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STANDBY REDUNDANCY ALLOCATIONS IN SERIES AND PARALLEL SYSTEMS

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Abstract

To enhance the performance of a system, a common practice employed by reliability engineers is to use redundant components in the system. In this paper we compare lifetimes of series (parallel) systems arising out of different allocations of one or two standby redundancies. These comparisons are made with respect to the increasing concave (convex) order, the hazard rate order, and the stochastic precedence order. The main results extend some related conclusions in the literature.

Keywords: Hazard rate order; IFR; increasing concave (convex) order; likelihood ratio order; reversed hazard rate order; stochastic precedence order; usual stochastic order

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1. Introduction

The problem of where and how to allocate redundant components in a system, in order to optimize its lifetime or some other performance characteristic, is interesting and important in reliability theory and its applications. It has posed many interesting theoretical problems in probability, which have attracted the attention of many researchers (see [2], [9], [10], and [12], among others).

Generally, there are two methods to allocate redundant components in a system: the active (or parallel) redundancy allocation, and the standby redundancy allocation. Parallel redundancy is used when replacement of components during the operation of the system is not possible. In this case redundant components are connected in parallel with the components of the system and function simultaneously with them (which leads to the consideration of the maximum of random variables). Standby redundancy is used when replacement of components during the operation of the system is possible. In this case a spare starts functioning immediately after the failure of the corresponding component in the system (which leads to the consideration of the convolution of random variables). Evidently, to achieve the desired system reliability, the standby redundancy is more economical than the parallel redundancy. Performances of various allocations can be compared through stochastic comparisons between the corresponding system lifetimes.

Let X and Y be random variables having common support $[0, \infty)$, distribution functions F and G, and Lebesgue density functions f and g, respectively. Let $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ be the survival functions. Also, assume that f and g take positive values on $[0, \infty)$.

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Definition 1.1. (*[11].*) The random variable X is said to be smaller than Y in the

(i) likelihood ratio order (written as $X \leq_{\mathrm{lr}} Y$) if $g(t)/f(t)$ is increasing in $t \in [0, \infty)$;

(ii) usual stochastic order (written as $X \leq_{\mathrm{st}} Y$) if $\bar{F}(t) \leq \bar{G}(t)$ for all $t \in [0, \infty)$;

(iii) hazard rate order (written as $X \leq_{\mathrm{hr}} Y$) if $\bar{G}(t)/\bar{F}(t)$ is increasing in $t \in [0, \infty)$;

(iv) reversed hazard rate order (written as $X \leq_{\mathrm{rh}} Y$) if $G(t)/F(t)$ is increasing in $t \in (0, \infty)$;

(v) increasing convex order (written as $X \leq_{\mathrm{icx}} Y$) if $\int_x^\infty \mathrm{P}(Y > t)\,\mathrm{d}t \geq \int_x^\infty \mathrm{P}(X > t)\,\mathrm{d}t$ for all $x \in [0, \infty)$;

(vi) increasing concave order (written as $X \leq_{\mathrm{icv}} Y$) if $\int_0^x \mathrm{P}(Y > t)\,\mathrm{d}t \geq \int_0^x \mathrm{P}(X > t)\,\mathrm{d}t$ for all $x \in [0, \infty)$;

(vii) stochastic precedence order (written as $X \leq_{\mathrm{sp}} Y$) if $\mathrm{P}(X > Y) \leq \mathrm{P}(X < Y)$.

Definition 1.2. (*[11].*) The random variable X is said to have an increasing failure rate (IFR) if $\bar{F}(\cdot)$ is log-concave on $[0, \infty)$.

For equivalent definitions and properties of various stochastic orders and ageing classes, we refer the reader to [4], [8], and [11].

Consider a series (or parallel) system consisting of components $C_1, C_2, \ldots, C_n$ having random lifetimes $X_1, X_2, \ldots, X_n$, respectively. Suppose that we also have spares R, R_1, and R_2 having random lifetimes X, Y_1, and Y_2, respectively. Assume that nonnegative random variables $X_1, X_2, \ldots, X_n, X, Y_1$, and Y_2 are statistically independent. Boland *et al.* [2], Singh and Misra [12], and Li and Hu [5] considered a model in which a spare R is available for standby redundancy. The available spare can be allocated either to component C_1 or to component C_2. We can decide which of these two allocations is better (with respect to some performance characteristics of the resulting systems) by making stochastic comparisons on

$$V_1 = \wedge\{X_1 + X, X_2, X_3, \ldots, X_n\} \quad \text{and} \quad V_2 = \wedge\{X_1, X_2 + X, X_3, \ldots, X_n\}$$

for the series system, and on

$$V_1' = \vee\{X_1 + X, X_2, X_3, \ldots, X_n\} \quad \text{and} \quad V_2' = \vee\{X_1, X_2 + X, X_3, \ldots, X_n\}$$

for the parallel system; here the symbols '$\wedge$' and '$\vee$' represent *min* and *max*, respectively. Boland *et al.* [2] proved that if $X_1 \leq_{\mathrm{hr}} X_2$ then $V_2 \leq_{\mathrm{st}} V_1$ and if $X_1 \leq_{\mathrm{rh}} X_2$ then $V_1' \leq_{\mathrm{st}} V_2'$. Singh and Misra [12] established that if $X_1 \leq_{\mathrm{st}} X_2$ then $V_2 \leq_{\mathrm{sp}} V_1$ and $V_1' \leq_{\mathrm{sp}} V_2'$. Li and Hu [5] proved that if $X_1 \leq_{\mathrm{icv}} X_2$ and $X_1, X_3, \ldots, X_n$ have convex survival functions, then $V_2 \leq_{\mathrm{sp}} V_1$. They also proved that if X_1 or X_2 has a convex survival function and if $X_1 \leq_{\mathrm{icv}} X_2$, then $\vee\{X_1 + X, X_2\} \leq_{\mathrm{sp}} \vee\{X_1, X_2 + X\}$.

Following the ideas of Valdés and Zequeira [13] and Romera *et al.* [9], we consider two models which are mathematically more general than the model considered in [2], [5], and [12]. In the first model we have two spares R_1 and R_2 (possibly identical), and, due to some constraints, we can use only one of them: either R_1 with C_1, or R_2 with C_2. To decide which of these two allocations is better, we make stochastic comparisons on

$$U_1 = \wedge\{X_1 + Y_1, X_2, X_3, \ldots, X_n\} \quad \text{and} \quad U_2 = \wedge\{X_1, X_2 + Y_2, X_3, \ldots, X_n\}$$

and on

$$U_1' = \vee\{X_1 + Y_1, X_2, X_3, \ldots, X_n\} \quad \text{and} \quad U_2' = \vee\{X_1, X_2 + Y_2, X_3, \ldots, X_n\}.$$

Note that if Y_1 and Y_2 are identically distributed then this model reduces to the model considered in [2], [5], and [12]. In Section 2 we provide conditions under which $U_2 \le_{\mathrm{icv}} U_1$, $U_2 \le_{\mathrm{hr}} U_1$, $U_2 \le_{\mathrm{sp}} U_1$, $U_1' \le_{\mathrm{icx}} U_2'$, and $U_1' \le_{\mathrm{sp}} U_2'$ hold. For allocation of active redundancy, this model was considered in [7], [9], and [13]. A practical situation where the above model may be of interest is described in the following example, which is similar to Example 3.2 discussed in [3].

Example 1.1. Two safety devices S_1 and S_2 run on batteries B_1 and B_2, respectively. The batteries B_1 and B_2 are of different types and have lifetimes X_1 and X_2, respectively. Suppose that two new types of battery, B_1' and B_2', are available in the market and that B_i' is compatible with only S_i, $i = 1, 2$. Let Y_i denote the lifetime of B_i', $i = 1, 2$. Owing to a limited budget, let us assume that we can afford a spare battery for one of the devices only. If our aim is to optimize the time during which both devices will function, stochastic comparisons on $U_1 = \wedge\{X_1 + Y_1, X_2\}$ and $U_2 = \wedge\{X_1, X_2 + Y_2\}$ may be of interest.

In the second model we have two spares R_1 and R_2 which can be used in one of the following two ways: R_1 with C_2 and R_2 with C_1, or R_1 with C_1 and R_2 with C_2. In Section 3 we compare these two methods of allocation of spares through stochastic comparisons on

$$Z_1 = \wedge\{X_1+Y_2, X_2+Y_1, X_3, \dots, X_n\} \quad \text{and} \quad Z_2 = \wedge\{X_1+Y_1, X_2+Y_2, X_3, \dots, X_n\} \tag{1.1}$$

for the series system, and on

$$Z_1' = \vee\{X_1+Y_2, X_2+Y_1, X_3, \dots, X_n\} \quad \text{and} \quad Z_2' = \vee\{X_1+Y_1, X_2+Y_2, X_3, \dots, X_n\} \tag{1.2}$$

for the parallel system. Note that if $\mathrm{P}(Y_2 = 0) = 1$ then this model reduces to the model considered in [2], [5], and [12]. In Section 3 we derive conditions under which $Z_2 \le_{\mathrm{sp}} Z_1$ and $Z_1' \le_{\mathrm{sp}} Z_2'$ hold. For allocation of active redundancy, this model was considered in [9] and [14]. The following example illustrates a situation where the above model may be of interest.

Example 1.2. In Example 1.1, suppose that each of B_1' and B_2' is compatible with both S_1 and S_2. Now suppose that we do not have a budgetary constraint and, thus, spare batteries of both types can be used, i.e. we can use either B_1' with S_1 and B_2' with S_2, or B_1' with S_2 and B_2' with S_1. For optimizing the time during which both devices will function, we may be interested in stochastic comparisons on $Z_1 = \wedge\{X_1 + Y_2, X_2 + Y_1\}$ and $Z_2 = \wedge\{X_1 + Y_1, X_2 + Y_2\}$.

Throughout this paper, increasing and decreasing are used to mean nondecreasing and nonincreasing, respectively. Moreover, the common support of all the random variables considered in the paper is assumed to be $[0, \infty)$, which we denote by $\mathbb{R}_+$. For random variables X and Y, we write $X =_{\mathrm{st}} Y$ to indicate that X and Y have the same distribution. For $x, y \in \mathbb{R} := (-\infty, \infty)$, we define $I(x > y) = 1$ if $x > y$ and $I(x > y) = 0$ if $x \le y$. Let F_i, $\bar{F}_i$, and f_i respectively denote the distribution function, the survival function, and the Lebesgue density function of X_i, $i = 1, 2, \dots, n$. Furthermore, let G_i, $\bar{G}_i$, and g_i respectively denote the distribution function, the survival function, and the Lebesgue density function of Y_i, $i = 1, 2$.

2. Allocation of one standby redundancy

In this section we will make stochastic comparisons between U_1 and U_2 and also between U_1' and U_2'. It may be worth mentioning here that any result on the stochastic comparison between U_1 and U_2 (or U_1' and U_2') yields, as a particular case, a result on the stochastic comparison between V_1 and V_2 (or V_1' and V_2').

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Theorem 2.1. *If $X_1 \leq_{\mathrm{st}} X_2$ and $Y_2 \leq_{\mathrm{icv}} Y_1$, then $U_2 \leq_{\mathrm{icv}} U_1$.*

Proof. Let $\bar{H}(t) = \prod_{i=3}^{n} \bar{F}_i(t),\ t \in \mathbb{R}_+$. Then, it suffices to prove that

$$\begin{aligned}\Delta_1(x) &= \int_0^x \mathrm{P}(U_1 > t)\,\mathrm{d}t - \int_0^x \mathrm{P}(U_2 > t)\,\mathrm{d}t \\ &= \int_0^x \bar{H}(t)\bar{F}_2(t)\,\mathrm{P}(X_1 + Y_1 > t)\,\mathrm{d}t - \int_0^x \bar{H}(t)\bar{F}_1(t)\,\mathrm{P}(X_2 + Y_2 > t)\,\mathrm{d}t\end{aligned}$$

is nonnegative for every $x \in \mathbb{R}_+$. Since the increasing concave order is closed under convolution (see Theorem 4.A.8(d) of [11]), we have $X_1 + Y_2 \leq_{\mathrm{icv}} X_1 + Y_1$, i.e.

$$\int_0^x \mathrm{P}(X_1 + Y_1 > t)\,\mathrm{d}t \geq \int_0^x \mathrm{P}(X_1 + Y_2 > t)\,\mathrm{d}t \quad \text{for all } x \in \mathbb{R}_+.$$

Now by using Theorem 7.3 of [1, Chapter 4] we obtain

$$\int_0^x \bar{H}(t)\bar{F}_2(t)\,\mathrm{P}(X_1 + Y_1 > t)\,\mathrm{d}t \geq \int_0^x \bar{H}(t)\bar{F}_2(t)\,\mathrm{P}(X_1 + Y_2 > t)\,\mathrm{d}t \quad \text{for all } x \in \mathbb{R}_+.$$

Therefore, for every $x \in \mathbb{R}_+$,

$$\Delta_1(x) \geq \int_0^x \bar{H}(t)\bar{F}_2(t)\,\mathrm{P}(X_1 + Y_2 > t)\,\mathrm{d}t - \int_0^x \bar{H}(t)\bar{F}_1(t)\,\mathrm{P}(X_2 + Y_2 > t)\,\mathrm{d}t.$$

Moreover, for all $x \in \mathbb{R}_+$,

$$\begin{aligned}&\int_0^x \bar{H}(t)\bar{F}_2(t)\,\mathrm{P}(X_1 + Y_2 > t)\,\mathrm{d}t - \int_0^x \bar{H}(t)\bar{F}_1(t)\,\mathrm{P}(X_2 + Y_2 > t)\,\mathrm{d}t \\ &\quad = \int_0^x \int_t^\infty \bar{H}(t)\,\mathrm{P}(X_1 + Y_2 > t) f_2(u)\,\mathrm{d}u\,\mathrm{d}t \\ &\qquad - \int_0^x \bar{H}(t)\bar{F}_1(t)\biggl(\int_0^t \bar{G}_2(t-u) f_2(u)\,\mathrm{d}u + \int_t^\infty f_2(u)\,\mathrm{d}u\biggr)\mathrm{d}t \\ &\quad = \int_0^\infty \biggl(\int_0^{u\wedge x} \bar{H}(t)[\mathrm{P}(X_1 + Y_2 > t) - \bar{F}_1(t)]\,\mathrm{d}t\biggr) f_2(u)\,\mathrm{d}u \\ &\qquad - \int_0^x \biggl(\int_u^x \bar{H}(t)\bar{F}_1(t)\bar{G}_2(t-u)\,\mathrm{d}t\biggr) f_2(u)\,\mathrm{d}u \quad \text{(using Fubini's theorem)} \\ &\quad = \mathrm{E}[\psi_1(X_2)],\end{aligned}$$

where

$$\psi_1(y) = \begin{cases} \displaystyle\int_0^y \bar{H}(t)[\mathrm{P}(X_1 + Y_2 > t) - \bar{F}_1(t)]\,\mathrm{d}t \\ \displaystyle\qquad - \int_0^{x-y} \bar{H}(t+y)\bar{F}_1(t+y)\bar{G}_2(t)\,\mathrm{d}t & \text{if } 0 \leq y \leq x, \\ \displaystyle\int_0^x \bar{H}(t)[\mathrm{P}(X_1 + Y_2 > t) - \bar{F}_1(t)]\,\mathrm{d}t & \text{if } y > x.\end{cases}$$

Clearly, for every fixed $x \in \mathbb{R}_+$, $\psi_1(y)$ is an increasing function of y on $\mathbb{R}_+$. Let $\hat{X}_1 =_{\mathrm{st}} X_1$ be such that $\hat{X}_1, X_1, X_2, \ldots, X_n, Y_1$, and Y_2 are independent random variables. Then $\hat{X}_1 \leq_{\mathrm{st}} X_2$,

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and, thus, for every $x \in \mathbb{R}_+$,

$$\begin{aligned}\Delta_1(x) &\geq \mathrm{E}[\psi_1(X_2)]\\ &\geq \mathrm{E}[\psi_1(\hat{X}_1)]\\ &= \int_0^x \bar{H}(t)\,\mathrm{P}(\hat{X}_1 > t)\,\mathrm{P}(X_1 + Y_2 > t)\,\mathrm{d}t - \int_0^x \bar{H}(t)\bar{F}_1(t)\,\mathrm{P}(\hat{X}_1 + Y_2 > t)\,\mathrm{d}t\\ &= 0.\end{aligned}$$

Next we will compare U_1' and U_2' with respect to the increasing convex order.

Theorem 2.2. *If $X_1 \leq_{\mathrm{st}} X_2$ and $Y_1 \leq_{\mathrm{icx}} Y_2$, then $U_1' \leq_{\mathrm{icx}} U_2'$.*

Proof. Let $K(t) = \prod_{i=3}^n F_i(t),\ t \in \mathbb{R}_+$. Then, it suffices to prove that

$$\begin{aligned}\Delta_2(x) &= \int_x^\infty [\mathrm{P}(U_2' > t) - \mathrm{P}(U_1' > t)]\,\mathrm{d}t\\ &= \int_x^\infty K(t)[F_2(t)\,\mathrm{P}(X_1 + Y_1 \leq t) - F_1(t)\,\mathrm{P}(X_2 + Y_2 \leq t)]\,\mathrm{d}t\end{aligned}$$

is nonnegative for every $x \in \mathbb{R}_+$. Preservation of the increasing convex order under convolution (see Theorem 4.A.8(d) of [11]) implies that $X_2 + Y_1 \leq_{\mathrm{icx}} X_2 + Y_2$, i.e.

$$\int_x^\infty \mathrm{P}(X_2 + Y_2 > t)\,\mathrm{d}t \geq \int_x^\infty \mathrm{P}(X_2 + Y_1 > t)\,\mathrm{d}t \quad \text{for all } x \in \mathbb{R}_+.$$

Now Theorem 7.4 of [1, Chapter 4] yields, for every $x \in \mathbb{R}_+$,

$$\int_x^\infty [1 - K(t)F_1(t)\,\mathrm{P}(X_2 + Y_2 \leq t)]\,\mathrm{d}t \geq \int_x^\infty [1 - K(t)F_1(t)\,\mathrm{P}(X_2 + Y_1 \leq t)]\,\mathrm{d}t,$$

i.e.

$$\Delta_2(x) \geq \int_x^\infty K(t)[F_2(t)\,\mathrm{P}(X_1 + Y_1 \leq t) - F_1(t)\,\mathrm{P}(X_2 + Y_1 \leq t)]\,\mathrm{d}t \quad \text{for all } x \in \mathbb{R}_+.$$

Moreover, for every $x \in \mathbb{R}_+$,

$$\begin{aligned}&\int_x^\infty K(t)[F_2(t)\,\mathrm{P}(X_1 + Y_1 \leq t) - F_1(t)\,\mathrm{P}(X_2 + Y_1 \leq t)]\,\mathrm{d}t\\ &\qquad = \int_x^\infty \int_0^t K(t)[\mathrm{P}(X_1 + Y_1 \leq t) - F_1(t)G_1(t-u)]f_2(u)\,\mathrm{d}u\,\mathrm{d}t\\ &\qquad = \int_0^\infty \bigg(\int_{u \vee x}^\infty K(t)[\mathrm{P}(X_1 + Y_1 \leq t) - F_1(t)G_1(t-u)]\,\mathrm{d}t\bigg) f_2(u)\,\mathrm{d}u\\ &\qquad = \mathrm{E}[\psi_2(X_2)],\end{aligned}$$

where

$$\psi_2(y) = \int_{y \vee x}^\infty K(t)[\mathrm{P}(X_1 + Y_1 \leq t) - F_1(t)G_1(t-y)]\,\mathrm{d}t, \qquad y \in \mathbb{R}_+.$$

Clearly, $\psi_2(y)$ is increasing in $y \in [0, x]$. Furthermore, for $y > x$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}\psi_2(y) &= \int_y^\infty K(t)F_1(t)g_1(t-y)\,\mathrm{d}t - K(y)\,\mathrm{P}(X_1+Y_1 \le y)\\ &\ge K(y)[F_1(y) - \mathrm{P}(X_1+Y_1 \le y)]\\ &\ge 0.\end{aligned}$$

Thus, $\psi_2(y)$ is increasing in $y \in \mathbb{R}_+$. Let $\hat{X}_1 =_{\text{st}} X_1$ be such that $\hat{X}_1, X_1, X_2, \dots, X_n, Y_1$, and Y_2 are independent random variables. Then $\hat{X}_1 \le_{\text{st}} X_2$ and, thus, for every $x \in \mathbb{R}_+$,

$$\begin{aligned}\Delta_2(x) &\ge \mathrm{E}[\psi_2(X_2)]\\ &\ge \mathrm{E}[\psi_2(\hat{X}_1)]\\ &= \int_x^\infty K(t)[\mathrm{P}(\hat{X}_1 \le t)\,\mathrm{P}(X_1+Y_1 \le t) - F_1(t)\,\mathrm{P}(\hat{X}_1+Y_1 \le t)]\,\mathrm{d}t\\ &= 0.\end{aligned}$$

The following lemma, which may be of independent interest, will be useful in the stochastic comparison of U_1 and U_2 with respect to the hazard rate ordering.

Lemma 2.1. *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}$, and let $g\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}_+$ and $\psi\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be such that*

(i) *$g(x, y)$ is TP$_2$ on $\mathcal{X} \times \mathcal{Y}$ and, for each $y \in \mathcal{Y}$, $\int_{\mathcal{X}} g(x, y)\,\mathrm{d}x > 0$;*

(ii) *for each $y \in \mathcal{Y}$, $\psi(x, y)$ is increasing in $x \in \mathcal{X}$;*

(iii) *for each $x \in \mathcal{X}$, $\psi(x, y)$ is increasing in $y \in \mathcal{Y}$.*

Then the function

$$K(y) = \frac{\int_{\mathcal{X}} \psi(x, y)g(x, y)\,\mathrm{d}x}{\int_{\mathcal{X}} g(x, y)\,\mathrm{d}x}$$

is increasing in $y \in \mathcal{Y}$. If $\psi(x, y)$ is decreasing in (ii) and (iii), then the function $K(y)$ is decreasing in $y \in \mathcal{Y}$.

Proof. In view of (i), (ii), and (iii) we have, for $y_1 \le y_2$,

$$\begin{aligned}&[K(y_2) - K(y_1)] \int_{\mathcal{X}}\int_{\mathcal{X}} g(x_1, y_1)g(x_2, y_2)\,\mathrm{d}x_2\,\mathrm{d}x_1\\ &\quad= \int_{\mathcal{X}}\int_{\mathcal{X}} \psi(x_2, y_2)g(x_2, y_2)g(x_1, y_1)\,\mathrm{d}x_2\,\mathrm{d}x_1\\ &\qquad - \int_{\mathcal{X}}\int_{\mathcal{X}} \psi(x_2, y_1)g(x_2, y_1)g(x_1, y_2)\,\mathrm{d}x_2\,\mathrm{d}x_1\\ &\quad\ge \int_{\mathcal{X}}\int_{\mathcal{X}} \psi(x_2, y_1)[g(x_2, y_2)g(x_1, y_1) - g(x_2, y_1)g(x_1, y_2)]\,\mathrm{d}x_2\,\mathrm{d}x_1\\ &\quad= \int_{\mathcal{X}}\int_{\mathcal{X}} [\psi(x_2, y_1) - \psi(x_1, y_1)][g(x_2, y_2)g(x_1, y_1) - g(x_2, y_1)g(x_1, y_2)]\\ &\qquad\qquad \times I(x_2 > x_1)\,\mathrm{d}x_2\,\mathrm{d}x_1\\ &\quad\ge 0,\end{aligned}$$

i.e. $K(y_2) \ge K(y_1)$. If $\psi(x, y)$ is decreasing in (ii) and (iii), then all the '$\ge$' above are replaced by '$\le$'.

Theorem 2.3. *Suppose that X_1 or X_2 has IFR, $\bar{F}_2(t)/\bar{F}_1(t)$ is log-convex in $t \in \mathbb{R}_+$, $X_1 \le_{\mathrm{hr}} X_2$, and that $Y_2 \le_{\mathrm{lr}} Y_1$. Then $U_2 \le_{\mathrm{hr}} U_1$.*

Proof. First suppose that X_2 has IFR. For $t \in \mathbb{R}_+$, we have

$$\frac{\mathrm{P}(U_1 > t)}{\mathrm{P}(U_2 > t)} = \frac{\mathrm{P}(X_1 + Y_1 > t)\bar{F}_2(t)}{\mathrm{P}(X_2 + Y_2 > t)\bar{F}_1(t)} = \frac{\int_0^\infty \psi(u,t) g(u,t)\,\mathrm{d}u}{\int_0^\infty g(u,t)\,\mathrm{d}u} = h_1(t), \quad \text{say,}$$

where, for $u, t \in \mathbb{R}_+$,

$$\psi(u,t) = \frac{\bar{F}_1(t-u)\bar{F}_2(t)g_1(u)}{\bar{F}_2(t-u)\bar{F}_1(t)g_2(u)} \quad \text{and} \quad g(u,t) = \frac{\bar{F}_2(t-u)}{\bar{F}_2(t)} g_2(u).$$

The IFR property of X_2 implies that $\bar{F}_2(t-u)/\bar{F}_2(t)$, and, hence, $g(u,t)$, is TP_2. Now we will show that, for each fixed $t \in \mathbb{R}_+$ or $u \in \mathbb{R}_+$, $\psi(u,t)$ is increasing in $u \in \mathbb{R}_+$ or, respectively, $t \in \mathbb{R}_+$. The assumptions that $X_1 \le_{\mathrm{hr}} X_2$ and $Y_2 \le_{\mathrm{lr}} Y_1$ respectively imply that $\bar{F}_1(t-u)/\bar{F}_2(t-u)$ and $g_1(u)/g_2(u)$ are increasing in $u \in \mathbb{R}_+$ for each fixed $t \in \mathbb{R}_+$. Therefore, for each fixed $t \in \mathbb{R}_+$, $\psi(u,t)$ is increasing in $u \in \mathbb{R}_+$. For fixed $u \in \mathbb{R}_+$, we have

$$\psi(u,t) = \begin{cases} \dfrac{\bar{F}_2(t)g_1(u)}{\bar{F}_1(t)g_2(u)} & \text{if } 0 \le t < u, \\[2ex] \dfrac{\bar{F}_2(t)\bar{F}_1(t-u)g_1(u)}{\bar{F}_1(t)\bar{F}_2(t-u)g_2(u)} & \text{if } t \ge u. \end{cases}$$

The assumption that $X_1 \le_{\mathrm{hr}} X_2$ implies that $\psi(u,t)$ is increasing in $t \in [0,u)$. Moreover, for $t \ge u$, we have

$$\ln \psi(u,t) = \ln \frac{\bar{F}_2(t)}{\bar{F}_1(t)} - \ln \frac{\bar{F}_2(t-u)}{\bar{F}_1(t-u)} + \ln \frac{g_1(u)}{g_2(u)}.$$

Since $\bar{F}_2(t)/\bar{F}_1(t)$ is log-convex in $t \in \mathbb{R}_+$, it follows that $\ln \psi(u,t)$ (and so $\psi(u,t)$) is increasing in $t \in [u,\infty)$. Now, on using Lemma 2.1, it follows that $h_1(t)$ is increasing in $t \in \mathbb{R}_+$.

Now consider the case when X_1 has IFR. We can write

$$\frac{1}{h_1(t)} = \int_0^\infty \left[\frac{\bar{F}_2(t-u)\bar{F}_1(t)g_2(u)}{\bar{F}_1(t-u)\bar{F}_2(t)g_1(u)}\right] \frac{\bar{F}_1(t-u)}{\bar{F}_1(t)} g_1(u)\,\mathrm{d}u \Bigg/ \int_0^\infty \frac{\bar{F}_1(t-u)}{\bar{F}_1(t)} g_1(u)\,\mathrm{d}u$$

for $t \in \mathbb{R}_+$. Proceeding as in the previous case, we can show that $1/h_1(t)$ is decreasing in $t \in \mathbb{R}_+$. This implies that $h_1(t)$ is increasing in $t \in \mathbb{R}_+$.

The following theorem deals with the stochastic comparison of U_1 and U_2 with respect to the stochastic precedence order.

Theorem 2.4. *Suppose that $X_1 \le_{\mathrm{icv}} X_2$, and that $X_1, X_3, X_4, \ldots, X_n$ (or $X_2, X_3, X_4, \ldots, X_n$) have convex survival functions on $\mathbb{R}_+$. Then $U_2 \le_{\mathrm{sp}} U_1$.*

Proof. Let $T = \wedge\{X_3, X_4, \ldots, X_n\}$. Then it is easy to verify that

$$[U_1 > U_2] = [X_2 > \wedge\{X_1, T\}] \cap [T > \wedge\{X_1, X_2 + Y_2\}] = [X_2 > X_1] \cap [T > X_1].$$

Let $\bar{H}$ be the survival function of T. By symmetry we have

$$\begin{aligned}\Delta_4 &\equiv \mathrm{P}(U_1 > U_2) - \mathrm{P}(U_2 > U_1) \\ &= \mathrm{P}(X_2 > X_1,\ T > X_1) - \mathrm{P}(X_1 > X_2,\ T > X_2) \\ &= \mathrm{E}[\bar{H}(X_1) I(X_2 > X_1) - \bar{H}(X_2) I(X_1 > X_2)]. \end{aligned} \tag{2.1}$$

Case I: $X_1, X_3, X_4, \dots, X_n$ have convex survival functions. Let $\hat{X}_1$ be an independent copy of X_1. Then, using (2.1), we can write $\Delta_4 = \mathrm{E}[\psi_3(X_2)]$, where

$$\psi_3(t) = \mathrm{E}[\bar{H}(\hat{X}_1) I(t > \hat{X}_1) - \bar{H}(t) I(\hat{X}_1 > t)], \qquad t \in \mathbb{R}_+. \tag{2.2}$$

Clearly, $\hat{X}_1 \le_{\mathrm{icv}} X_2$ and $\mathrm{E}[\psi_3(\hat{X}_1)] = 0$. Using (2.2), we have

$$\psi_3(t) = \int_0^t \bar{H}(s) f_1(s)\,\mathrm{d}s - \bar{H}(t)\bar{F}_1(t), \qquad t \in \mathbb{R}_+,$$

and, therefore,

$$\psi_3'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\psi_3(t) = 2\bar{H}(t) f_1(t) + \bar{F}_1(t) \sum_{i=3}^{n} f_i(t) \prod_{\substack{j=3 \\ j \neq i}}^{n} \bar{F}_j(t) \ge 0 \quad \text{for all } t \in \mathbb{R}_+.$$

Since $X_1, X_3, X_4, \dots, X_n$ have convex survival functions on $\mathbb{R}_+$ (i.e. $f_i(\cdot)$, $i = 1, 3, 4, \dots, n$ are decreasing on $\mathbb{R}_+$), it follows that $\psi_3'(\cdot)$ is a decreasing function on $\mathbb{R}_+$. Thus, $\psi_3(\cdot)$ is an increasing concave function on $\mathbb{R}_+$. Now, on using $\hat{X}_1 \le_{\mathrm{icv}} X_2$ we obtain

$$\Delta_4 = \mathrm{E}[\psi_3(X_2)] \ge \mathrm{E}[\psi_3(\hat{X}_1)] = 0,$$

i.e. $U_2 \le_{\mathrm{sp}} U_1$.

Case II: $X_2, X_3, X_4, \dots, X_n$ have convex survival functions. Let $\hat{X}_2$ be an independent copy of X_2. Then, using (2.1), we can write $\Delta_4 = -\mathrm{E}[\psi_4(X_1)]$, where

$$\psi_4(t) = \mathrm{E}[\bar{H}(\hat{X}_2) I(t > \hat{X}_2) - \bar{H}(t) I(\hat{X}_2 > t)], \qquad t \in \mathbb{R}_+.$$

Then, $X_1 \le_{\mathrm{icv}} \hat{X}_2$ and $\mathrm{E}[\psi_4(\hat{X}_2)] = 0$. Proceeding as in case I, we can show that $\psi_4(\cdot)$ is an increasing concave function on $\mathbb{R}_+$. Therefore,

$$\Delta_4 = -\mathrm{E}[\psi_4(X_1)] \ge -\mathrm{E}[\psi_4(\hat{X}_2)] = 0,$$

i.e. $U_2 \le_{\mathrm{sp}} U_1$.

Theorem 2.5. *Suppose that $X_1 \le_{\mathrm{icx}} X_2$, $Y_1 \le_{\mathrm{icx}} Y_2$, and that $X_1, X_3, X_4, \dots, X_n$ (or $X_2, X_3, X_4, \dots, X_n$) have concave survival functions on $\mathbb{R}_+$. Then $U_1' \le_{\mathrm{sp}} U_2'$.*

Proof. Let $T_1 = \vee\{X_3, X_4, \dots, X_n\}$. Then it can be verified that

$$[U_2' > U_1'] = [W_2 > W_1] \cap [W_2 > T_1],$$

where $W_i = X_i + Y_i$, $i = 1, 2$.

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Let K be the distribution function of T_1, and let $\pi_i(\cdot)$ and $\Pi_i(\cdot)$ respectively denote the density function and the distribution function of W_i, $i = 1, 2$. Using symmetry, we have

$$\begin{aligned}\Delta_5 &\equiv \mathrm{P}(U_2' > U_1') - \mathrm{P}(U_1' > U_2') \\ &= \mathrm{P}(W_2 > W_1,\ W_2 > T_1) - \mathrm{P}(W_1 > W_2,\ W_1 > T_1) \\ &= \mathrm{E}[K(W_2)I(W_2 > W_1) - K(W_1)I(W_1 > W_2)]. \end{aligned} \tag{2.3}$$

Case I: $X_1, X_3, X_4, \dots, X_n$ have concave survival functions. Let $\hat{W}_1$ be an independent copy of W_1. Then (2.3) can be written as $\Delta_5 = \mathrm{E}[\psi_5(W_2)]$, where

$$\begin{aligned}\psi_5(t) &= \mathrm{E}[K(t)I(t > \hat{W}_1) - K(\hat{W}_1)I(\hat{W}_1 > t)] \\ &= K(t)\Pi_1(t) - \int_t^\infty K(s)\pi_1(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}_+. \end{aligned} \tag{2.4}$$

Evidently, $\mathrm{E}[\psi_5(\hat{W}_1)] = 0$ and from (2.4) we obtain

$$\psi_5'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\psi_5(t) = 2K(t)\pi_1(t) + \Pi_1(t)\sum_{i=3}^{n} f_i(t)\prod_{\substack{j=3\\ j\neq i}}^{n} F_j(t) \geq 0 \quad \text{for all } t \in \mathbb{R}_+.$$

Using the concavity of the survival function of X_1, it can be easily verified that $\pi_1(\cdot)$ is increasing on $\mathbb{R}_+$. Also, the concavity of the survival function of X_i implies that $f_i(\cdot)$ is increasing on $\mathbb{R}_+$, $i = 3, 4, \dots, n$. Therefore, it follows that $\psi_5(\cdot)$ is an increasing convex function on $\mathbb{R}_+$. Since the increasing convex order is closed under convolution (see Theorem 4.A.8(d) of [11]), we have $W_1 \leq_{\mathrm{icx}} W_2$ (or $\hat{W}_1 \leq_{\mathrm{icx}} W_2$). Consequently,

$$\Delta_5 = \mathrm{E}[\psi_5(W_2)] \geq \mathrm{E}[\psi_5(\hat{W}_1)] = 0,$$

i.e. $U_1' \leq_{\mathrm{sp}} U_2'$.

Case II: $X_2, X_3, X_4, \dots, X_n$ have concave survival functions. Let $\hat{W}_2$ be an independent copy of W_2 so that we have $\Delta_5 = -\,\mathrm{E}[\psi_6(W_1)]$, where

$$\psi_6(t) = \mathrm{E}[K(t)I(t > \hat{W}_2) - K(\hat{W}_2)I(\hat{W}_2 > t)], \qquad t \in \mathbb{R}_+.$$

Then $\mathrm{E}[\psi_6(\hat{W}_2)] = 0$. Proceeding as in case I, we can show that

$$\Delta_5 = -\,\mathrm{E}[\psi_6(W_1)] \geq -\,\mathrm{E}[\psi_6(\hat{W}_2)] = 0,$$

i.e. $U_1' \leq_{\mathrm{sp}} U_2'$.

Corollary 2.1 below follows from the above theorems on taking $Y_i =_{\mathrm{st}} X$, $i = 1, 2$.

Corollary 2.1. (i) *If $X_1 \leq_{\mathrm{st}} X_2$ then $V_2 \leq_{\mathrm{icv}} V_1$ and $V_1' \leq_{\mathrm{icx}} V_2'$.*

(ii) *Suppose that X_1 or X_2 has IFR, $\bar{F}_2(t)/\bar{F}_1(t)$ is log-convex in $t \in \mathbb{R}_+$, and that $X_1 \leq_{\mathrm{hr}} X_2$. Then $V_2 \leq_{\mathrm{hr}} V_1$.*

(iii) *Suppose that $X_1 \leq_{\mathrm{icv}} X_2$ and that $X_1, X_3, X_4, \dots, X_n$ have convex survival functions on $\mathbb{R}_+$. Then $V_2 \leq_{\mathrm{sp}} V_1$.*

(iv) *Suppose that $X_1 \leq_{\mathrm{icx}} X_2$ and that $X_2, X_3, X_4, \dots, X_n$ have concave survival functions on $\mathbb{R}_+$. Then $V_1' \leq_{\mathrm{sp}} V_2'$.*

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Remark 2.1. (i) On recursively using Theorem 2.1 and Corollary 2.1(i), we have the following assertions.

(a) If $X_1 \leq_{\text{st}} X_2 \leq_{\text{st}} \cdots \leq_{\text{st}} X_n$ and $Y_n \leq_{\text{icv}} Y_{n-1} \leq_{\text{icv}} \cdots \leq_{\text{icv}} Y_1$, then $U_n \leq_{\text{icv}} U_{n-1} \leq_{\text{icv}} \cdots \leq_{\text{icv}} U_1$, where $U_i = \wedge\{X_1, \ldots, X_{i-1}, X_i + Y_i, X_{i+1}, \ldots, X_n\}$, $i = 1, 2, \ldots, n$.

(b) If $X_1 \leq_{\text{st}} X_2 \leq_{\text{st}} \cdots \leq_{\text{st}} X_n$ then $V_n \leq_{\text{icv}} V_{n-1} \leq_{\text{icv}} \cdots \leq_{\text{icv}} V_1$, where $V_i = \wedge\{X_1, \ldots, X_{i-1}, X_i + X, X_{i+1}, \ldots, X_n\}$, $i = 1, 2, \ldots, n$.

(ii) On using Theorem 2.2 and Corollary 2.1(i), we have the following assertions.

(a) If $X_1 \leq_{\text{st}} X_2 \leq_{\text{st}} \cdots \leq_{\text{st}} X_n$ and $Y_1 \leq_{\text{icx}} Y_2 \leq_{\text{icx}} \cdots \leq_{\text{icx}} Y_n$, then $U_1' \leq_{\text{icx}} U_2' \leq_{\text{icx}} \cdots \leq_{\text{icx}} U_n'$, where $U_i' = \vee\{X_1, \ldots, X_{i-1}, X_i + Y_i, X_{i+1}, \ldots, X_n\}$, $i = 1, 2, \ldots, n$.

(b) If $X_1 \leq_{\text{st}} X_2 \leq_{\text{st}} \cdots \leq_{\text{st}} X_n$ then $V_1' \leq_{\text{icx}} V_2' \leq_{\text{icx}} \cdots \leq_{\text{icx}} V_n'$, where $V_i' = \vee\{X_1, \ldots, X_{i-1}, X_i + X, X_{i+1}, \ldots, X_n\}$, $i = 1, 2, \ldots, n$.

(iii) Li and Hu [5] proved Corollary 2.1(iii).

(iv) Li and Hu [5] proved that if X_1 or X_2 has a convex survival function and $X_1 \leq_{\text{icv}} X_2$, then $\vee\{X_1 + X, X_2\} \leq_{\text{sp}} \vee\{X_1, X_2 + X\}$. Corollary 2.1(iv) may be viewed as a supplement of the result proved in [5].

3. Allocation of two standby redundancies

In this section we deal with stochastic comparisons between Z_1 and Z_2 (defined by (1.1)) and also between Z_1' and Z_2' (defined by (1.2)). Let $\bar{H}$ and h respectively denote the survival function and the Lebesgue density function of $T = \wedge\{X_3, X_4, \ldots, X_n\}$. Then

$$h(t) = \sum_{i=3}^{n} f_i(t) \prod_{\substack{j=3 \\ j \neq i}}^{n} \bar{F}_j(t), \qquad t \in \mathbb{R}_+. \tag{3.1}$$

Theorem 3.1. *Suppose that $X_1 \leq_{\text{icv}} X_2$, $Y_1 \leq_{\text{hr}} Y_2$, and that $X_1, X_3, X_4, \ldots, X_n$ (or $X_2, X_3, X_4, \ldots, X_n$) have convex survival functions on $\mathbb{R}_+$. Then $Z_2 \leq_{\text{sp}} Z_1$.*

Proof. It is easy to verify that

$$[Z_1 > Z_2] = [X_1 > X_2,\ Y_1 > Y_2,\ T > X_2 + Y_2] \cup [X_2 > X_1,\ Y_2 > Y_1,\ T > X_1 + Y_1].$$

Using symmetry, we can write

$$\begin{aligned}
\Delta_6 &\equiv \mathrm{P}(Z_1 > Z_2) - \mathrm{P}(Z_2 > Z_1) \\
&= \mathrm{P}(X_1 > X_2,\ Y_1 > Y_2,\ T > X_2 + Y_2) + \mathrm{P}(X_2 > X_1,\ Y_2 > Y_1,\ T > X_1 + Y_1) \\
&\quad - \mathrm{P}(X_1 > X_2,\ Y_2 > Y_1,\ T > X_2 + Y_1) - \mathrm{P}(X_2 > X_1,\ Y_1 > Y_2,\ T > X_1 + Y_2) \\
&= \mathrm{E}\bigg[\int_0^\infty [\bar{G}_2(y) g_1(y) - \bar{G}_1(y) g_2(y)] \\
&\qquad\qquad \times [\bar{H}(X_1 + y) I(X_2 > X_1) - \bar{H}(X_2 + y) I(X_1 > X_2)]\,\mathrm{d}y\bigg].
\end{aligned} \tag{3.2}$$

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Case I: $X_1, X_3, X_4, \dots, X_n$ have convex survival functions. Let $\hat{X}_1$ be an independent copy of X_1. Then, using (3.2), we can write $\Delta_6 = \mathrm{E}[\psi_7(X_2)]$, where

$$\psi_7(t) = \mathrm{E}\bigg[\int_0^\infty [\bar{G}_2(y)g_1(y) - \bar{G}_1(y)g_2(y)] \times [\bar{H}(\hat{X}_1 + y)I(t > \hat{X}_1) - \bar{H}(t+y)I(\hat{X}_1 > t)]\,\mathrm{d}y\bigg], \qquad t \in \mathbb{R}_+.$$

Obviously, $\hat{X}_1 \le_{\mathrm{icv}} X_2$, $\mathrm{E}[\psi_7(\hat{X}_1)] = 0$, and

$$\psi_7(t) = \int_0^t \int_0^\infty [\bar{G}_2(y)g_1(y) - \bar{G}_1(y)g_2(y)]\bar{H}(s+y)f_1(s)\,\mathrm{d}y\,\mathrm{d}s - \int_0^\infty [\bar{G}_2(y)g_1(y) - \bar{G}_1(y)g_2(y)]\bar{H}(t+y)\bar{F}_1(t)\,\mathrm{d}y, \qquad t \in \mathbb{R}_+.$$

The assumption that $Y_1 \le_{\mathrm{hr}} Y_2$ (i.e. $\bar{G}_2(y)g_1(y) \ge \bar{G}_1(y)g_2(y)$ for all $y \in \mathbb{R}_+$) implies that $\psi_7(t)$ is increasing in $t \in \mathbb{R}_+$. Since $X_1, X_3, X_4, \dots, X_n$ have convex survival functions on $\mathbb{R}_+$ (i.e. $f_i(\cdot)$, $i = 1, 3, 4, \dots, n$, are decreasing on $\mathbb{R}_+$), it can be easily verified that $\psi_7'(t)$ is decreasing in $t \in \mathbb{R}_+$. Thus, $\psi_7(\cdot)$ is an increasing concave function on $\mathbb{R}_+$. Now, on using $\hat{X}_1 \le_{\mathrm{icv}} X_2$, it follows that $\Delta_6 = \mathrm{E}[\psi_7(X_2)] \ge \mathrm{E}[\psi_7(\hat{X}_1)] = 0$, i.e. $Z_2 \le_{\mathrm{sp}} Z_1$.

Case II: $X_2, X_3, X_4, \dots, X_n$ have convex survival functions. Let $\hat{X}_2$ be an independent copy of X_2. Then we can write $\Delta_6 = -\mathrm{E}[\psi_8(X_1)]$, where

$$\psi_8(t) = \mathrm{E}\bigg[\int_0^\infty [\bar{G}_2(y)g_1(y) - \bar{G}_1(y)g_2(y)] \times [\bar{H}(\hat{X}_2 + y)I(t > \hat{X}_2) - \bar{H}(t+y)I(\hat{X}_2 > t)]\,\mathrm{d}y\bigg], \qquad t \in \mathbb{R}_+.$$

Clearly, $X_1 \le_{\mathrm{icv}} \hat{X}_2$ and $\mathrm{E}[\psi_8(\hat{X}_2)] = 0$. Proceeding as in case I, we can show that $\Delta_6 = -\mathrm{E}[\psi_8(X_1)] \ge -\mathrm{E}[\psi_8(\hat{X}_2)] = 0$, i.e. $Z_2 \le_{\mathrm{sp}} Z_1$.

Theorem 3.2. *Suppose that $X_1 \le_{\mathrm{icv}} X_2$, $Y_1 \le_{\mathrm{st}} Y_2$, X_1 or X_2 has convex survival function on $\mathbb{R}_+$, and that X_i, $i = 3, 4, \dots, n$, has a log-convex density on $\mathbb{R}_+$. Then $Z_2 \le_{\mathrm{sp}} Z_1$.*

Proof. We will follow the line of the proof of Theorem 3.1. From case I (when X_1 has convex survival function), we have

$$\begin{aligned}\psi_7'(t) &= \int_0^\infty [\bar{G}_2(y)g_1(y) - \bar{G}_1(y)g_2(y)][2\bar{H}(t+y)f_1(t) + \bar{F}_1(t)h(t+y)]\,\mathrm{d}y \\ &= \mathrm{E}[\bar{G}_2(Y_1)\varphi(t, Y_1)] - \mathrm{E}[\bar{G}_1(Y_2)\varphi(t, Y_2)], \qquad t \in \mathbb{R}_+,\end{aligned}$$

where

$$\varphi(t, y) = 2\bar{H}(t+y)f_1(t) + \bar{F}_1(t)h(t+y), \qquad t, y \in \mathbb{R}_+.$$

Log-convexity of $f_i(\cdot)$ implies log-convexity of $\bar{F}_i(\cdot)$, $i = 3, \dots, n$ (see Proposition B.8 of [6, p. 101]). Since the product of log-convex functions is log-convex and log-convexity implies convexity, it follows that $\bar{H}(\cdot)$ is convex on $\mathbb{R}_+$. Using the abovementioned observations along with the property that the sum of convex functions is convex, we conclude (using (3.1)) that $h(\cdot)$ is a decreasing and convex function on $\mathbb{R}_+$. Therefore, for each fixed $t \in \mathbb{R}_+$, $\bar{G}_2(y)\varphi(t, y)$ is decreasing in $y \in \mathbb{R}_+$. Now, using $Y_1 \le_{\mathrm{st}} Y_2$, we obtain

$$\mathrm{E}[\bar{G}_2(Y_1)\varphi(t, Y_1)] \ge \mathrm{E}[\bar{G}_2(Y_2)\varphi(t, Y_2)] \ge \mathrm{E}[\bar{G}_1(Y_2)\varphi(t, Y_2)],$$

i.e. $\psi_7'(t) \ge 0$ for all $t \in \mathbb{R}_+$.

Let $0 \le t_1 < t_2 < \infty$. Consider

$$\begin{aligned}\Delta_7 &= \psi_7'(t_1) - \psi_7'(t_2)\\ &= \mathrm{E}[\bar{G}_2(Y_1)\{\varphi(t_1, Y_1) - \varphi(t_2, Y_1)\}] - \mathrm{E}[\bar{G}_1(Y_2)\{\varphi(t_1, Y_2) - \varphi(t_2, Y_2)\}].\end{aligned}$$

Convexity of $\bar{H}(\cdot)$ and $h(\cdot)$ on $\mathbb{R}_+$ implies that $\bar{H}(t_2 + y_1) - \bar{H}(t_2 + y_2) \le \bar{H}(t_1 + y_1) - \bar{H}(t_1 + y_2)$ and $h(t_2+y_1)-h(t_2+y_2) \le h(t_1+y_1)-h(t_1+y_2)$ whenever $0 \le t_1 < t_2 < \infty$ and $0 \le y_1 < y_2 < \infty$. Also, convexity of the survival function of X_1 implies that $f_1(\cdot)$, and, hence, $\varphi(\cdot, y)$, is decreasing on $\mathbb{R}_+$. Using these observations, it can be verified that, for each fixed $0 \le t_1 < t_2 < \infty$, $\varphi(t_1, y) - \varphi(t_2, y)$ (and, hence, $\bar{G}_2(y)\{\varphi(t_1, y) - \varphi(t_2, y)\}$) is decreasing in $y \in \mathbb{R}_+$, i.e. $\varphi(t_1, y_1) - \varphi(t_2, y_1) \ge \varphi(t_1, y_2) - \varphi(t_2, y_2)$ whenever $0 \le y_1 < y_2 < \infty$. Now, using $Y_1 \le_{\mathrm{st}} Y_2$, we obtain

$$\begin{aligned}\mathrm{E}[\bar{G}_2(Y_1)\{\varphi(t_1, Y_1) - \varphi(t_2, Y_1)\}] &\ge \mathrm{E}[\bar{G}_2(Y_2)\{\varphi(t_1, Y_2) - \varphi(t_2, Y_2)\}]\\ &\ge \mathrm{E}[\bar{G}_1(Y_2)\{\varphi(t_1, Y_2) - \varphi(t_2, Y_2)\}],\end{aligned}$$

i.e. $\Delta_7 \ge 0$. Thus, $\psi_7(\cdot)$ is an increasing concave function on $\mathbb{R}_+$. Since $\hat{X}_1 \le_{\mathrm{icv}} X_2$, we obtain $\Delta_6 = \mathrm{E}[\psi_7(X_2)] \ge \mathrm{E}[\psi_7(\hat{X}_1)] = 0$, i.e. $Z_2 \le_{\mathrm{sp}} Z_1$.

Using similar techniques, we can show that $Z_2 \le_{\mathrm{sp}} Z_1$ holds for case II (when X_2 has convex survival function).

Theorem 3.3. *Suppose that $X_1 \le_{\mathrm{icx}} X_2$, $Y_1 \le_{\mathrm{rh}} Y_2$, and that $X_1, X_3, X_4, \dots, X_n$ (or $X_2, X_3, X_4, \dots, X_n$) have concave survival functions on $\mathbb{R}_+$. Then $Z_1' \le_{\mathrm{sp}} Z_2'$.*

Proof. We will prove the result for the case when $X_1, X_3, X_4, \dots, X_n$ have concave survival functions on $\mathbb{R}_+$. The proof for the other case follows along similar lines. Let $T_1 = \vee\{X_3, X_4, \dots, X_n\}$. Then it can be easily verified that

$$[Z_2' > Z_1'] = [X_1 > X_2,\ Y_1 > Y_2,\ X_1 + Y_1 > T_1] \cup [X_2 > X_1,\ Y_2 > Y_1,\ X_2 + Y_2 > T_1].$$

Let K be the distribution function of T_1. Using symmetry, we obtain

$$\begin{aligned}\Delta_8 &\equiv \mathrm{P}(Z_2' > Z_1') - \mathrm{P}(Z_1' > Z_2')\\ &= \mathrm{P}(X_1 > X_2,\ Y_1 > Y_2,\ X_1 + Y_1 > T_1) + \mathrm{P}(X_2 > X_1,\ Y_2 > Y_1,\ X_2 + Y_2 > T_1)\\ &\quad - \mathrm{P}(X_1 > X_2,\ Y_2 > Y_1,\ X_1 + Y_2 > T_1) - \mathrm{P}(X_2 > X_1,\ Y_1 > Y_2,\ X_2 + Y_1 > T_1)\\ &= \mathrm{E}\bigg[\int_0^\infty [G_2(y)g_1(y) - G_1(y)g_2(y)]\\ &\qquad\qquad \times [K(X_1 + y)I(X_1 > X_2) - K(X_2 + y)I(X_2 > X_1)]\,\mathrm{d}y\bigg]\\ &= \mathrm{E}[\psi_9(X_2)],\end{aligned}$$

where

$$\begin{aligned}\psi_9(t) = \mathrm{E}\bigg[\int_0^\infty &[G_2(y)g_1(y) - G_1(y)g_2(y)]\\ &\times [K(\hat{X}_1 + y)I(\hat{X}_1 > t) - K(t + y)I(t > \hat{X}_1)]\,\mathrm{d}y\bigg], \qquad t \in \mathbb{R}_+, \quad (3.3)\end{aligned}$$

and $\hat{X}_1$ is an independent copy of X_1. Evidently, $\hat{X}_1 \leq_{\text{icx}} X_2$ and $\mathrm{E}[\psi_9(\hat{X}_1)] = 0$. From (3.3) we have

$$\psi_9(t) = \int_t^\infty \int_0^\infty [G_2(y)g_1(y) - G_1(y)g_2(y)]K(s+y)f_1(s)\,\mathrm{d}y\,\mathrm{d}s \\ - \int_0^\infty [G_2(y)g_1(y) - G_1(y)g_2(y)]K(t+y)F_1(t)\,\mathrm{d}y, \qquad t \in \mathbb{R}_+.$$

Using $Y_1 \leq_{\text{rh}} Y_2$ and the concavity of the survival functions of $X_1, X_3, X_4, \dots, X_n$, it can be verified that $\psi_9(\cdot)$ is an increasing convex function on $\mathbb{R}_+$. Now, using $\hat{X}_1 \leq_{\text{icx}} X_2$, we obtain $\Delta_8 = \mathrm{E}[\psi_9(X_2)] \geq \mathrm{E}[\psi_9(\hat{X}_1)] = 0$, i.e. $Z_1' \leq_{\text{sp}} Z_2'$.

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