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LETTER TO THE EDITOR

Dear Editor,

A note on invariance principles for iterated random functions

In this letter, we refer to the papers of Benda (1998) and Wu and Woodroofe (2000). In each paper, a central limit theorem is proved, one for contractive stochastic dynamical systems and the other for iterated random functions, which amount to the same mathematical model. Wu and Woodroofe show that, by slightly strengthening the moment condition on the function g appearing in Benda's central limit theorem, the continuity condition on g can be relaxed essentially. So, for example, the indicator functions of balls are allowed as g. Moreover, using work by Durrett and Resnick (1978), they prove an invariance principle for the central limit theorem.

We show that, using work by Heyde and Scott (1973) and Scott (1973), the central results of Benda (1998) and Wu and Woodroofe (2000) can be easily derived. Moreover, we show that, along with an invariance principle for the central limit theorem, such a principle also holds for the law of the iterated logarithm. To illustrate our results, we show that they can be applied to autoregressive processes with an ARCH(1)-noise sequence.

1. Preliminaries

Throughout, let $\mathcal{X}$ be a Polish (i.e. complete and separable) metric space with metric ρ, endowed with the σ-algebra of its Borel sets. Consider a discrete-parameter strictly stationary $\mathcal{X}$-valued Markov process $X = (X_n)_{n\in\mathbb{N}}$ on an underlying probability space $(\Omega, \mathcal{K}, \mathrm{P})$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ as usual. Let E denote the mean-value operator associated with P. Assume that X is ergodic as a strictly stationary process, that is, the left-shift transformation on $\mathcal{X}^{\mathbb{N}}$ is ergodic for the probability measure $\mathrm{P}\,X^{-1}$. Note also that, under our assumption on $\mathcal{X}$, there always exists a strictly stationary process $\tilde{X} = (\tilde{X}_n)_{n\in\mathbb{Z}}$ (where $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$) which extends X into the past, that is, X and $\tilde{X}$ have the same finite-dimensional distributions. See Lemma 1 of Elton (1990).

Let Q denote the transition probability function of X, and consider the transition operator of X, denoted by the same letter Q, which is defined by

$$Qf(x) = \int_{\mathcal{X}} Q(x, \mathrm{d}y) f(y), \qquad x \in \mathcal{X},$$

for any bounded measurable (complex- or real-valued) function f on $\mathcal{X}$.

For any real-valued measurable function g on $\mathcal{X}$, we define

$$\|g\| := \left(\int_{\Omega} g^2(X_0)\,\mathrm{d}\,\mathrm{P}\right)^{1/2} = \left(\int_{\mathcal{X}} g^2 \mathrm{d}\pi\right)^{1/2} = \mathrm{E}^{1/2}\, g^2,$$

where $\pi := \mathrm{P}\,X_0^{-1}$, and put $g_n = g(X_n)$, $S_0 = 0$, $S_n = \sum_{k=1}^{n} g_k$ for $n \in \mathbb{N}_+ = \{1, 2, \dots\}$.

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Assuming that $\mathrm{E}\, g(X_0) = \int_{\mathcal{X}} g \,\mathrm{d}\pi = 0$, for any $n \in \mathbb{N}_+$ define the stochastic processes η_n^C and η_n^D by

$$\eta_n^C(t) = \frac{1}{\sigma\sqrt{n}}(S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) g_{\lfloor nt \rfloor + 1}), \qquad t \in [0, 1],$$

$$\eta_n^D(t) = \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor}, \qquad t \in [0, 1],$$

where $\lfloor \cdot \rfloor$ is the integer-part function and σ^2 a positive number to be precisely defined later. Clearly, $\eta_n^C(t)$ and $\eta_n^D(t)$ are random variables on Ω for any $t \in [0, 1]$ and, consequently, η_n^C and η_n^D are random variables on Ω which take values in C (the metric space of real-valued continuous functions on [0, 1] with the uniform metric) and D (the metric space of real-valued functions on [0, 1] which are right continuous and have left limits, with the Skorokhod metric) respectively (see Billingsley (1968)).

We can now state the result of Heyde and Scott (see Heyde and Scott (1973) and Scott (1973)) in the present special case as follows.

Theorem 1. *Assume that the Markov process X satisfies the conditions above (i.e. is stationary and ergodic) and that $\|g\| < \infty$, $\int_{\mathcal{X}} g \,\mathrm{d}\pi = 0$, and*

$$\sum_{n \in \mathbb{N}_+} \|\mathrm{E}(g_n \mid X_0)\| < \infty. \tag{1}$$

Then the limit $\lim_{n\to\infty} \mathrm{E}(S_n^2/n) = \sigma^2 \geq 0$ exists. If $\sigma > 0$, then the measure $\mathrm{P}\,\eta_n^{-1}$ converges weakly to the Wiener measure, where η_n stands for either η_n^C or η_n^D, and the sequence

$$\left(\frac{\eta_n^C}{\sqrt{2 \log\log n}}\right)_{n \geq 3},$$

viewed as a subset of C, is a relatively compact set whose derived set coincides P*-almost surely with the set* $\{\boldsymbol{x} \in C : \boldsymbol{x}$ *is absolutely continuous,* $\boldsymbol{x}(0) = 0$, *and* $\int_0^1 [\mathrm{d}\boldsymbol{x}/\mathrm{d}t]^2 \,\mathrm{d}t \leq 1\}$.

Note that, since

$$\mathrm{E}(g_n \mid X_0) = Q^n g(X_0) \qquad \mathrm{P}\text{-a.s.}, \ n \in \mathbb{N}_+,$$

the condition (1) amounts to

$$\sum_{n \in \mathbb{N}_+} \|Q^n g\| < \infty.$$

2. Two invariance principles for iterated random functions

On an underlying probability space $(\Omega, \mathcal{K}, \mathrm{P}_0)$, we consider an $\mathcal{X}$-valued Markov process $X = (X_n)_{n\in\mathbb{N}}$ given by an iterated random function, i.e. by means of the recursive equation $X_n = F(X_{n-1}, \theta_n)$, $n \in \mathbb{N}_+$. We adopt here the notation and definitions from Wu and Woodroofe (2000). As is mentioned there and shown by Diaconis and Freedman (1999), under the conditions

$$\int_\Theta \log(L_\theta) H(\mathrm{d}\theta) < 0, \tag{2}$$

$$\int_\Theta L_\theta^\alpha H(\mathrm{d}\theta) < \infty, \tag{3}$$

$$\int_\Theta \rho^\alpha(x_0, F(x_0, \theta)) H(\mathrm{d}\theta) < \infty \tag{4}$$

for some $\alpha > 0$ and $x_0 \in \mathcal{X}$ (hence for all $x_0 \in \mathcal{X}$), there is a unique stationary distribution π for X, and the strictly stationary process obtained by letting X_0 have distribution π is ergodic. Let P denote the probability measure on $(\Omega, \mathcal{K})$ for which $\mathrm{P}\, X_0^{-1} = \pi$. Finally, let Ψ be the collection of nondecreasing concave functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(t) > 0$ for $t > 0$ and

$$\int_0^1 t^{-1}\sqrt{\psi(t)}\, \mathrm{d}t < \infty.$$

For $\psi \in \Psi$, $x \in \mathcal{X}$, and any real-valued measurable function g on $\mathcal{X}$ with $\|g\| < \infty$, define

$$K(g, \psi, x) = \sup_{\{x' : 0 < d(x, x') < 1\}} \frac{|g(x') - g(x)|}{\sqrt{\psi(d(x, x'))}}$$

and

$$\kappa(g, \psi) = \left(\int_{\mathcal{X}} K^2(g, \psi, x)\pi(\mathrm{d}x) \right)^{1/2}.$$

Theorem 2. *Assume that (2), (3), and (4) hold and let π be the stationary distribution for X. If g is a real-valued measurable function on $\mathcal{X}$ with $\int_{\mathcal{X}} g\, \mathrm{d}\pi = 0$ and $\int_{\mathcal{X}} |g|^p\, \mathrm{d}\pi < \infty$ for some $p > 2$, and there is a $\psi \in \Psi$ for which $\kappa(g, \psi) < \infty$, then the conclusions of Theorem 1 all hold.*

Proof. Under the same assumptions, Wu and Woodroofe (2000) showed that the series $\sum_{n \in \mathbb{N}_+} \|Q^n g\|$ is convergent. Hence, $h = \sum_{n \in \mathbb{N}_+} Q^n g$ converges in $L^2_\pi(\mathcal{X})$ and, moreover, the term σ^2 occurring in Theorem 1 is equal to $\|h\|^2 - \|Qh\|^2$.

Remark 1. The condition $\kappa(g, \psi) < \infty$ is related to similar conditions considered by Lasota and Yorke (1994) (see also Szarek (1997)). In particular, it holds if g is a Lipschitz function. Moreover, Wu and Woodroofe (2000, Theorem 3) showed that this condition holds if g is the indicator function of a ball.

Remark 2. The conclusions of Theorem 2 still hold under a probability P_0 which corresponds to either an initial distribution $\nu \ll \pi$ (see Billingsley (1968, Chapter 3)) or to one concentrated at $x_0 \in \mathcal{X}$ for all $x_0 \in \mathcal{X}$ not belonging to an exceptional set of P-measure equal to 0 (see Durrett and Resnick (1978)). Clearly, for the iterated logarithm case this is obvious.

Remark 3. The convergence of the series $\sum_{n \in \mathbb{N}_+} \|Q^n g\|$ can be easily shown in the case where g is a bounded Lipschitz function. See Herkenrath *et al.* (2003).

3. An application

In addition to the applications which are discussed by Wu and Woodroofe (2000), we refer to autoregressive processes with an ARCH(1)-noise sequence $(\varepsilon_n)_{n \in \mathbb{N}_+}$, which are described by the iterated random function

$$X_n = F(X_{n-1}, \varepsilon_n) = \alpha X_{n-1} + \sqrt{\alpha_0 + \alpha_1 X_{n-1}^2}\,\varepsilon_n, \qquad n \in \mathbb{N}_+.$$

Here $\mathcal{X} = \Theta = \mathbb{R}$, $\rho(x, x') = |x - x'|$ for $x, x' \in \mathbb{R}$, $\alpha \ge 0$, $\alpha_0 > 0$, $\alpha_1 > 0$ are real parameters, and $(\varepsilon_n)_{n \in \mathbb{N}_+}$ stands for $(\theta_n)_{n \in \mathbb{N}_+}$; see e.g. Borkovec and Klüppelberg (2001). In Herkenrath *et al.* (2003) it is shown that, if the common distribution H of the ε_n, $n \in \mathbb{N}_+$, has a finite first absolute moment, i.e. $\mathrm{E}(|\varepsilon_1|) = \int_{\mathbb{R}} |\varepsilon| H(\mathrm{d}\varepsilon) < \infty$, and $\mathrm{E}(\log(\alpha + \sqrt{\alpha_1}|\varepsilon_1|)) < 0$,

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then the conditions (2), (3), and (4) hold with $\alpha = 1$ so that a unique stationary distribution π for $(X_n)_{n\in\mathbb{N}}$ should exist.

Theorem 2 therefore holds for such time series with appropriate functions g.

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Yours sincerely,

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