}\colon I\,\mathrm{gr}(-)\otimes- \xrightarrow{\sim} \mathrm{id}_{\mathbb{G}}, \quad \mathbf{r}\colon -\otimes I\,\mathrm{gr}(-) \xrightarrow{\sim} \mathrm{id}_{\mathbb{G}} \end{gathered} \tag{2}$$

such that for any objects $X, Y, Z, T \in \mathbb{G}$, the following two coherence conditions hold:

$$\mathbf{a}_{X,Y,Z\otimes T}\mathbf{a}_{X\otimes Y,Z,T} = (X \otimes \mathbf{a}_{Y,Z,T})\mathbf{a}_{X,Y\otimes Z,T}(\mathbf{a}_{X,Y,Z} \otimes T), \tag{3}$$

$$(X \otimes \mathbf{l}_Y)\mathbf{a}_{X,I,Y} = \mathbf{r}_X \otimes Y. \tag{4}$$

If $\mathbb{G}$, $\mathbb{H}$ are Γ-graded monoidal categories, then a *graded monoidal functor*

$$F = (F, \Phi, \Phi_*)\colon \mathbb{G} \to \mathbb{H},$$

consists of a graded functor $F\colon \mathbb{G} \to \mathbb{H}$, natural isomorphisms of grade 1,

(5) $$\Phi = \Phi_{X,Y} : FX \otimes FY \xrightarrow{\sim} F(X \otimes Y),$$

and an isomorphism of grade 1 (natural with respect to the elements $\sigma \in \Gamma$)

(6) $$\Phi_* : I \xrightarrow{\sim} FI$$

(where $I = I(*)$) such that, for all objects $X, Y, Z \in \mathbb{G}$, the following coherence conditions hold:

(7) $$\Phi_{X,Y\otimes Z}(FX \otimes \Phi_{Y,Z})\mathbf{a}_{FX,FY,FZ} = F(\mathbf{a}_{X,Y,Z})\Phi_{X\otimes Y,Z}(\Phi_{X,Y} \otimes FZ),$$

(8) $$F(\mathbf{r}_X)\Phi_{X,I}(FX \otimes \Phi_*) = \mathbf{r}_{FX}, \quad F(\mathbf{l}_X)\Phi_{I,X}(\Phi_* \otimes FX) = \mathbf{l}_{FX}.$$

Suppose $F'\colon \mathbb{G} \to \mathbb{H}$ is also a graded monoidal functor. A *homotopy* (or *graded monoidal natural equivalence*) $\theta\colon F \to F'$ *of graded monoidal functors* is a graded natural equivalence $\theta\colon F \xrightarrow{\sim} F'$ such that, for all objects $X, Y \in \mathbb{G}$, the following coherence conditions hold:

(9) $$\Phi'_{X,Y}(\theta_X \otimes \theta_Y) = \theta_{X\otimes Y}\Phi_{X,Y}, \quad \theta_I\Phi_* = \Phi'_*.$$

For later use, we prove here the lemma below.

Lemma 1.1 *Every graded monoidal functor $F = (F, \Phi, \Phi_*)\colon \mathbb{G} \to \mathbb{H}$ is homotopic to a graded monoidal functor $F' = (F', \Phi', \Phi'_*)$ with $F'I = I$ and $\Phi'_* = \mathrm{id}_I$.*

Proof Consider the family of 1-graded isomorphisms in $\mathbb{H}$, $\theta_X = \begin{cases} \mathrm{id}_{FX} & \text{if } X \neq I \\ \Phi_*^{-1} & \text{if } X = I \end{cases}$, $X \in \mathbb{G}$. Then, F can be deformed to a new graded monoidal functor, say F', in a unique way such that $\theta\colon F \to F'$ becomes a homotopy. Namely,

$$F'X = \begin{cases} FX & \text{if } X \neq I \\ I & \text{if } X = I \end{cases}, F'(f\colon X \to Y) = \left(\theta_Y F(f)\theta_X^{-1} : F'X \to F'Y\right),$$

$$\Phi'_{X,Y} = \theta_{X\otimes Y}\Phi_{X,Y}(\theta_X \otimes \theta_Y)^{-1}, \ \Phi'_* = \theta_I\Phi_* = \mathrm{id}_I. \qquad \blacksquare$$

In a monoidal category, an object X is *regular* if the endofunctors $X \otimes -$ and $- \otimes X$ are equivalences. A *categorical group* is a monoidal category in which every arrow is invertible and every object is regular, that is, a monoidal groupoid that is compact, in the sense that for each object X there is another object X' with an arrow $X \otimes X' \to I$ (*cf.* [17, 16, 9, 12, 1]).

If $\mathbb{G}$ is a Γ-graded monoidal category, then the subcategory $\operatorname{Ker} \mathbb{G}$ inherits a monoidal structure, whose tensor product $\bigotimes\colon \operatorname{Ker} \mathbb{G} \times \operatorname{Ker} \mathbb{G} \to \operatorname{Ker} \mathbb{G}$ is the restriction of the graded tensor product (1), the unit object is $I = I(*)$ and the associativity, left and right constraints are the same as $\mathbb{G}$. When $\operatorname{Ker}\mathbb{G}$ is a categorical group, then $\mathbb{G}$ is said to be a Γ-*graded categorical group*. Since $\operatorname{Ker}\mathbb{G}$ is a groupoid if and only if $\mathbb{G}$ is, a Γ-graded categorical group can be defined as a Γ-graded monoidal groupoid such that for any object X there is an object X' with an 1-arrow $X \otimes X' \to I$.

We write

$$_{\Gamma}\mathcal{CG}$$

for the *category of* Γ-*graded categorical groups*, whose morphisms are the graded monoidal functors between them. Indeed, $_{\Gamma}\mathcal{CG}$ is a 2-category, whose deformations are the homotopies between graded monoidal functors. Every homotopy is invertible, so we can define the *homotopy category of* Γ-*graded categorical groups* to be the quotient category with the same objects, but morphisms are homotopy classes of graded monoidal functors. We write $\operatorname{Hom}_{_{\Gamma}\mathcal{CG}}[\mathbb{G},\mathbb{H}]$ for the homsets of the homotopy category, that is,

$$\operatorname{Hom}_{_{\Gamma}\mathcal{CG}}[\mathbb{G},\mathbb{H}] = \frac{\operatorname{Hom}_{_{\Gamma}\mathcal{CG}}(\mathbb{G},\mathbb{H})}{\text{homotopies}}. \tag{10}$$

A graded monoidal functor inducing an isomorphism in the homotopy category is said to be a *graded monoidal equivalence* and two graded categorical groups are *equivalent* if they are isomorphic in the homotopy category.

The homotopy classification of Γ-graded categorical groups is our major objective. For, we will associate to each Γ-graded categorical group $\mathbb{G}$ the algebraic data $\pi_0\mathbb{G}$, $\pi_1\mathbb{G}$ and $k(\mathbb{G})$, which are invariant under graded monoidal equivalences. We next introduce the first two.

Let us recall that a Γ-*group* G means a group G enriched with a left Γ-action by automorphisms, and that an *equivariant module* over a Γ-group G is a Γ-module A, that is, an abelian Γ-group, endowed with a G-module structure such that ${}^{\sigma}({}^{x}a) = {}^{({}^{\sigma}x)}({}^{\sigma}a)$ for all $\sigma \in \Gamma$, $x \in G$ and $a \in A$ [7, Definition 2.1].

Suppose $\mathbb{G} = (\mathbb{G}, \mathrm{gr}, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$ is a Γ-graded categorical group. Then we take the invariant made up of:

- $\pi_0\mathbb{G}$ = the set of 1-isomorphism classes of the objects in $\mathbb{G}$.
- $\pi_1\mathbb{G}$ = the set of 1-automorphisms of the unit object I.

Thus, $\pi_i\mathbb{G} = \pi_i \operatorname{Ker}\mathbb{G}$, $i = 0, 1$, the first invariants of the categorical group $\operatorname{Ker}\mathbb{G}$ considered by Sinh in [17, Chapter I, Sections 1, 2, Definition 1]. Therefore, we know that $\pi_0\mathbb{G}$ is a group, where multiplication is induced by tensor product, $[X][Y] = [X \otimes Y]$, and that $\pi_1\mathbb{G}$ is an abelian group, where operation is composition, with a canonical structure of $\pi_0\mathbb{G}$-module as follows [17, Chapter I, Sections 1, 2, Proposition 4]: if $s \in \pi_0\mathbb{G}$ and $a \in \pi_1\mathbb{G}$, then ${}^{s}a$ is determined by the formula

$$\delta_X(a) = \gamma_X({}^{s}a) \quad \text{for } X \in s, \tag{11}$$

where

$$\operatorname{Aut}_1(X) \xleftarrow[\sim]{\delta_X} \pi_1\mathbb{G} \xrightarrow[\sim]{\gamma_X} \operatorname{Aut}_1(X) \tag{12}$$

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are the isomorphisms defined respectively by $\delta_X(a) = \mathbf{r}_X(X \otimes a)\mathbf{r}_X^{-1}$ and $\gamma_X(a) = \mathbf{l}_X(a \otimes X)\mathbf{l}_X^{-1}$. The proof that (11) defines a $\pi_0\mathbb{G}$-module structure on $\pi_1\mathbb{G}$ uses the following equalities by [15, Chapter I, Sections 2, (2.2.6.1)] or [9, Theorem 2.2]:

$$\begin{aligned} &\delta_{X\otimes Y}(a) = X \otimes \delta_Y(a), \quad \gamma_{X\otimes Y}(a) = \gamma_X(a) \otimes Y, \\ &\delta_X(a) \otimes Y = X \otimes \gamma_Y(a), \quad \delta_I(a) = a = \gamma_I(a), \end{aligned} \tag{13}$$

for all objects $X, Y \in \mathbb{G}$ and $a \in \pi_1\mathbb{G}$. In addition, we shall remark the followings equalities:

$$g\gamma_X(a) = \gamma_Y({}^{\sigma}a)g, \quad g\delta_X(a) = \delta_Y({}^{\sigma}a)g, \tag{14}$$

for all $\sigma \in \Gamma$, $g\colon X \to Y$ a morphism in $\mathbb{G}$ of grade σ and $a \in \pi_1\mathbb{G}$, where

$${}^{\sigma}a = I(\sigma)aI(\sigma)^{-1}. \tag{15}$$

These equalities (14) can be easily proved: since $\mathbf{l}_X$ is an isomorphism and we have

$$\begin{aligned} \gamma_Y({}^{\sigma}a)g\mathbf{l}_X &\overset{\text{nat}}{=\!=} \gamma_Y({}^{\sigma}a)\mathbf{l}_Y\big(I(\sigma)\otimes g\big) \overset{\text{def}}{=\!=} \mathbf{l}_Y({}^{\sigma}a \otimes Y)\big(I(\sigma)\otimes g\big) \\ &\overset{(15)}{=\!=} \mathbf{l}_Y\big(I(\sigma)\otimes g\big)(a \otimes X) \overset{\text{nat}}{=\!=} g\mathbf{l}_X(a\otimes X) \\ &\overset{\text{def}}{=\!=} g\gamma_X(a)\mathbf{l}_X, \end{aligned}$$

we deduce the left equality in (14). The proof of the other equality is similar.

It is clear that (15) defines a Γ-module structure on $\pi_1\mathbb{G}$. Next we observe that $\pi_0\mathbb{G}$ is a Γ-group: if $\sigma \in \Gamma$ and $s \in \pi_0\mathbb{G}$, then we write

$${}^{\sigma}s = s', \tag{16}$$

whenever there exists a morphism of grade σ, $X \to X'$, with $X \in s$ and $X' \in s'$. Since the grading on $\mathbb{G}$ is stable, ${}^{\sigma}s$ is defined for all $\sigma \in \Gamma$ and $s \in \pi_0\mathbb{G}$. The map $\Gamma \times \pi_0\mathbb{G} \to \pi_0\mathbb{G}$, $s \mapsto {}^{\sigma}s$, is well defined since every morphism in $\mathbb{G}$ is invertible: if $g\colon X \to X'$ and $g'\colon Y \to Y'$ are both σ-morphisms and $h\colon X \to Y$ is an 1-morphism, then $h' = g'hg^{-1}\colon X' \to Y'$ is an 1-morphism. Therefore, $[X'] = [Y'] \in \pi_0\mathbb{G}$.

If $s, t \in \pi_0\mathbb{G}$, $X \in s$, $Y \in t$ and $g\colon X \to X'$ and $h\colon Y \to Y'$ are two σ-morphisms, then $g \otimes h\colon X \otimes X' \to Y \otimes Y'$ is also a σ-morphism; whence,

$${}^{\sigma}(st) = {}^{\sigma}[X \otimes Y] = [X' \otimes Y'] = [X'][Y'] = {}^{\sigma}s{}^{\sigma}t.$$

Furthermore, if $\sigma, \tau \in \Gamma$, then for any $s = [X] \in \pi_0\mathbb{G}$, any τ-morphism $h\colon X \to Y$ and any σ-morphism $g\colon Y \to Z$, since the composition fg is a $\sigma\tau$-morphism, we have

$${}^{\sigma\tau}s = [Z] = {}^{\sigma}[Y] = {}^{\sigma}({}^{\tau}s).$$

Hence, $\pi_0\mathbb{G}$ is a Γ-group.

We now are ready for the following proposition which prepares the way for the classification of graded categorical groups.

Proposition 1.2 *If $\mathbb{G} = (\mathbb{G}, gr, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$ is a Γ-graded categorical group, then $\pi_0\mathbb{G}$ is a Γ-group, with Γ-action (16), and $\pi_1\mathbb{G}$ is a Γ-equivariant $\pi_0\mathbb{G}$-module, with Γ-action (15) and $\pi_0\mathbb{G}$-action (11).*

Proof It only remains to prove that the $\pi_0\mathbb{G}$-action map on $\pi_1\mathbb{G}$, $(s, a) \mapsto {}^s a$, is Γ-equivariant, that is, ${}^\sigma({}^s a) = {}^{({}^\sigma s)}({}^\sigma a)$, for each $\sigma \in \Gamma$, $s \in \pi_0\mathbb{G}$ and $a \in \pi_1\mathbb{G}$. For, suppose $X \in s$ and let $g\colon X \to Y$ be any morphism in $\mathbb{G}$ of grade σ. Then $Y \in {}^\sigma s$ and

$$\gamma_Y\left({}^\sigma({}^s a)\right) g \overset{(14)}{=\!=} g\gamma_X({}^s a) \overset{(11)}{=\!=} g\delta_X(a) \overset{(14)}{=\!=} \delta_Y({}^\sigma a)g.$$

Since g is invertible, it follows that $\gamma_Y\left({}^\sigma({}^s a)\right) = \delta_Y({}^\sigma a)$, whence ${}^\sigma({}^s a) = {}^{({}^\sigma s)}({}^\sigma a)$, as required. ∎

We shall note that the correspondence

$$\mathbb{G} \mapsto (\pi_0\mathbb{G}, \pi_1\mathbb{G})$$

is the mapping on objects of a functor from ${}_\Gamma\mathcal{CG}$. Let us introduce the symbol (G, A) (and call it a Γ-*pair*) to signify that G is a Γ-group and A is a Γ-equivariant G-module. Suppose that (G', A') is another Γ-pair. If we have an equivariant homomorphism $p\colon G \to G'$ (so that A' becomes a Γ-equivariant G-module via p) and a homomorphism of Γ-equivariant G-modules $q\colon A \to A'$ (*i.e.*, a homomorphism which is both of Γ- and G-modules), then the composite object (p, q) is called a *morphism of* Γ-*pairs*; symbolically,

$$(p, q)\colon (G, A) \to (G', A'). \tag{17}$$

Thus, with the obvious definition for the composition of morphisms, $(p, q)(p', q') = (pp', qq')$, we have the *category of* Γ-*pairs*, denoted by ${}_\Gamma\,\mathrm{Pairs}$.

Proposition 1.3

(i) *Every graded monoidal functor between Γ-graded categorical groups $F\colon \mathbb{G} \to \mathbb{H}$ induces a morphism of Γ-pairs*

$$(\pi_0 F, \pi_1 F)\colon (\pi_0\mathbb{G}, \pi_1\mathbb{G}) \longrightarrow (\pi_0\mathbb{H}, \pi_1\mathbb{H}), \tag{18}$$
$$\pi_0 F\colon [X] \mapsto [FX],$$
$$\pi_1 F\colon a \mapsto \Phi_*^{-1}F(a)\Phi_*.$$

(ii) *The mapping $F \mapsto (\pi_0 F, \pi_1 F)$ defines a functor*

$$(\pi_0, \pi_1)\colon {}_\Gamma\mathcal{CG} \to {}_\Gamma\,\mathrm{Pairs}\,.$$

(iii) *Two homotopic graded monoidal functors induce the same morphism of Γ-pairs.*

(iv) *A graded monoidal functor is a graded monoidal equivalence if and only if the induced Γ-pair morphism is an isomorphism.*

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Proof (i) Since the restriction $F\colon \operatorname{Ker}\mathbb{G} \to \operatorname{Ker}\mathbb{H}$ is a monoidal functor between categorical groups, it follows from [17, Chapter I, Section 1, 1, Proposition 9] that $\pi_0 F\colon \pi_0\mathbb{G} \to \pi_0\mathbb{H}$ is a group homomorphism and that $\pi_1 F\colon \pi_1\mathbb{G} \to \pi_1\mathbb{H}$ is a $\pi_0\mathbb{G}$-module homomorphism, considering $\pi_1\mathbb{H}$ as $\pi_0\mathbb{G}$-module via $\pi_0 F$. Therefore, it only remains to prove that $\pi_0 F$ and $\pi_1 F$ are Γ-equivariant maps.

Suppose $\sigma \in \Gamma$ and $s \in \pi_0\mathbb{G}$ with ${}^{\sigma}s = s'$. This means that there exists an arrow in $\mathbb{G}$ of grade σ, say $g\colon X \to X'$, with $X \in s$ and $X' \in s'$. Then, $\pi_0 F({}^{\sigma}[X]) = \pi_0 F([X']) = [FX'] = {}^{\sigma}[FX] = {}^{\sigma}\big(\pi_0 F(s)\big)$, because the arrow $Ff\colon FX \to FX'$. Therefore, $\pi_0 F$ is a Γ-group homomorphism.

Suppose now that $\sigma \in \Gamma$ and $a \in \pi_1\mathbb{G}$. We have the following diagram,

$$\begin{tikzcd}
FI \arrow[rrr, "FI(\sigma)"] \arrow[dr, "F(a)"'] & & & FI \arrow[dl, "F({}^{\sigma}a)"] \\
& FI \arrow[r, "FI(\sigma)"'] \arrow[r, phantom, shift left=4ex, "\mathbf{3}"] & FI & \\
& I \arrow[u, "\Phi_*"'] \arrow[r, "I(\sigma)"] \arrow[ur, phantom, "\mathbf{4}"] & I \arrow[u, "\Phi_*"] & \\
I \arrow[uuu, "\Phi_*"] \arrow[ur, "\pi_1 F(a)"'] \arrow[rrr, "I(\sigma)"'] \arrow[uur, phantom, "\mathbf{1}"] \arrow[rrr, phantom, shift left=4ex, "?"] & & & I \arrow[uuu, "\Phi_*"'] \arrow[ul, "\pi_1 F({}^{\sigma}a)"] \arrow[uul, phantom, "\mathbf{2}"]
\end{tikzcd}$$

in which region **4** and the outside commute by the naturalness of Φ_*, regions **1** and **2** commute by definition of $\pi_1 F$, and region **3** commutes since F is a functor. It follows that the region distinguished by the question mark commutes, which tells us that $\pi_1 F({}^{\sigma}a) = {}^{\sigma}\big(\pi_1 F(a)\big)$, that is, $\pi_1 F$ is equivariant.

(ii) It is straightforward.

(iii) Suppose $F, F'\colon \mathbb{G} \to \mathbb{H}$ are two graded monoidal functors made homotopic by $\theta\colon F \to F'$. Then, for all object $X \in \mathbb{G}$, $\theta_X\colon FX \to F'X$ has grade 1 and therefore $[FX] = [F'X]$, that is, $\pi_0 F([X]) = \pi_0 F'([X])$. Furthermore, for any $a \in \pi_1\mathbb{G}$, in the following diagram,

$$\begin{tikzcd}
& FI \arrow[r, "Fa"] \arrow[dd, "\theta_I"'] & FI \arrow[dd, "\theta_I"'] & \\
I \arrow[ur, "\Phi_*"] \arrow[dr, "\Phi'_*"'] & & & I, \arrow[ul, "\Phi_*"'] \arrow[dl, "\Phi'_*"] \\
& F'I \arrow[r, "F'a"] & F'I &
\end{tikzcd}$$

the triangles commute by (9) and the square in the middle by naturalness. Hence,

$$\pi_1 F(a) = \Phi_*^{-1} F(a) \Phi_* = {\Phi'_*}^{-1} F(a) \Phi'_* = \pi_1 F'(a).$$

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(iv) Let $F\colon \mathbb{G} \to \mathbb{H}$ be a graded monoidal functor between graded Γ-graded categorical groups. We know that F is a graded equivalence if and only if its restriction $F\colon \operatorname{Ker}\mathbb{G} \to \operatorname{Ker}\mathbb{H}$ is an equivalence, which occurs if and only if $\pi_0 F$ and $\pi_1 F$ are isomorphisms by [17, Chapter I, Sections 1, 2, Proposition 9]. Then, to complete the proof it is enough to observe that if F is a graded equivalence, then it is actually a graded monoidal equivalence.

Assume F is a Γ-equivalence. Then there exists a graded functor $F'\colon \mathbb{H} \to \mathbb{G}$ and graded natural equivalences $\theta\colon FF' \xrightarrow{\sim} \mathrm{id}_{\mathbb{H}}$ and $\theta'\colon F'F \xrightarrow{\sim} \mathrm{id}_{\mathbb{G}}$ such that $F'\theta = \theta' F'$ and $F\theta' = \theta F$. Now, observe that F' can be provided in a unique way with a monoidal structure such that θ and θ' become homotopies, by means of the isomorphisms $\Phi'_{X,Y}\colon F'X \otimes F'Y \to F'(X \otimes Y)$ and $\Phi'_*\colon I \to F'I$, which make the following diagrams commutative:

$$\begin{tikzcd}
F(F'X \otimes F'Y) \arrow[r, "(\Phi'_{X,Y})"] \arrow[d, "\Phi_{F'X,F'Y}"'] & FF'(X \otimes Y) \arrow[d, "\theta_{X\otimes Y}"] \\
FF'X \otimes FF'Y \arrow[r, "\theta_X \otimes \theta_Y"] & X \otimes Y
\end{tikzcd} \quad , \quad
\begin{tikzcd}
FI \arrow[rr, "F(\Phi'_*)"] & & FF'I \\
& I \arrow[ul, "\Phi_*"] \arrow[ur, "\Phi_I"'] &
\end{tikzcd} \quad .$$

■

2 The Category of 3-Cocycles of Γ-Groups

In the previous section, we showed that every Γ-graded categorical group has associated a Γ-pair (G, A), that is, a Γ-group G and a Γ-equivariant G-module A, whose cohomology groups $H^n_\Gamma(G, A)$ are studied in [7]. The homotopy classification of Γ-graded categorical groups will be done, in the next section, by showing an equivalence of categories with the homotopy category of the *category 3-cocycles of Γ-groups*, $\mathcal{Z}^3_\Gamma$, whose brief study this section is dedicated to.

Before introducing the category $\mathcal{Z}^3_\Gamma$, we shall recall from [7] that the cohomology groups $H^n_\Gamma(G, A)$, which are a kind of cotriple cohomology for the algebraic category of Γ-groups, for $n \leq 3$, can be computed as the cohomology groups of the truncated cochain complex

$$(19) \qquad \tilde{C}_\Gamma(G, A)\colon 0 \to C^1_\Gamma(G, A) \xrightarrow{\partial} C^2_\Gamma(G, A) \xrightarrow{\partial} Z^3_\Gamma(G, A) \to 0,$$

in which $C^1_\Gamma(G, A)$ consists of normalized maps $f\colon G \to A$, $C^2_\Gamma(G, A)$ consists of normalized maps $g\colon G^2 \cup (G \times \Gamma) \to A$ and $Z^3_\Gamma(G, A)$ consists of all normalized maps $h\colon G^3 \cup (G^2 \times \Gamma) \cup (G \times \Gamma^2) \to A$ satisfying the following 3-cocycle conditions:

$$(20) \qquad h(x, y, zt) + h(xy, z, t) = {}^x h(y, z, t) + h(x, yz, t) + h(x, y, z),$$

(21)
$${}^\sigma h(x, y, z) + h(xy, z, \sigma) + h(x, y, \sigma) = h({}^\sigma x, {}^\sigma y, {}^\sigma z) + {}^{({}^\sigma x)} h(y, z, \sigma) + h(x, yz, \sigma),$$

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(22)
$$^{\sigma}h(x,y,\tau)+h(^{\tau}x,{}^{\tau}y,\sigma)+h(x,\sigma,\tau)+{}^{(^{\sigma\tau}x)}h(y,\sigma,\tau)=h(x,y,\sigma\tau)+h(xy,\sigma,\tau),$$

(23)
$$^{\sigma}h(x,\tau,\gamma)+h(x,\sigma,\tau\gamma)=h(x,\sigma\tau,\gamma)+h(^{\gamma}x,\sigma,\tau),$$

for $x,y,z,t\in G$, $\sigma,\tau,\gamma\in\Gamma$. For each $f\in C^1_{\Gamma}(G,A)$, the coboundary ∂f is given by

(24)
$$(\partial f)(x,y)={}^{x}f(y)-f(xy)+f(x),$$

(25)
$$(\partial f)(x,\sigma)={}^{\sigma}f(x)-f(^{\sigma}x),$$

and for $g\in C^2_{\Gamma}(G,A)$, ∂g is given by

(26)
$$(\partial g)(x,y,z)={}^{x}g(y,z)-g(xy,z)+g(x,yz)-g(x,y),$$

(27)
$$(\partial g)(x,y,\sigma)={}^{\sigma}g(x,y)-g(^{\sigma}x,{}^{\sigma}y)-{}^{(^{\sigma}x)}g(y,\sigma)+g(xy,\sigma)-g(x,\sigma),$$

(28)
$$(\partial g)(x,\sigma,\tau)={}^{\sigma}g(x,\tau)-g(x,\sigma\tau)+g(^{\tau}x,\sigma).$$

Let us remark that any morphism of Γ-pairs $(p,q)\colon(G,A)\to(G',A')$ yields morphisms of cochain complexes

$$\tilde{C}_{\Gamma}(G',A')\xrightarrow{p^*}\tilde{C}_{\Gamma}(G,A')\xleftarrow{q_*}\tilde{C}_{\Gamma}(G,A),$$

where A' is considered an equivariant G-module via $p\colon G\to G'$.

We now define the *category of* 3*-cocycles of* Γ*-groups*, $\mathcal{Z}^3_{\Gamma}$.

An *object* of $\mathcal{Z}^3_{\Gamma}$ is a triple $T=(G,A,h)$, where (G,A) is a Γ-pair and $h\in Z^3_{\Gamma}(G,A)$. Suppose $T'=(G',A',h')$ is another object in $\mathcal{Z}^3_{\Gamma}$. A *morphism* $P\colon T\to T'$ is a triple $P=(p,q,g)$, where $(p,q)\colon(G,A)\to(G',A')$ is a Γ-pair morphism and $g\in C^2_{\Gamma}(G,A')$ is a 2-cochain of G in A', such that $q_*(h)=p^*(h')+\partial g$. The composite of P with the morphism $P'=(p',q',g')\colon T'\to T''$ is defined as the triple $P'P=\big(p'p,q'q,p^*(g')+q'_*(g)\big)$. It is easy to verify that $P'P\colon T\to T''$ is a morphism, that the composition of morphisms is associative and that identity morphisms exist (namely $\mathrm{id}_{(G,A,h)}=(\mathrm{id}_G,\mathrm{id}_A,0)$). Thus we have indeed constructed a category.

The above notion of morphism is, however, a little too rigid for most purposes and we conclude by showing how to relax it. We say that two morphisms, say (p,q,g), $(p',q',g')\colon(G,A,h)\to(G',A',h')$, are *homotopic* if $p=p'$, $q=q'$ and there exists a 1-cochain of G in A', $f\in C^1_{\Gamma}(G,A')$, such that $g'=g+\partial f$. Homotopy is an equivalence relation among morphisms, and it is also easy to see that it is compatible with the composition in $\mathcal{Z}^3_{\Gamma}$. We can therefore define the *homotopy category of* 3*-cocycles of* Γ*-groups*, to be the corresponding quotient category.

Closely related to the category $\mathcal{Z}^3_{\Gamma}$ is the *category of* 3*-cohomology classes of* Γ*-groups*, $\mathcal{H}^3_{\Gamma}$, which plays a fundamental role to state our classification theorem for Γ-graded categorical groups. It is defined below.

An *object* of $\mathcal{H}^3_{\Gamma}$ is a triple (G,A,c), where (G,A) is a Γ-pair and $c\in H^3_{\Gamma}(G,A)$. An arrow $(p,q)\colon(G,A,c)\to(G',A',c')$ in $\mathcal{H}^3_{\Gamma}$ is a morphism of Γ-pairs $(p,q)\colon(G,A)\to(G',A')$ such that $p^*(c')=q_*(c)\in H^3_{\Gamma}(G,A')$. The composition in $\mathcal{H}^3_{\Gamma}$ is the composition of Γ-pair morphisms: $(p',q')(p,q)=(p'p,q'q)$.

We have the *cohomology class functor*

$$\mathrm{cl}\colon \mathcal{Z}^3_\Gamma \to \mathcal{H}^3_\Gamma, \tag{29}$$

$$(G,A,h) \mapsto \big(G,A,\mathrm{cl}(h)\big)\,, (p,q,g) \mapsto (p,q),$$

where $\mathrm{cl}(h) \in H^3_\Gamma(G,A)$ denotes the cohomology class of $h \in Z^3_\Gamma(G,A)$. This functor clearly carries two homotopic morphisms of $\mathcal{Z}^3_\Gamma$ to the same morphism in $\mathcal{H}^3_\Gamma$; it is surjective on objects, and it is full: if $(p,q)\colon\ \mathrm{cl}(G,A,h) \to \mathrm{cl}(G',A',h')$ is any morphism in $\mathcal{H}^3_\Gamma$, then $p^*(h')$ and $q_*(h)$ both represent the same class in $H^3_\Gamma(G,A')$; so there is a 2-cochain $g \in C^2_\Gamma(G,A')$ such that $q_*(h) = p^*(h') + \partial g$, that is, $(p,q,g)\colon (G,A,h) \to (G',A',h')$ is a morphism in $\mathcal{Z}^3_\Gamma$ with $\mathrm{cl}(p,q,g) = (p,q)$. Observe that if the maps p and q are invertible, then, for any $f \in C^1_\Gamma(G',A)$, the morphism of $\mathcal{Z}^3_\Gamma$

$$\big(p^{-1}, q^{-1}, -(p^{-1})^*(q^{-1})_*(g) + \partial f\big) : (G',A',h') \to (G,A,h)$$

is a homotopy inverse of (p,q,g). The next proposition now becomes quite obvious.

Proposition 2.1

(i) *For any object $(G,A,c) \in \mathcal{H}^3_\Gamma$, there is an object T of $\mathcal{Z}^3_\Gamma$ with an isomorphism* $\mathrm{cl}(T) \cong (G,A,c)$.

(ii) *For any isomorphism $(p,q)\colon\ \mathrm{cl}(T) \cong \mathrm{cl}(T')$, there is a homotopy equivalence $P\colon T \xrightarrow{\sim} T'$ such that* $\mathrm{cl}(P) = (p,q)$.

(iii) $\mathrm{cl}(P)$ *is an isomorphism if and only if P is a homotopy equivalence.*

We also have *underlying functors* into the category of Γ-pairs

$$\mathcal{Z}^3_\Gamma \xrightarrow{U} {}_\Gamma\,\mathrm{Pairs} \xleftarrow{U} \mathcal{H}^3_\Gamma \tag{30}$$

$$\begin{array}{ccccc} (G,A,h) & \longmapsto & (G,A) & \longleftarrow\!\shortmid & (G,A,c) \\ (p,q,h) & \longmapsto & (p,q) & \longleftarrow\!\shortmid & (p,q), \end{array} \tag{31}$$

which are clearly surjective on objects, but neither is full and $U\colon \mathcal{Z}^3_\Gamma \to {}_\Gamma\,\mathrm{Pairs}$ is not faithful. Suppose $T = (G,A,h)$, $T' = (G',A',h')$ are objects in $\mathcal{Z}^3_\Gamma$. If $(p,q)\colon (G,A) \to (G',A')$ is any morphism of Γ-pairs, then we will refer to the cohomology class

$$\mathrm{Obs}(p,q) = \mathrm{cl}\big(p^*(h') - q_*(h)\big) \in H^3_\Gamma(G,A') \tag{32}$$

as the *obstruction* of (p,q).

Vanishing $\mathrm{Obs}(p,q)$ means that $(p,q)\colon\ \mathrm{cl}(T) \to \mathrm{cl}(T)$ is a morphism in $\mathcal{H}^3_\Gamma$ and therefore, since cl is full, that there exists a morphism in $\mathcal{Z}^3_\Gamma$, say $P = (p,q,g)\colon T \to T'$, which *realizes* (p,q) by the forgetful functor (30). In such a case, observe that any other realization of (p,q), $P'\colon T \to T'$, is necessarily written in the form $P' = (p,q,g+g')$ with $g' \in Z^2_\Gamma(G,A')$ and, moreover, that P and P' are homotopic if and only if $g' = \partial f$ for some $f \in C^1_\Gamma(G,A')$. Hence, we have the proposition below for later use.

Proposition 2.2 *Let $T = (G, A, h)$, $T' = (G', A', h')$ be objects of $\mathcal{Z}^3_\Gamma$, and let $(p, q)\colon (G, A) \to (G', A')$ be a morphism of Γ-pairs. Then,*

(i) *The obstruction $\mathrm{Obs}(p, q) \in H^3_\Gamma(G, A')$, (32), depends only on the homotopy type of T and T' in $\mathcal{Z}^3_\Gamma$.*
(ii) *There exists a morphism $T \to T'$ in $\mathcal{Z}^3_\Gamma$ that realizes (p, q) by the forgetful functor U, (30), if and only if its obstruction vanishes.*
(iii) *If $\mathrm{Obs}(p, q) = 0$, then the set of homotopy classes of arrows $T \to T'$ in $\mathcal{Z}^3_\Gamma$ that realize (p, q) are in bijection with the elements of the group $H^2_\Gamma(G, A')$.*

3 The Classification Theorems

In this section we describe and study the device, namely a functor $\int_\Gamma \colon \mathcal{Z}^3_\Gamma \colon \to {}_\Gamma\mathcal{CG}$, which allows us to prove precise homotopy classification results for Γ-graded categorical groups. We have summarized these results in two theorems stated below.

Theorem 3.1 *There is a classifying functor*

$$\begin{aligned} \mathrm{cl}\colon {}_\Gamma\mathcal{CG} &\longrightarrow \mathcal{H}^3_\Gamma, \\ \mathbb{G} &\mapsto \big(\pi_0\mathbb{G}, \pi_1\mathbb{G}, k(\mathbb{G})\big), \\ F &\mapsto (\pi_0 F, \pi_1 F), \end{aligned} \tag{33}$$

which has the following properties:

(i) *For any object $(G, A, c) \in \mathcal{H}^3_\Gamma$, there exists a Γ-graded categorical group $\mathbb{G}$ with an isomorphism $\mathrm{cl}(\mathbb{G}) \cong (G, A, c)$.*
(ii) *For any isomorphism $(p, q)\colon\ \mathrm{cl}(\mathbb{G}) \cong \mathrm{cl}(\mathbb{H})$, there is a graded monoidal equivalence $F\colon \mathbb{G} \xrightarrow{\sim} \mathbb{H}$ such that $\mathrm{cl}(F) = (p, q)$.*
(iii) $\mathrm{cl}(F)$ *is an isomorphism if and only if F is a graded monoidal equivalence.*

Theorem 3.2 *Let $\mathbb{G}, \mathbb{H}$ be Γ-graded categorical groups.*

(i) *There is a canonical partition of the set of homotopy classes of graded monoidal functors from $\mathbb{G}$ to $\mathbb{H}$,*

$$\mathrm{Hom}_{{}_\Gamma\mathcal{CG}}[\mathbb{G}, \mathbb{H}] = \bigsqcup_{(p,q)} \mathrm{Hom}_{(p,q)}[\mathbb{G}, \mathbb{H}], \tag{34}$$

where for each morphism of Γ-pairs $(p, q)\colon (\pi_0\mathbb{G}, \pi_1\mathbb{G}) \to (\pi_0\mathbb{H}, \pi_1\mathbb{H})$, $\mathrm{Hom}_{(p,q)}[\mathbb{G}, \mathbb{H}]$ is the set of homotopy classes of those $F\colon \mathbb{G} \to \mathbb{H}$ which realize (p, q), in the sense that $\pi_0 F = p$ and $\pi_1 F = q$.

(ii) *Each morphism of Γ-pairs $(p, q)\colon (\pi_0\mathbb{G}, \pi_1\mathbb{G}) \to (\pi_0\mathbb{H}, \pi_1\mathbb{H})$ determines a three-dimensional cohomology class*

$$\mathrm{Obs}(p, q) = p^*\big(k(\mathbb{H})\big) - q_*\big(k(\mathbb{G})\big) \in H^3_\Gamma(\pi_0\mathbb{G}, \pi_1\mathbb{H}), \tag{35}$$

which depends only on the graded equivalence classes of $\mathbb{G}$ and $\mathbb{H}$. This invariant is called the obstruction of (p, q).

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(iii) *A morphism of* Γ*-pairs* $(p,q)\colon(\pi_0\mathbb{G},\pi_1\mathbb{G})\to(\pi_0\mathbb{H},\pi_1\mathbb{H})$ *is realizable, that is,* $\mathrm{Hom}_{(p,q)}[\mathbb{G},\mathbb{H}]\neq\emptyset$*, if and only if its obstruction vanishes.*
(iv) *If the obstruction of a morphism of* Γ*-pairs* $(p,q)\colon(\pi_0\mathbb{G},\pi_1\mathbb{G})\to(\pi_0\mathbb{H},\pi_1\mathbb{H})$ *vanishes, then there is a (non-natural) bijection*

$$\mathrm{Hom}_{(p,q)}[\mathbb{G},\mathbb{H}]\cong H^2_\Gamma(\pi_0\mathbb{G},\pi_1\mathbb{H}). \tag{36}$$

After Propositions 2.1 and 2.2, both Theorems 3.1 and 3.2 are immediate consequence of the one below.

Theorem 3.3 *There is a faithful functor*

$$\textstyle\int_\Gamma\colon\mathcal{Z}^3_\Gamma\longrightarrow{}_\Gamma\mathcal{CG},$$

which makes the diagram below commutative:

$$\begin{tikzcd}
\mathcal{Z}^3_\Gamma \arrow[rr, "\int_\Gamma"] \arrow[dr, "U"'] & & {}_\Gamma\mathcal{CG} \arrow[dl, "(\pi_0{,}\pi_1)"] \\
& {}_\Gamma\mathrm{Pairs} &
\end{tikzcd}, \tag{37}$$

and induces an equivalence of categories over ${}_\Gamma$ Pairs *between the corresponding homotopy categories.*

Proof This is given in five successive stages.

1. The definition of the functor $\int_\Gamma:\mathcal{Z}^3_\Gamma\to{}_\Gamma\,\mathcal{CG}$.

Every 3-cocycle of a Γ-group G in a Γ-equivariant G-module A, $h\in Z^3_\Gamma(G,A)$, gives rise to a Γ-graded categorical group

$$\int_\Gamma(G,A,h)=\Big(\int_\Gamma(G,A,h),\mathrm{gr},\otimes,I,\mathbf{a},\mathbf{l},\mathbf{r}\Big),$$

which is defined as follows: the objects are the elements $x\in G$ and their arrows are pairs $(a,\sigma)\colon x\to y$ consisting of an element $a\in A$ and an element $\sigma\in\Gamma$ with ${}^\sigma x=y$; the composition of two morphisms $x\xrightarrow{(a,\sigma)}y\xrightarrow{(b,\tau)}z$ is defined by

$$(b,\tau)(a,\sigma)=\big(b+{}^\tau a+h(x,\tau,\sigma),\tau\sigma\big). \tag{38}$$

This composition is associative and unitary owing to the 3-cocycle condition (23) and the normalization condition of h. Since every arrow is invertible (observe that $(a,\sigma)^{-1}=\big(-{}^{\sigma^{-1}}a-h(x,\sigma^{-1},\sigma),\sigma^{-1}\big)$), $\int_\Gamma(G,A,h)$ is a groupoid. The stable Γ-grading is given by $\mathrm{gr}(a,\sigma)=\sigma$.

The graded tensor product $\int_\Gamma(G,A,h)\times_\Gamma\int_\Gamma(G,A,h)\xrightarrow{\otimes}\int_\Gamma(G,A,h)$ is defined by

$$(x\xrightarrow{(a,\sigma)}y)\otimes(x'\xrightarrow{(b,\sigma)}y')=(xx'\xrightarrow{(a+{}^{y}b+h(x,x',\sigma),\sigma)}yy'),\tag{39}$$

which is a functor thanks to the 3-cocycle condition (22) and the normalization of h.

The associativity isomorphisms are

$$\mathbf{a}_{x,y,z}=\big(h(x,y,z),1\big):(xy)z\to x(yz),\tag{40}$$

which satisfy the coherence condition (3) because of the 3-cocycle condition (20) of h, and define a graded natural equivalence $\mathbf{a}\colon(-\otimes-)\otimes-\xrightarrow{\sim}-\otimes(-\otimes-)$ thanks to the 3-cocycle condition (21).

The unit graded functor $I\colon\Gamma\to\int_\Gamma(G,A,h)$ is defined by

$$I(*\xrightarrow{\sigma}*)=(1\xrightarrow{(0,\sigma)}1),\tag{41}$$

and the unit constraints are identities: $\mathbf{l}_x=(0,1)=\mathbf{r}_x\colon x\to x$.

Since for any object $x\in G$, we have $x\otimes x^{-1}=xx^{-1}=1=I$, $\int_\Gamma(G,A,h)$ is actually a Γ-graded categorical group.

If $(p,q,g)\colon(G,A,h)\longrightarrow(G',A',h')$ is a morphism in $\mathcal{Z}^3_\Gamma(G,A)$, then the associated graded monoidal functor

$$\int_\Gamma(p,q,g)\colon\int_\Gamma(G,A,h)\to\int_\Gamma(G',A',h'),$$

is given by

$$(x\xrightarrow{(a,\sigma)}y)\mapsto\big(p(x)\xrightarrow{(q(a)+g(x,\sigma),\sigma)}p(y)\big),\tag{42}$$

together with the isomorphisms

$$\Phi_{x,y}=(g(x,y),1)\colon p(x)p(y)\to p(xy),\tag{43}$$

and

$$\Phi_*=(0,1)=\mathrm{id}_I\colon p(1)\to 1.\tag{44}$$

So defined, and taking in account that $(p,q)\colon(G,A)\to(G',A')$ is a morphism of Γ-pairs such that $q_*(h)=p^*(h')+\partial g$, it is ot hard to see that $\int_\Gamma(p,q,g)$ is a functor because of the coboundary condition (28) and because g is normalized, that the isomorphisms (43) define a graded natural equivalence owing to coboundary condition (27) and that the coherence condition (7) follows from (26) and the normalization of g.

For $(G,A,h)\overset{(p,q,g)}{\longrightarrow}(G',A',h')\overset{(p',q',g')}{\longrightarrow}(G'',A'',h'')$ two composable morphisms in $\mathcal{Z}^3_\Gamma$, it is straightforward to verify that $\int_\Gamma(p',q',g')\int_\Gamma(p,q,g)=$

$\int_\Gamma \big(p'p, q'q, p^*(g') + q'_*(g) \big)$, and that $\int_\Gamma (\mathrm{id}_G, \mathrm{id}_A, 0) = \mathrm{id}_{\int_\Gamma (G,A,h)}$. Hence, $\int_\Gamma \colon \mathcal{Z}^3_\Gamma \to {}_\Gamma\mathcal{CG}$ is indeed a functor, which is plainly recognized to be faithful.

2. The commutativity of (37).

Let us note that, for any object $(G, A, h) \in \mathcal{Z}^3_\Gamma$, Ker $\int_\Gamma (G, A, k)$ can be identified (writing $x \xrightarrow{(a,1)} x$ by $x \xrightarrow{a} x$) with the categorical group whose objects are the elements of the group G, whose hom-sets are given by

$$\mathrm{Hom}_1(x, y) = \begin{cases} A & \text{for } x = y \\ \varnothing & \text{for } x \neq y, \end{cases}$$

whose composition is addition in A, whose tensor product is given by

$$(x \xrightarrow{a} x) \otimes (x' \xrightarrow{b} wx') = (xx' \xrightarrow{a+{}^y b} xx''),$$

whose associativity isomorphism is

$$h(x, y, z) \colon (xy)z \longrightarrow x(yz),$$

and whose (strict) unit object is the 1 of G. Then, we see that

$$\Big(\pi_0 \int_\Gamma (G, A, h), \pi_1 \int_\Gamma (G, A, h) \Big) = (G, A) \tag{45}$$

as pairs of groups. Moreover, for any $\sigma \in \Gamma$ and $x \in G$, $(0, \sigma) \colon x \to {}^\sigma x$ is a morphism of grade σ in $\int_\Gamma (G, A, h)$, so the Γ-action (16) coincides with the one originally given. Since, for any $\sigma \in \Gamma$ and $a = (a, 1) \in \mathrm{Aut}_1(1)$, we have

$$I(\sigma) a I(\sigma)^{-1} = (0, \sigma)(a, 1)(0, \sigma^{-1}) \overset{(37)}{=\!=} ({}^\sigma a, 1) = {}^\sigma a,$$

it follows that the Γ-action on A, (15), is the given one. Moreover, the group isomorphisms (12) for our graded categorical group $\int_\Gamma (G, A, h)$ are

$$A \xrightarrow[\sim]{\delta_x} A \xrightarrow[\sim]{\gamma_x} A, \quad x \in G,$$

where $\delta_x(1 \xrightarrow{a} 1) = (x \xrightarrow{0} x) \otimes (1 \xrightarrow{a} 1) = (x \xrightarrow{{}^x a} x)$ and $\gamma_x(1 \xrightarrow{a} 1) = (1 \xrightarrow{a} 1) \otimes (x \xrightarrow{0} x) = (x \xrightarrow{a} x)$, whence the G-action on A defined by (11) is also the given G-module structure. Therefore, equality (45) is of Γ-pairs, from which it is now easy to complete the proof that the composite $\mathcal{Z}^3_\Gamma \xrightarrow{\int_\Gamma} \mathcal{CG} \xrightarrow{(\pi_0, \pi_1)} {}_\Gamma \mathrm{Pairs}$ is the same as the forgetful functor U (30), $(G, A, h) \mapsto (G, A)$.

3. Morphisms in $\mathcal{Z}^3_\Gamma$ are homotopic if and only if their associated morphisms by $\int_\Gamma$ are homotopic.

Let $(p,q,g),(p',q',g')\colon (G,A,h)\longrightarrow (G',A',h')$ be two morphisms in $\mathcal{Z}^3_\Gamma$, and suppose first that they are homotopic, so that $g = g' + \partial f$ for some $f\in C^1_\Gamma(G,A')$. Then, the family of isomorphisms of grade 1 in $\int_\Gamma (G',A',h')$,

$$\theta_x = \big(f(x),1\big) : p(x)\longrightarrow p(x),\quad x\in G,$$

is easily recognized as a graded equivalence $\theta\colon \int_\Gamma(p,q,g)\longrightarrow \int_\Gamma(p',q',g')$, thanks to coboundary condition (27), which, owing to (26) and the normalization of f also satisfies the coherence conditions (9). That is, θ is a homotopy of graded monoidal functors. And conversely, if we suppose $\theta\colon \int_\Gamma(p,q,g)\longrightarrow \int_\Gamma(p',q',g')$ is any homotopy and we write $\theta_x = (f(x),1)\colon p(x)\to p(x)$ for a map $f\colon G\to A'$, then we can see that $f\in C^1_\Gamma(G,A')$ and $g = g'+\partial f$ amount to the conditions of θ being a graded monoidal equivalence. Therefore, (p,q,g) and (p',q',g') are homotopic morphisms in $\mathcal{Z}^3_\Gamma$.

4. The induced functor by $\int_\Gamma$ between the homotopy categories is full.

Suppose that $F=(F,\Phi,\Phi_*)\colon \int_\Gamma(G,A,h)\longrightarrow \int_\Gamma(G',A',h')$ is any graded monoidal functor, where (G,A,h) and (G',A',h') are objects of $\mathcal{Z}^3_\Gamma$. By Lemma 1.1, there is no loss of generality in assuming that F satisfies that $\Phi_* = \mathrm{id}_1$ and note that, then, by coherence condition (7), $\Phi_{x,1}=\mathrm{id}_1=\Phi_{1,x}$ for all $x\in G$.

Let $(p,q) = ((\pi_0 F,\pi_1 F)\colon (G,A)\longrightarrow (G',A')$ be the induced morphism of Γ-pairs (see Proposition 1.3). Then,

$$p(x)=F(x),\quad F(1\xrightarrow{(a,1)}1) = 1\xrightarrow{(q(a),1)}1,$$

for all $x\in G$ and $a\in A$. Further, since every morphism of grade 1, say $x\xrightarrow{(a,1)}x$, can be expressed in the form

$$(x\xrightarrow{(a,1)}x) = (1\xrightarrow{(a,1)}1)\otimes(x\xrightarrow{(0,1)}x),$$

where $(0,1)=\mathrm{id}_x$, we deduce by naturalness that

$$\begin{aligned}F(x\xrightarrow{(a,1)}x) &= F(1\xrightarrow{(a,1)}1)\otimes F(x\xrightarrow{(0,1)}x) = (1\xrightarrow{(q(a),1)}1)\otimes\big(p(x)\xrightarrow{(0,1)}p(x)\big)\\ &= p(x)\xrightarrow{(q(a),1)}p(x).\end{aligned}$$

Let us write

$$F(x\xrightarrow{(0,\sigma)}y) = p(x)\xrightarrow{(g(x,\sigma),\sigma)}p(y),$$

with $g(x,\sigma)\in A'$, for each $\sigma\in\Gamma$ and $x\in G$, where $y={}^\sigma x$, and

$$\Phi_{x,y} = p(x)p(y)\xrightarrow{(g(x,y),1)}p(xy),$$

with $g(x,y)\in A'$, for each $x,y\in G$.

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Thus we get a 2-cochain $g \in C^2_\Gamma(G, A')$, which, together with p and q, determines completely F. In fact, for any morphism in $\int_\Gamma(G, A, h)$, say $(a, \sigma)\colon x \to y$, where $y = {}^\sigma x$, we have

$$\begin{aligned} F(x \xrightarrow{(a,\sigma)} y) &= F(x \xrightarrow{(0,\sigma)} y)F(y \xrightarrow{(a,1)} y) \\ &= \big(p(x) \xrightarrow{(g(x,\sigma),\sigma)} p(y)\big)\big(p(y) \xrightarrow{(q(a),1)} p(y)\big) \\ &= p(x) \xrightarrow{(q(a)+g(x,\sigma),\sigma)} p(y). \end{aligned}$$

It is straightforward to see that the equality $q_*(h) = p^*(h') + \partial g$ amounts to the properties of F being a graded monoidal functor. More precisely, $qh(x, y, z) = h'\big(p(x), p(y), p(z)\big) + \partial g(x, y, z)$ follows from the coherence condition (7); $qh(x, y, \sigma) = h'\big(p(x), p(y), \sigma\big) + \partial g(x, y, \sigma)$ owing to the naturalness of the isomorphisms $\Phi_{x,y}$ and $qh(x, \sigma, \tau) = h'\big(p(x), \sigma, \tau\big) + \partial g(x, \sigma, \tau)$ is a direct consequence of F being a functor.

Thus, $(p, q, h)\colon (G, A, h) \longrightarrow (G', A', h')$ is a morphism in $\mathcal{Z}^3_\Gamma$ and, by construction (see (42), (43) and (44)), it is clear that $\int_\Gamma(p, q, g) = F$.

5. For any graded categorical group $\mathbb{G}$, there exists a 3-cocycle

$$h^{\mathbb{G}} \in Z^3_\Gamma(\pi_0\mathbb{G}, \pi_1\mathbb{G}), \tag{46}$$

with a graded monoidal equivalence

$$F\colon \int_\Gamma(\pi_0\mathbb{G}, \pi_1\mathbb{G}, h^{\mathbb{G}}) \xrightarrow{\sim} \mathbb{G}$$

such that

$$\pi_i F = \mathrm{id}_{\pi_i\mathbb{G}}, \quad i = 0, 1.$$

A graded category $(\mathcal{C}, \mathrm{gr})$ is *skeletal* when $\mathrm{Ker}\,\mathcal{C}$ is a skeletal category, that is, when any two objects isomorphic by an isomorphism of grade 1 are equal. The graded categorical group $\mathbb{G}$ is equivalent to a skeletal one, say $\hat{\mathbb{G}}$, which can be constructed as follows: for each each $s \in \pi_0\mathbb{G}$, let us choose an object

$$X_s \in s, \quad \text{with} \quad X_1 = I,$$

and for any other $X \in s$, we fix an arrow of grade 1

$$\Gamma_X\colon X \to X_s, \quad \text{with} \quad \Gamma_{X_s} = \mathrm{id}_{X_s},\ \Gamma_{I\otimes X_s} = \mathbf{l}_{X_s} \text{ and } \Gamma_{X_s\otimes I} = \mathbf{r}_{X_s}.$$

Let $\hat{\mathbb{G}}$ be the full subcategory of $\mathbb{G}$ whose objects are all X_s, $s \in \pi_0\mathbb{G}$. Then, $\hat{\mathbb{G}}$ is stably Γ-graded with $\widehat{\mathrm{gr}} = \mathrm{gr}|_{\hat{\mathbb{G}}}$; the inclusion functor $\mathrm{in}\colon \widehat{\mathbb{G}} \hookrightarrow \mathbb{G}$ is a graded equivalence and clearly $\hat{\mathbb{G}}$ is a skeletal graded category. Now, the graded categorical group structure of $\mathbb{G}$ can be transported to a graded categorical group structure on $\hat{\mathbb{G}}$ in a unique way such that the inclusion functor in, together with the isomorphisms $\Phi_{X_s,Y_s} = \Gamma_{X_s\otimes X_t}\colon X_s\otimes X_t \to X_s\hat{\otimes} X_t = X_{st}$, $\Phi_* = \mathrm{id}_I$, turns out to be a graded monoidal equivalence (see (7)). Note that in the resulting skeletal graded categorical group $\hat{\mathbb{G}} = (\hat{\mathbb{G}}, \hat{\mathrm{gr}}, \hat{\otimes}, \hat{I}, \hat{\mathbf{a}}, \hat{\mathbf{l}}, \hat{\mathbf{r}})$, the unit $\hat{I} = I$ is strict in the sense that $\hat{\mathbf{l}} = \mathrm{id} = \hat{\mathbf{r}}$.

Hence, $\hat{\mathbb{G}}$ has the following properties:

P1. $\hat{I}(\sigma)\hat{\otimes} f = f = f\hat{\otimes}\hat{I}(\sigma)$ for any morphism f in $\hat{\mathbb{G}}$ of grade σ, (due to the naturalness of the unit constraints).

P2. $\hat{\mathbf{a}}_{X_r,X_s,X_t} = \mathrm{id}_{X_{rst}}$, whenever one of r, s or t is 1, (because of the coherence condition (4) and [12, Proposition 1.1]).

P3. For any $\sigma\in\Gamma$, there exists a morphism of grade σ in $\hat{\mathbb{G}}$, $X_s\to X_t$, if and only if ${}^{\sigma}s = t$, (according to (16)).

P4. For any $s\in\pi_0\mathbb{G}$, the group isomorphisms (12),

$$\mathrm{Aut}_1(X_s) \xleftarrow{\hat{\delta}} \pi_1\hat{\mathbb{G}} = \pi_1\mathbb{G} \xrightarrow{\hat{\gamma}} \mathrm{Aut}_1(X_s),$$

are given by $\hat{\delta}(a) = X_s\hat{\otimes} a$, $\hat{\gamma}(a) = a\hat{\otimes} X_s$.

P5. For any $s\in\pi_0\hat{\mathbb{G}} = \pi_0\mathbb{G}$ and $a\in\pi_1\hat{\mathbb{G}} = \pi_1\mathbb{G}$,

$$X_s\hat{\otimes} a = {}^{s}a\hat{\otimes} X_s,$$

(according to (11)).

P6. If $\Upsilon\colon X_r\to X_s$ is any morphism in $\hat{\mathbb{G}}$ of grade σ, then, for any $a\in\pi_1\hat{\mathbb{G}} = \pi_1\mathbb{G}$,

$$\Upsilon(X_r\hat{\otimes} a) = ({}^{\sigma}a\hat{\otimes} X_s)\Upsilon,$$

(by (14)).

We are now ready to build the 3-cocycle $h^{\mathbb{G}}$ (46). To do so, we begin by choosing, for each $\sigma\in\Gamma$ and each $s\in\pi_0\hat{\mathbb{G}} = \pi_0\mathbb{G}$, an arrow in $\hat{\mathbb{G}}$ with domain X_s and grade σ, say

$$\Upsilon_{(s,\sigma)}\colon X_s\to X_{{}^{\sigma}s},\quad \text{with } \Upsilon_{(1,\sigma)} = \hat{I}(\sigma) \text{ and } \Upsilon_{(s,1)} = \mathrm{id}_{X_s}.$$

Then, using the group isomorphisms $\hat{\gamma}$ described in P4, we determine a 3-cochain $h^{\mathbb{G}}\in C^3_{\Gamma}(\pi_0\mathbb{G},\pi_1\mathbb{G})$ by the equations

$$\hat{\mathbf{a}}_{X_r,X_s,X_t} = h^{\mathbb{G}}(r,s,t)\hat{\otimes} X_{rst},\tag{47}$$

$$\Upsilon_{(rs,\sigma)} = \left(h^{\mathbb{G}}(r,s,\sigma)\hat{\otimes} X_{{}^{\sigma}(rs)}\right)\left(\Upsilon_{(r,\sigma)}\hat{\otimes}\Upsilon_{(s,\sigma)}\right),\tag{48}$$

$$\Upsilon_{({}^{\tau}r,\sigma)}\Upsilon_{(r,\tau)} = \left(h^{\mathbb{G}}(r,\sigma,\tau)\hat{\otimes} X_{{}^{(\sigma\tau)}r}\right)\Upsilon_{(r,\sigma\tau)}.\tag{49}$$

Observe that $h^{\mathbb{G}}$ satisfies the normalization conditions as a consequence of the above properties, P1 and P2. To prove that $h^{\mathbb{G}}$ is indeed a 3-cocycle, note that the coherence equations for the associativity constraint (3) say that, for any $r,s,t,u\in\pi_0\mathbb{G}$,

$$\begin{aligned}&\left(h^{\mathbb{G}}(r,s,tu)\hat{\otimes}X_{rstu}\right)\left(h^{\mathbb{G}}(rs,t,u)\hat{\otimes} X_{rstu}\right)\\ &\qquad = \left(X_r\hat{\otimes} h^{\mathbb{G}}(s,t,u)\hat{\otimes} X_{stu}\right)\left(h^{\mathbb{G}}(r,st,u)\hat{\otimes} X_{rstu}\right)\left(h^{\mathbb{G}}(r,s,t)\hat{\otimes} X_{rstu}\right).\end{aligned}$$

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Then, by property P5, we deduce the equality

$$\big(h^{\mathbb{G}}(r,s,tu)h^{\mathbb{G}}(rs,t,u)\big)\mathbin{\hat{\otimes}}X_{rstu}=\big({}^{r}h^{\mathbb{G}}(s,t,u)h^{\mathbb{G}}(r,st,u)h^{\mathbb{G}}(r,s,t)\big)\mathbin{\hat{\otimes}}X_{rstu},$$

from which follows the cocycle condition (20) (written multiplicatively).

The cocycle condition (21) is a consequence of the naturalness of the associativity constraint: for any arrow $\sigma\in\Gamma$ and any $r,s,t\in\pi_0\mathbb{G}$, we have the diagram

$$\begin{tikzcd}
 & X_\sigma \arrow[r, "{{}^{\sigma}h(r,s,t)\hat{\otimes}X_\sigma}"] & X_\sigma & \\
X_\sigma \arrow[ur, "{h(rs,t,\sigma)\hat{\otimes}X_\sigma}"] \arrow[r, phantom, "\mathbf{3}", yshift=3ex] & X \arrow[l, "\Upsilon_{rs}\hat{\otimes}\Upsilon_t"'] \arrow[u, "\Upsilon_{rst}"'] \arrow[r, "{h(r,s,t)\hat{\otimes}X}"] \arrow[d, "(\Upsilon_r\hat{\otimes}\Upsilon_s)\hat{\otimes}\Upsilon_t"] \arrow[r, phantom, "\mathbf{2}", yshift=6ex] \arrow[r, phantom, "\mathbf{1}", yshift=-6ex] & X \arrow[u, "\Upsilon_{rst}"'] \arrow[r, "\Upsilon_r\hat{\otimes}\Upsilon_{st}"] \arrow[d, "\Upsilon_r\hat{\otimes}(\Upsilon_s\hat{\otimes}\Upsilon_t)"'] \arrow[r, phantom, "\mathbf{5}", yshift=3ex] \arrow[r, phantom, "\mathbf{6}", yshift=-3ex] & X_\sigma \arrow[ul, "{h(r,st,\sigma)\hat{\otimes}X_\sigma}"'] \\
 & X_\sigma \arrow[ul, "{h(r,s,\sigma)\hat{\otimes}X_\sigma}"] \arrow[r, "{h({}^{\sigma}r,{}^{\sigma}s,{}^{\sigma}t)\hat{\otimes}X_\sigma}"'] \arrow[ul, phantom, "\mathbf{4}", xshift=3ex] & X_\sigma \arrow[ur, "{{}^{({}^{\sigma}r)}h(s,t,\sigma)\hat{\otimes}X_\sigma}"'] &
\end{tikzcd}$$

where we have written X_{rst} by X, $X_{{}^{\sigma}r{}^{\sigma}s{}^{\sigma}t}$ by X_σ, $\Upsilon_{(x,\sigma)}$ by Υ_x, for $x=r,s,t$, and h by $h^{\mathbb{G}}$. In this diagram, region **1** commutes by the naturalness of **a** and region **2** is commutative thanks to the property P6. Since $h(r,s,\sigma)\mathbin{\hat{\otimes}}X_\sigma=\big(h(r,s,\sigma)\mathbin{\hat{\otimes}}X_{{}^{\sigma}r}\big)\mathbin{\hat{\otimes}}X_{{}^{\sigma}s{}^{\sigma}t}$ and $X_{{}^{\sigma}r}\mathbin{\hat{\otimes}}\big(h(s,t,\sigma)\mathbin{\hat{\otimes}}X_{{}^{\sigma}s{}^{\sigma}t}\big)\overset{\text{P5}}{=\!=}{}^{({}^{\sigma}r)}h(s,t,\sigma)\mathbin{\hat{\otimes}}X_\sigma$, the regions **3**–**6** commute by the definition of $h=h^{\mathbb{G}}$, (48). Then, the outside region is also commutative, since $(\Upsilon_r\mathbin{\hat{\otimes}}\Upsilon_s)\mathbin{\hat{\otimes}}\Upsilon_t$ is invertible, and the cocycle condition (21) follows.

Cocycle condition (22) can be deduced from the fact that $\hat{\otimes}\colon\hat{\mathbb{G}}\times_\Gamma\hat{\mathbb{G}}\to\hat{\mathbb{G}}$ is a functor: for every $\tau,\sigma\in\Gamma$ and $r,s\in\pi_0\hat{\mathbb{G}}=\pi_0\mathbb{G}$, we can compute the composition morphism

$$J=(X_{rs}\overset{\Upsilon_{(rs,\tau)}}{\longrightarrow}X_{{}^{\tau}(rs)}\overset{\Upsilon_{({}^{\tau}(rs),\sigma)}}{\longrightarrow}X_{{}^{\sigma\tau}(rs)})$$

in two ways. Writing $X_{{}^{\sigma\tau}(rs)}$ by X, on the one hand, we have

$$\begin{aligned}
J&\overset{(48)}{=\!=}\big(h^{\mathbb{G}}(rs,\sigma,\tau)\mathbin{\hat{\otimes}}X\big)\,\Upsilon_{(rs,\sigma\tau)}\\
&\overset{(47)}{=\!=}\big(h^{\mathbb{G}}(rs,\sigma,\tau)\mathbin{\hat{\otimes}}X\big)\big(h^{\mathbb{G}}(r,s,\sigma\tau)\mathbin{\hat{\otimes}}X\big)\big(\Upsilon_{(r,\sigma\tau)}\mathbin{\hat{\otimes}}\Upsilon_{(s,\sigma\tau)}\big)\\
&=\big(h^{\mathbb{G}}(rs,\sigma,\tau)h^{\mathbb{G}}(r,s,\sigma\tau)\mathbin{\hat{\otimes}}X\big)\big(\Upsilon_{(r,\sigma\tau)}\mathbin{\hat{\otimes}}\Upsilon_{(s,\sigma\tau)}\big),
\end{aligned}$$

and on the other hand,

$$
\begin{aligned}
J &\overset{(47)}{=\!=} \Upsilon_{({}^{\tau}(rs),\sigma)}\big(h^{\mathbb{G}}(r,s,\tau)\,\hat{\otimes}\,X_{{}^{\tau}(rs)}\big)\,(\Upsilon_{(r,\tau)}\,\hat{\otimes}\,\Upsilon_{(s,\tau)})\\
&\overset{\mathrm{P6}}{=\!=} \big({}^{\sigma}h^{\mathbb{G}}(r,s,\tau)\,\hat{\otimes}\,X\big)\,\Upsilon_{({}^{\tau}(rs),\sigma)}(\Upsilon_{(r,\tau)}\,\hat{\otimes}\,\Upsilon_{(s,\tau)})\\
&\overset{(47)}{=\!=} \big({}^{\sigma}h^{\mathbb{G}}(r,s,\tau)\,\hat{\otimes}\,X\big)\,\big(h^{\mathbb{G}}({}^{\tau}r,{}^{\tau}s,\sigma)\,\hat{\otimes}\,X\big)\,\big(\Upsilon_{({}^{\tau}r,\sigma)}\,\hat{\otimes}\,\Upsilon_{({}^{\tau}s,\sigma)}\big)\,(\Upsilon_{(r,\tau)}\,\hat{\otimes}\,\Upsilon_{(s,\tau)})\\
&= \big({}^{\sigma}h^{\mathbb{G}}(r,s,\tau)h^{\mathbb{G}}({}^{\tau}r,{}^{\tau}s,\sigma)\,\hat{\otimes}\,X\big)\,(\Upsilon_{({}^{\tau}r,\sigma)}\Upsilon_{(r,\tau)}\,\hat{\otimes}\,\Upsilon_{({}^{\tau}s,\sigma)}\Upsilon_{(s,\tau)})\\
&\overset{(48)}{=\!=} \big({}^{\sigma}h^{\mathbb{G}}(r,s,\tau)h^{\mathbb{G}}({}^{\tau}r,{}^{\tau}s,\sigma)\,\hat{\otimes}\,X\big)\\
&\qquad \Big[\big(h^{\mathbb{G}}(r,\sigma,\tau)\,\hat{\otimes}\,X_{{}^{\sigma\tau}r}\big)\,\hat{\otimes}\big(h^{\mathbb{G}}(s,\sigma,\tau)\,\hat{\otimes}\,X_{{}^{\sigma\tau}s}\big)\Big]\,(\Upsilon_{(r,\sigma\tau)}\,\hat{\otimes}\,\Upsilon_{(s,\sigma\tau)})\\
&= \big({}^{\sigma}h^{\mathbb{G}}(r,s,\tau)h^{\mathbb{G}}({}^{\tau}r,{}^{\tau}s,\sigma)\,\hat{\otimes}\,X\big)\,\Big[\big(h^{\mathbb{G}}(r,\sigma,\tau)\,\hat{\otimes}\,X_{{}^{\sigma\tau}r}\big)\,\hat{\otimes}\,X_{{}^{\sigma\tau}s}\Big]\\
&\qquad \Big[X_{{}^{\sigma\tau}r}\,\hat{\otimes}\big(h^{\mathbb{G}}(s,\sigma,\tau)\,\hat{\otimes}\,X_{{}^{\sigma\tau}s}\big)\Big]\,(\Upsilon_{(r,\sigma\tau)}\,\hat{\otimes}\,\Upsilon_{(s,\sigma\tau)})\\
&\overset{\mathrm{P5}}{=\!=} \big({}^{\sigma}h^{\mathbb{G}}(r,s,\tau)h^{\mathbb{G}}({}^{\tau}r,{}^{\tau}s,\sigma)\,\hat{\otimes}\,X\big)\,\big(h^{\mathbb{G}}(r,\sigma,\tau)\,\hat{\otimes}\,X\big)\,\big({}^{({}^{\sigma\tau}r)}h^{\mathbb{G}}(s,\sigma,\tau)\,\hat{\otimes}\,X\big)\\
&\qquad (\Upsilon_{(r,\sigma\tau)}\,\hat{\otimes}\,\Upsilon_{(s,\sigma\tau)})\\
&= \big({}^{\sigma}h^{\mathbb{G}}(r,s,\tau)h^{\mathbb{G}}({}^{\tau}r,{}^{\tau}s,\sigma)h^{\mathbb{G}}(r,\sigma,\tau)^{({}^{\sigma\tau}r)}h^{\mathbb{G}}(s,\sigma,\tau)\,\hat{\otimes}\,X\big)\\
&\qquad (\Upsilon_{(r,\sigma\tau)}\,\hat{\otimes}\,\Upsilon_{(s,\sigma\tau)}).
\end{aligned}
$$

Hence, by comparison and taking into account that $\Upsilon_{(r,\sigma\tau)}\,\hat{\otimes}\,\Upsilon_{(s,\sigma\tau)}$ is invertible, we arrive at (22).

To prove (23) let $\gamma,\tau,\sigma\in\Gamma$ and $r\in\pi_0\hat{\mathbb{G}}=\pi_0\mathbb{G}$. We have the diagram

$$
\begin{tikzcd}
X \arrow[rrr, "h^{\mathbb{G}}(r,\sigma\tau,\gamma)\,\hat{\otimes}\,X"] \arrow[ddd, "h^{\mathbb{G}}(r,\sigma,\tau\gamma)\,\hat{\otimes}\,X"'] & & & X \arrow[ddd, "h^{\mathbb{G}}({}^{\gamma}r,\sigma,\tau)\,\hat{\otimes}\,X"] \\
& X_r \arrow[ul, "\Upsilon_{(r,\sigma\tau\gamma)}"'] \arrow[r, "\Upsilon_{(r,\gamma)}"'] \arrow[d, "\Upsilon_{(r,\tau\gamma)}"] \arrow[ur, phantom, "3"] \arrow[dr, phantom, "1"] \arrow[dl, phantom, "2"] & X_{{}^{\gamma}r} \arrow[ur, "\Upsilon_{({}^{\gamma}r,\sigma\tau)}"'] \arrow[d, "\Upsilon_{({}^{\gamma}r,\tau)}"'] \arrow[dr, phantom, "4"] & \\
& X_{{}^{\tau\gamma}r} \arrow[r, "h^{\mathbb{G}}(r,\tau,\gamma)\,\hat{\otimes}\,X_{{}^{\tau\gamma}r}"] \arrow[dl, "\Upsilon_{({}^{\tau\gamma}r,\sigma)}"] \arrow[dr, phantom, "5"] & X_{{}^{\tau\gamma}r} \arrow[dr, "\Upsilon_{({}^{\tau\gamma}r,\sigma)}"'] & \\
X \arrow[rrr, "{}^{\sigma}h^{\mathbb{G}}(r,\tau,\gamma)\,\hat{\otimes}\,X"'] & & & X
\end{tikzcd}
$$

where we have written X by $X_{{}^{\sigma\tau\gamma}r}$. In this diagram, regions 1–4 are commutative by (49) and region 5 commutes by the property P6. Hence, the outside region is commutative (since $\Upsilon_{(r,\sigma\tau\gamma)}$ is invertible) and (23) follows.

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Since $F = (F, \Phi = \mathrm{id}, \Phi_* = \mathrm{id})\colon \int_\Gamma(\pi_0\mathbb{G}, \pi_1\mathbb{G}, h^{\mathbb{G}}) \to \hat{\mathbb{G}}$, where

$$F(r \xrightarrow{(a,\sigma)} s) = (X_r \xrightarrow{(a\,\hat{\otimes}\, X_s)\Upsilon_{(r,\sigma)}} X_s),$$

is easily recognized as an equivalence (actually an isomorphism) of graded categorical groups with $\pi_i F = \mathrm{id}_{\pi_i\mathbf{G}}$, $i = 0, 1$, the proof of the theorem is complete. ∎

It is a consequence of Theorem 3.3 that the cohomology class of cocycle (46), $h^{\mathbb{G}} \in Z^3_\Gamma(\pi_0\mathbb{G}, \pi_1\mathbb{G})$, depends only on the homotopy equivalence class of $\mathbb{G}$. We denote it by

$$k(\mathbb{G}) = \mathrm{cl}(h^{\mathbb{G}}) \in H^3_\Gamma(\pi_0\mathbb{G}, \pi_1\mathbb{G}).$$

The classifying functor (33) in Theorem 3.1,

$$\mathrm{cl}\colon {}_\Gamma\mathcal{CG} \to \mathcal{H}^3_\Gamma, \quad \mathbb{G} \mapsto \big(\pi_0\mathbb{G}, \pi_1\mathbb{G}, k(\mathbb{G})\big),$$

is the composition of the functors over the category ${}_\Gamma$ Pairs,

$$\begin{tikzcd}
{}_\Gamma\mathcal{CG} \arrow[r, two heads] \arrow[drr, "{(\pi_0,\pi_1)}"'] & \mathrm{Ho}_\Gamma\,\mathcal{CG} \arrow[rr, "\int_\Gamma^{-1}", "\sim"'] \arrow[dr, "{(\pi_0,\pi_1)}"] & & \mathrm{Ho}\,\mathcal{Z}^3_\Gamma \arrow[r, "cl"] \arrow[dl, "U"'] & \mathcal{H}^3_\Gamma, \arrow[dll, "U"] \\
& & {}_\Gamma\,\mathrm{Pairs} & &
\end{tikzcd}$$

where ${}_\Gamma\mathcal{CG} \to \mathrm{Ho}\,{}_\Gamma\mathcal{CG}$ is the *homotopy class functor*, $\int_\Gamma^{-1}$ denotes a quasi-inverse, over ${}_\Gamma$ Pairs, of the equivalence induced by $\int_\Gamma$ between the homotopy categories, and $\mathrm{cl}\colon \mathcal{Z}^3_\Gamma \to \mathcal{H}^3_\Gamma$ is the cohomology class functor (29).

Therefore, Theorems 3.1 and 3.2 follow from Theorem 3.3 and Propositions 2.1 and 2.2.

4 Equivariant Group Extensions

For G and N two given Γ-groups, by an *equivariant group extension of G by N* we mean a short exact sequence of Γ-groups and equivariant homomorphisms

$$\underline{E}\colon N \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G; \tag{50}$$

thus, N can be identified with a normal Γ-subgroup of E and $E/N \cong G$ as Γ-groups. If $\underline{E}$ and $\underline{E}'$ represent two such equivariant group extensions, then we say that they are equivalent if there exists a Γ-group homomorphism $\mathbf{g}\colon E \to E'$ such that $\mathbf{g}i = i'$ and $p'\mathbf{g} = p$. Note then that $\mathbf{g}$ must be an isomorphism. We denote by

$$\mathrm{Ext}_\Gamma(G, N) \tag{51}$$

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the set of equivalence classes of equivariant group extensions of G by N.

In this section, we show a cohomological solution to the problem of classifying all equivariant group extensions of any prescribed pair of Γ-groups, (G, N). The treatment is parallel to the known theory [8] of extensions of groups, which appears now as the particular case in which $\Gamma = \mathbf{1}$, the trivial group. However, the proofs here bear only an incidental similarity with those by Eilenberg and MacLane, since we derive the results on equivariant group extensions from the results obtained on the classification of graded categorical groups.

Our conclusion here can be summarized as follows.

If N is a Γ-group, then the group $\mathrm{Aut}(N)$ of all (group) automorphisms of N is also a Γ-group under the diagonal Γ-action, ${}^{\sigma}f\colon n \mapsto {}^{\sigma}f({}^{\sigma^{-1}}n)$, and the map $C\colon N \to \mathrm{Aut}(N)$ sending each element $n \in N$ into the inner automorphism given by conjugation with n, $C_n\colon n' \mapsto nn'n^{-1}$, is a Γ-group homomorphism. Then, the center of N, $\mathrm{Z}(N) = \mathrm{Ker}(C)$, and the group of automorphism classes $\mathrm{Out}(N) = \mathrm{Aut}(N)/\mathrm{In}(N) = \mathrm{coker}(C)$, are both Γ-groups. Furthermore, $\mathrm{Z}(N)$ is a Γ-equivariant $\mathrm{Out}(H)$-module with action ${}^{[f]}a = f(a)$.

If (50) is an equivariant extension of G by N, then the assignment to each $e \in E$ of the operation of conjugation by e in N induces an equivariant homomorphism $p_{\underline{E}}\colon G \to \mathrm{Out}(N)$. A pair (N, p), where $p\colon G \to \mathrm{Out}(N)$ is a homomorphism of Γ-groups, is what we call an *equivariant G-kernel* (*cf.* [8]).

We state the following theorem.

Theorem 4.1 *Let G, N be two Γ-groups.*

(i) *There is a canonical partition*

$$\mathrm{Ext}_{\Gamma}(G, N) = \bigsqcup_{p} \mathrm{Ext}_{\Gamma}\big(G, (N, p)\big),$$

where, for each equivariant homomorphism $p\colon G \to \mathrm{Out}(H)$, $\mathrm{Ext}_{\Gamma}\big(G, (N, p)\big)$ is the set of classes of equivariant extensions $\underline{E}\colon N \rightarrowtail E \twoheadrightarrow G$, of G by N, which realize p, that is, with $p_{\underline{E}} = p$.

(ii) *Each equivariant G-kernel (N, p) invariably determines a 3-dimensional cohomology class*

$$\mathrm{Obs}(p) \in H^3_{\Gamma}\big(G, Z(N)\big),$$

of G with coefficients in the center of N (with respect to the equivariant G-module structure on $Z(N)$ obtained via p). This invariant is called the obstruction of (N, p).

(iii) *An equivariant G-kernel, (N, p) is realizable, that is, $\mathrm{Ext}_{\Gamma}\big(G, (N, p)\big) \neq \varnothing$, if and only if its obstruction vanishes.*

(iv) *If the obstruction of an equivariant G-kernel, (N, p), vanishes, then there is a bijection*

$$\mathrm{Ext}_{\Gamma}\big(G, (N, p)\big) \cong H^2_{\Gamma}\big(G, Z(N)\big).$$

This theorem is in fact an application of Theorem 3.2 for two particular Γ-graded categorical groups, $\mathbb{G} = \underline{\mathrm{dis}}_{\Gamma} G$ and $\mathbb{H} = \underline{\mathrm{Hol}}_{\Gamma} N$, canonically built from the Γ-groups G and N as follows:

The *discrete* Γ-graded categorical group $\underline{\mathrm{dis}}_\Gamma G$ defined by a Γ-group G, has the elements of G as objects and their arrows $\sigma\colon x \to y$ are the elements $\sigma \in \Gamma$ with ${}^\sigma x = y$. Composition is multiplication in Γ and the grading $\mathrm{gr}\colon \underline{\mathrm{dis}}_\Gamma G \to \Gamma$ is the obvious map $\mathrm{gr}(\sigma) = \sigma$. The graded tensor product is given by

$$(x \xrightarrow{\sigma} y) \otimes (x \xrightarrow{\sigma} y) = (xx \xrightarrow{\sigma} yy), \tag{52}$$

and the graded unit $I\colon \Gamma \to \underline{\mathrm{dis}}_\Gamma G$ by

$$I(* \xrightarrow{\sigma} *) = (1 \xrightarrow{\sigma} 1); \tag{53}$$

the associativity and unit isomorphisms are identities.

The *holomorph* Γ-graded categorical group of a Γ-group N, $\underline{\mathrm{Hol}}_\Gamma N$, has the elements of the Γ-group $\mathrm{Aut}(N)$ as objects. An arrow of grade σ, $\sigma \in \Gamma$, is a pair $(n,\sigma)\colon f \to g$, where $n \in N$, with ${}^\sigma f = C_n g$. The composition of two arrows $f \xrightarrow{(n,\sigma)} g \xrightarrow{(m,\tau)} h$ is given by

$$(m,\tau)\cdot(n,\sigma) = ({}^\tau n m, \tau\sigma), \tag{54}$$

the graded tensor product is

$$(f \xrightarrow{(n,\sigma)} g) \otimes (f' \xrightarrow{(n',\sigma)} g') = (ff' \xrightarrow{(n^g n',\sigma),\sigma)} gg'), \tag{55}$$

and the graded unit $I\colon \Gamma \to \underline{\mathrm{Hol}}_\Gamma N$ is defined by

$$I(* \xrightarrow{\sigma} *) = \mathrm{id}_N \xrightarrow{(0,\sigma)} \mathrm{id}_N\,. \tag{56}$$

The associativity and unit constraints are identities.

Observe that the complete invariants (33) of these Γ-graded categorical groups are of the form

$$\begin{aligned} \mathrm{cl}(\underline{\mathrm{dis}}_\Gamma G) &= (G,0,0),\\ \mathrm{cl}(\underline{\mathrm{Hol}}_\Gamma N) &= \big(\mathrm{Out}(N), \mathrm{Z}(N), k(N)\big)\,, \end{aligned} \tag{57}$$

where we are writing $k(N)$ for $k(\underline{\mathrm{Hol}}_\Gamma N) \in H^3_\Gamma\big(\mathrm{Out}(N),\mathrm{Z}(N)\big)$. Hence, a morphism of Γ-pairs $(p,q)\colon (\pi_0 \underline{\mathrm{dis}}_\Gamma G, \pi_1 \underline{\mathrm{dis}}_\Gamma G) \to (\pi_0 \underline{\mathrm{Hol}}_\Gamma N, \pi_1 \underline{\mathrm{Hol}}_\Gamma N)$ is the same as an equivariant G-kernel, $p\colon G \to \mathrm{Out}(N)$.

To apply Theorem 3.2 in order to get Theorem 4.1, we develop next the device of *factor sets* for equivariant group extensions, such as Schreier [16] did for ordinary group extensions. This allows us to show how the graded monoidal functors $\underline{\mathrm{dis}}_\Gamma G \to \underline{\mathrm{Hol}}_\Gamma N$ are the appropriate systems of dates to construct the manifold of all equivariant group extensions of G by N.

Theorem 4.2 **(Schreier Theory for Equivariant Group Extensions)** *For any Γ-groups G, N, there is a bijection*

$$\mathrm{Hom}_{\Gamma\mathcal{CG}}[\underline{\mathrm{dis}}_\Gamma G, \underline{\mathrm{Hol}}_\Gamma N] \cong \mathrm{Ext}_\Gamma(G,N), \tag{58}$$

between the set of homotopy classes of graded monoidal functors from $\underline{\mathrm{dis}}_\Gamma G$ to $\underline{\mathrm{Hol}}_\Gamma N$ and the set of equivalence classes of equivariant extensions of G by N.

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Proof First, let us recall from Lemma 1.1 that every graded monoidal functor is homotopic to one $F = (F, \Phi, \Phi_*)$ in which $FI = I$ and $\Phi_* = \mathrm{id}_I$. Hence, we can restrict our attention to this kind of graded monoidal functors $F = (F, \Phi)\colon \underline{\mathrm{dis}}_\Gamma G \to \underline{\mathrm{Hol}}_\Gamma N$.

Second, observe that the system of data describing such a graded monoidal functor $F = (F, \Phi)$ consists of a pair of maps (f, φ), where

$$f\colon G \to \mathrm{Aut}(N), \quad \varphi\colon (G \times G) \cup (G \times \Gamma) \to N, \tag{59}$$

such that we write

$$F(x \xrightarrow{\ \sigma\ } {}^{\sigma}x) = f_x \xrightarrow{(\phi(x,\sigma),\sigma)} f_{{}^{\sigma}x}, \quad \Phi_{x,y} = f_x f_y \xrightarrow{(\phi(x,\sigma),1)} f_{xy}, \tag{60}$$

for all $x, y \in G$ and $\sigma \in \Gamma$. When we try to write the conditions of (F, Φ) being a graded monoidal functor in terms of (f, φ), then we find the following conditions for (f, φ):

$$f_1 = \mathrm{id}_N, \varphi(x, 1) = 1 = \varphi(1, y), \tag{61}$$

$$f_x f_y = C_{\varphi(x,y)} f_{xy}, \tag{62}$$

$$^{\sigma} f_x = C_{\varphi(x,\sigma)} f_{{}^{\sigma}x}, \tag{63}$$

$$\varphi(x, y)\varphi(xy, z) = f_x\big(\varphi(y, z)\big)\, \varphi(x, yz), \tag{64}$$

$$^{\sigma}\varphi(x, y)\varphi(xy, \sigma) = \varphi(x, \sigma) f_{({}^{\sigma}x)}\big(\varphi(y, \sigma)\big)\, \varphi({}^{\sigma}x {}^{\sigma}y), \tag{65}$$

$$\varphi(x, \sigma\tau) = {}^{\sigma}\varphi(x, \tau)\varphi({}^{\tau}x, \sigma), \tag{66}$$

for all $x, y, z \in G$, $\sigma, \tau \in \Gamma$. To prove this in full, several verifications are needed, but they are straightforward: conditions (62) and (63) say that $\big(\varphi(x, y), 1\big)$ and $\big(\varphi(x, \sigma), \sigma\big)$ in (60) are, respectively, morphisms in $\underline{\mathrm{Hol}}_\Gamma N$; (64) expresses the coherence condition (7), while (65) means that the isomorphisms $\Phi_{x,y}$ are natural and (66) that F preserves the composition of morphisms. The normalization condition (61) says that F preserves both identities as the unit object.

Let us note that when $\Gamma = \mathbf{1}$, the trivial group, then a pair (f, φ) describing a graded monoidal functor from $\underline{\mathrm{dis}} G$ to $\underline{\mathrm{Hol}} N$ is just a Schreier system of factor sets for a group extension of G by N.

Suppose (f', φ') describes another monoidal functor $F' = (F', \Phi')\colon \underline{\mathrm{dis}}_\Gamma G \to \underline{\mathrm{Hol}}_\Gamma N$. Then, any homotopy $\theta\colon F \to F'$ is given by a map

$$g\colon G \to H, \tag{67}$$

such that one writes

$$\theta_x = f_x \xrightarrow{(g(x),1)} f'_x, \tag{68}$$

for all $x \in G$. In terms of map g, the conditions for θ to be a homotopy are:

$$g(1) = 1 \tag{69}$$

$$f_x = C_{g(x)} f'_x, \tag{70}$$

$$\varphi(x, y)g(xy) = g(x) f'_x\big(g(y)\big)\, \varphi'(x, y), \tag{71}$$

$$\varphi(x, \sigma)g({}^{\sigma}x) = {}^{\sigma}g(x)\varphi'(x, \sigma), \tag{72}$$

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for all $x, y \in G$, $\sigma \in \Gamma$. Condition (70) expresses that θ_x is a morphism in $\underline{\mathrm{Hol}}_\Gamma N$ from f_x to f'_x, (72) is the naturalness of θ and (71) and (69) say that the coherence conditions (9) hold.

We now are ready to prove bijection (58).

Every graded monoidal functor $(f, \varphi)\colon \underline{\mathrm{dis}}_\Gamma G \to \underline{\mathrm{Hol}}_\Gamma N$ gives rise to an equivariant Γ-group extension

$$\underline{\Delta}(f,\varphi)\colon N \overset{i}{\rightarrowtail} N\times_{(f,\varphi)}G \overset{p}{\twoheadrightarrow} G, \tag{73}$$

that we call a *crossed product equivariant extension*, in which: Γ-group $N \times_{(f,\varphi)} G$ has the same elements as $N \times G$, multiplication according to the rule

$$(n, x)(m, y) = \big(nf_x(m)\varphi(x, y), xy\big)\,, \tag{74}$$

and Γ-action given by

$$^{\sigma}(n, x) = (^{\sigma}n\varphi(x, \sigma), {}^{\sigma}x). \tag{75}$$

The maps i and p are defined by

$$i(n) = (n, 1), \quad p(n, x) = x. \tag{76}$$

It is well known (see [13, Lemma 8.1]) that operation (74) in fact defines a group structure on $N\times_{(f,\varphi)}G$ thanks to conditions (61), (62) and (64). Furthermore, a routine calculation shows that (63) and (65) yield $^{\sigma}[(n, x)(m, y)] = {}^{\sigma}(n, x)^{\sigma}(m, y)$, and that (66) yields $^{\tau}\big(^{\sigma}(n, x)\big) = {}^{\tau\sigma}(n, x)$. Hence, $N \times_{(f,\varphi)} G$ is a Γ-group and it is easy to check that then $\underline{\Delta}(f, \varphi)$ is actually an equivariant group extension of G by N.

Suppose $(f', \varphi')\colon \underline{\mathrm{dis}}_\Gamma G \to \underline{\mathrm{Hol}}_\Gamma N$ is also a graded monoidal functor. If there is a Γ-group isomorphism, say $\mathbf{g}\colon N\times_{(f',\varphi')}G \to N\times_{(f,\varphi)}G$, establishing an equivalence between the corresponding crossed product equivariant extensions, then we can write $\mathbf{g}$ in the form $\mathbf{g}(n, x) = \mathbf{g}(n, 1)\mathbf{g}(1, x) = (n, 1)\big(g(x), x\big) = \big(ng(x), x\big)$ for a map $g\colon G \to N$. Since $\mathbf{g}\big((1, x)(n, 1)\big) = \mathbf{g}\big(f'_x(n), x\big) = \big(f'_x(n)g(x), x\big)$, while $\mathbf{g}(1, x)\mathbf{g}(n, 1) = \big(g(x)f_x(n), x\big)$, then $f'_x = C_{g(x)}f_x$; that is, (70) holds. Because $\mathbf{g}\big(^{\sigma}(1, x)\big) = \big(\varphi'(x, \sigma)g(^{\sigma}x), {}^{\sigma}x\big)$, while $^{\sigma}\mathbf{g}(1, x) = \big(^{\sigma}g(x)\varphi(x, \sigma), {}^{\sigma}x\big)$, we see that (72) holds. Therefore, g defines a homotopy between (the graded monoidal functors defined by) (f', φ') and (f, φ).

Conversely, if (f', φ') and (f, φ) are made homotopic by a $g\colon G \to N$, thus satisfying (69)–(72), then they lead to isomorphic crossed product equivariant extensions, just by the map $\mathbf{g}\colon (n, x) \mapsto \big(ng(x), x\big)$, as we see by retracing our steps.

Finally, we prove that any equivariant group extension of G by N, (50), has an associated factor set, that is, it is equivalent to a crossed product extension $\underline{\Delta}(f, \varphi)$ for some graded monoidal functor $(f, \varphi)\colon \underline{\mathrm{dis}}_\Gamma G \to \underline{\mathrm{Hol}}_\Gamma N$.

Let $\underline{E}\colon N \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G$ be an equivariant group extension of G by N. There is no loss of generality in assuming that i is the inclusion map.

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For each $x \in G$, let us choose a representative $u_x \in E$, so that $p(u_x) = x$, with $u_1 = 1$. Since $p(u_x u_y) = xy = p(u_{xy})$ and $p({}^\sigma u_x) = {}^\sigma x = p(u_{{}^\sigma x})$, there are unique elements $\varphi(x, y), \varphi(x, \sigma) \in N$, for each $x, y \in G$, and $\sigma \in \Gamma$, such that

$$u_x u_y = \varphi(x, y) u_{xy}, \tag{77}$$

$$ {}^\sigma u_x = \varphi(x, \sigma) u_{{}^\sigma x}. \tag{78}$$

Moreover, each $x \in G$ induces an automorphism f_x of N:

$$f_x \colon n \mapsto u_x n u_x^{-1}. \tag{79}$$

It is not true that $f \colon x \mapsto f_x$ defines a Γ-group homomorphism $f \colon G \to \operatorname{Aut}(N)$ (recall that Γ acts diagonally on $\operatorname{Aut}(N)$), but by (77) and (78) we have $f_x f_y = C_{\varphi(x,y)} f_{xy}$ and ${}^\sigma f_x = C_{\varphi(x,\sigma)} f_{{}^\sigma x}$; that is, the pair of maps

$$\bigl(f \colon G \to \operatorname{Aut}(N), \varphi \colon (G \times G) \cup (G \times \Gamma) \to N\bigr)$$

satisfies conditions (62) and (63) for being a graded monoidal functor. To observe the remaining conditions (64)–(66), note that every element $e \in E$ has a unique expression of the form $e = n u_x$, with $n \in N$ and $x \in G$. Because (79) can be written as $u_x n = f_x(n) u_x$, it follows that the Γ-group structure of E can be described in terms of the Γ-group structures of N and G and the pair (f, φ) by

$$(n u_x)(m u_y) = n f_x(m) \varphi(x, y) u_{xy}, \tag{80}$$

$$ {}^\sigma(n u_x) = {}^\sigma n \varphi(x, \sigma) u_{{}^\sigma x}. \tag{81}$$

It is well known that (64) follows from the associative law $u_x(u_y u_z) = (u_x u_y) u_z$ in E; and similarly it is not hard to see that (65) follows from the equality ${}^\sigma(u_x u_y) = {}^\sigma u_x {}^\sigma u_y$, while (66) is a consequence of the equality ${}^\tau({}^\sigma u_x) = {}^{\tau\sigma} u_x$.

Hence, (f, φ) defines a graded monoidal functor, and we recognize that it is a factor set for the given equivariant group extensions by the existence of the Γ-group isomorphism

$$N \times_{(f,\varphi)} G \xrightarrow{\cong} E, \quad (n, x) \mapsto n u_x.$$

Thus, we have proved that the map $(f, \varphi) \mapsto \underline{\Delta}(f, \varphi)$ induces the announced bijection (58), whose inverse is induced by the correspondence mapping any of its associated factor sets to an equivariant extension of G by N, and the proof of the theorem is complete.

As an additional comment, let us observe that our Schreier theory, for equivariant group extensions, may be viewed as an special case of Grothendieck's general theory of extensions of groups in a topos [11], when it is applied to the topos of Γ-sets, that is, to Γ-groups. To be more specific, recall that a group extension $N \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G$ in any topos consists in the underlying N-bitorsor $E \twoheadrightarrow G$ in the topos, together with a bitorsor map $E \wedge^{N} E \to E$ (where $E \wedge^{N} E$ is the quotient of the fiber product $E \times_{G} E$

by the relation $(en, e') \sim (e, ne')$), corresponding to the group multiplication in E, and for which the associativity diagram is commutative.

This geometric point of view would make more intelligible the various cocycle conditions for a pair (f, φ), (59). The bitorsor $E = N \times G \overset{\mathrm{pr}}{\twoheadrightarrow} G$ is defined by the map f (by putting $n' \cdot (n, x) = (n'n, x)$ and $(n, x) \cdot n' = \big(nf_x(n'), x\big)$). The fact that the bitorsor is in the topos of Γ-sets is expressed by the elements $\varphi(x, \sigma)$, such as in (75), and equations (63) and (66). Similarly, the group multiplication bitorsor morphism is given by the elements $\varphi(x, y)$ satisfying equation (62), and the fact that this multiplication is equivariant translates to equation (65). Finally, the commutativity of the associativity diagram is given by (64). ∎

The bijection (58) is all one needs to obtain the classification of equivariant group extensions as stated in Theorem 4.1 from Theorem 3.2.

Some readers may be interested in seeing an explicit construction of a 3-cocycle $h^p \in Z^3_\Gamma\big(G, \mathrm{Z}(N)\big)$, representing the cohomology class

$$\mathrm{Obs}(p) = p^*\big(k(N)\big) = [p^* h^{\underline{\mathrm{Hol}}_\Gamma N}] \in H^3_\Gamma\big(G, \mathrm{Z}(N)\big),$$

for a given equivariant G-kernel $p\colon G \to \mathrm{Out}(N)$. As a final comment, let us observe how the general construction of 3-cocycles $h^{\mathcal{G}}$, (46), specializes to $\underline{\mathrm{Hol}}_\Gamma N$, leading to the following construction of $h^p = p^*(h^{\underline{\mathrm{Hol}}_\Gamma N})$: in each automorphism class $p(x)$, $x \in G$, let us choose an automorphism f_x of N; in particular, select $f_1 = \mathrm{id}_N$. Since $p(xy) = p(x)p(y)$ and ${}^\sigma p(x) = p({}^\sigma x)$, for $x, y \in G$, and $\sigma \in \Gamma$, we can select elements $\varphi(x, y), \varphi(x, \sigma) \in N$, such that $f_x f_y = C_{\varphi(x,y)} f_{xy}$ and ${}^\sigma f_x = C_{\varphi(x,\sigma)} f_{{}^\sigma x}$, with $\varphi(x, 1) = 1 = \varphi(1, y) = \varphi(1, \sigma)$. The pair of maps

$$\big(f\colon G \to \mathrm{Aut}(N), \varphi\colon (G \times G) \cup (G \times \Gamma) \to N\big),$$

satisfies conditions (61), (62) and (63), but (64)–(66) need not be satisfied. The measure of such a deficiency is given by the map

$$h^p\colon G^3 \cup (G^2 \times \Gamma) \cup (G \times \Gamma^2) \to ZN$$

determined by the equations:

$$h^p(x, y, z)\varphi(x, y)\varphi(xy, z) = f_x\big(\varphi(y, z)\big)\,\varphi(x, yz),$$
$$h^p(x, y, \sigma)^\sigma\varphi(x, y)\varphi(xy, \sigma) = \varphi(x, \sigma) f_{{}^\sigma x}\big(\varphi(y, \sigma)\big)\,\varphi({}^\sigma x, {}^\sigma y),$$
$$h^p(x, \tau, \sigma)\varphi(x, \tau, \sigma) = {}^\tau\varphi(x, \sigma)\varphi({}^\sigma x, \tau),$$

for $x, y, z \in G$, $\sigma, \tau \in \Gamma$. This h^p is a 3-cocycle such that $\mathrm{cl}(h^p) = \mathrm{Obs}(p)$ (*cf.* [8]).

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