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# A CHARACTERIZATION OF $(-1,-1)$-FREUDENTHAL–KANTOR TRIPLE SYSTEMS

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**Abstract.** In this paper, we discuss a connection between $(-1,-1)$-Freudenthal–Kantor triple systems, anti-structurable algebras, quasi anti-flexible algebras and give examples of such structures. The paper provides the correspondence and characterization of a bilinear product corresponding a triple product.

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**1. Introduction.** The history of nonassociative algebras, the subject of this paper, started with Hamilton, Cayley and Hurwitz and further with Artin and Zorn, who studied alternative and nearly associative algebras. Thereafter, Freudenthal [**10**], Tits [**53**], Kantor [**34**, **35**, **36**] and Koecher [**39**, **40**] studied constructions of Lie algebras from nonassociative algebras and triple systems, in particular Jordan algebras, while Allison [**1**, **2**] defined the concept of structurable algebras, containing Jordan algebras. Recently, we have studied constructions of Lie superalgebras as well as Lie algebras from triple systems [**25**, **27**, **30–32**]. As a continuation of [**30**, **31**], we are interested in characterizing the structure properties [**7**, **20**, **28**] of the subspace $L_{-1}$ of the five graded Lie (super)algebra $L(\varepsilon,\delta) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, $[L_i, L_j] \subseteq L_{i+j}$, associated with an $(\varepsilon,\delta)$-Freudenthal–Kantor triple system. In particular, we deal with a characterization of triple systems from the point of view of a bilinear product by means of anti-structurable algebras or balanced property. Specially, Jordan and Lie (super)algebras [**9**, **13**, **52**] play an important role in many mathematical and physical subjects [**5**, **11–14**, **16**, **26**, **29**, **37**, **47**, **48**, **55**, **56**]). We also note that the construction and characterization of these algebras can be expressed in terms of triple systems [**20**, **23**, **24**, **28**, **38**, **49**] by using the standard embedding method [**22**, **41**, **42**, **50**, **54**].

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As for $\delta$-structurable algebras, the motivation for the study of such nonassociative algebras is as follows. The existence of the class of structurable algebras is an important generalization of Jordan algebras giving a construction of Lie algebras. Hence, from our concept, by means of triple products, we define a generalization of such class to construct Lie (super)algebras.

Summarizing the content, we give the introduction in Section 1. In Section 2, we present definitions and preamble. In Section 3, we present the study of $(-1,-1)$-Freudenthal–Kantor triple systems and anti-structurable algebras and the correspondence and characterization of a bilinear product corresponding a triple product and study the structure of a broader class of algebras, i.e. anti-structurable and quasi-flexible algebras, which generalize structurable and flexible algebras.

## 2. Definitions.

**2.1.** $(\varepsilon, \delta)$**-Freudenthal–Kantor triple systems.** We are concerned in this paper with triple systems which have finite dimension over a field $\Phi$ of characteristic $\neq 2$ or 3.

In order to render this paper as self-contained as possible, we recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order).

A vector space $V$ over a field $\Phi$ endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto (xyz)$ is said to be a *GJTS of 2nd order* if the following conditions are fulfilled:

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \tag{2.1}$$

$$K(K(a,b)x, y) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0, \tag{2.2}$$

where $L(a,b)c := (abc)$ and $K(a,b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (2.1) and the identity $(abc) = (cba)$.

We can generalize the concept of GJTS of 2nd order as follows (see [**14**, **15**, **18–22**, **54**] and the earlier references therein). For $\varepsilon = \pm 1$, $\delta = \pm 1$, a triple product that satisfies

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \tag{2.3}$$

$$K(K(a,b)x, y) - L(y,x)K(a,b) + \varepsilon K(a,b)L(x,y) = 0, \tag{2.4}$$

where

$$L(a,b)c := (abc), \quad K(a,b)c := (acb) - \delta(bca), \tag{2.5}$$

is called an $(\varepsilon, \delta)$-*Freudenthal–Kantor triple system* (for short $(\varepsilon, \delta)$-FKTS).

Remark. We note that the concept of GJTS of 2nd order coincides with that of $(-1, 1)$-FKTS. Thus, we can construct the simple Lie algebras by means of the standard embedding method [**6**, **14–18**, **22**, **25**, **27**, **36**, **54**].

For an $(\varepsilon, \delta)$-FKTS $U$, we denote

$$A(a,b) := L(a,b) - \varepsilon L(b,a), \tag{2.6}$$

where $L(a, b)$ is defined by (2.5). Then $A(a, b))$ is an anti-derivation of $U$ [**31**], that is

$$[A(a, b), L(c, d)] = L(A(a, b)c, d) - L(c, A(a, b)d). \tag{2.7}$$

An $(\varepsilon, \delta)$-FKTS $U$ is called *unitary* if the identity map $Id$ is contained in $\kappa := K(U, U)$ i.e., if there exist $a_i, b_i \in U$, such that $\Sigma_i K(a_i, b_i) = Id$.

REMARK. We note that a balanced triple system (i.e. it fulfils $K(x, y) =< x\,|\,y >' Id$, where $<|>'$ is a symmetric bilinear form) is unitary, since $Id \in \kappa = K(U, U)$.

We show in the following remark the equivalence between the balanced notion defined above and the one of [**7**].

REMARK. We note that for a triple system $U$ with product ( ), the notion of balanced $(-1, -1)$-FKTS is equivalent to saying that the triple system satisfies (2.3) and

$$(xxy) = (xyx) =< x\,|\,x > y, \quad x, y \in U, \tag{2.8}$$

where $<|>$ is a symmetric bilinear form.

Indeed, if $U$ is a $(-1, 1)$-FKTS then (2.3) is fulfilled and, by (2.4),

$$K(K(a, b)x, y) - L(y, x)K(a, b) - K(a, b)L(x, y) = 0, \quad a, b, x, y \in U,$$

that is $K(x, y) = L(y, x) + L(x, y) =< x\,|\,y >' Id$, since $K(x, y) =< x\,|\,y >' Id$, hence

$$(xwy) + (ywx) = (xyw) + (yxw) =< x\,|\,y >' w, \quad x, y, w \in U.$$

If we put now $x = y$ in the last line, it follows $2(xwx) = 2(xxw) =< x\,|\,x >' w$, that is, (2.8) is valid for the symmetric form $<|>= \frac{1}{2} <|>'$.

In contrast, by linearizing (2.8), we have

$$(xzy) + (yzx) = (xyz) + (zyx) =< x\,|\,z > y+ < z\,|\,x > y, \quad x, y, z \in U,$$

hence, by (2.5), $K(x, z) = 2 < x\,|\,z >$, $x, z \in U$, so $K(x, y) =< x\,|\,y >' Id$, $x, y \in U$.

**2.2. $\delta$-structurable algebras.** Within the general framework of $(\varepsilon, \delta)$-FKTSs $(\varepsilon, \delta = \pm 1)$ and the standard embedding Lie (super)algebra construction studied in [**6**, **7**, **14–16**, **27**], we define $\delta$-structurable algebras as a class of nonassociative algebras with involution which coincides with the class of structurable algebras for $\delta = 1$ as introduced and studied in [**1**, **2**]. Structurable algebras are a class of nonassociative algebras with involution that include Jordan algebras (with trivial involution), associative algebras with involution, and alternative algebras with involution. They are related to $(-1, 1)$-FKTSs as introduced and studied in [**34**, **35**] (and further studied in [**3**, **4**, **33**, **43–46**, **51**]). Their importance lies with constructions of five graded Lie algebras $L(-1, 1)$. For $\delta = -1$, the anti-structurable algebras [**30**] are a new class of nonassociative algebras that may similarly shed light on the notion of $(-1, -1)$-FKTSs, hence (by [**6**, **7**]) on the construction of Lie superalgebras and Jordan algebras.

Let $(\mathcal{A}, ^-)$ be a finite-dimensional nonassociative unital algebra with involution (involutive anti-automorphism, i.e. $\overline{\overline{x}} = x$, $\overline{xy} = \overline{y}\,\overline{x}$, $x, y \in \mathcal{A}$) over $\Phi$. Since char$\Phi \neq 2$, by [**1**], we have $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$, where $\mathcal{H} = \{a \in \mathcal{A}\,|\,\overline{a} = a\}$ and $\mathcal{S} = \{a \in \mathcal{A}\,|\,\overline{a} = -a\}$.

Suppose $x, y, z \in \mathcal{A}$. Put $[x, y] := xy - yx$ and $[x, y, z] := (xy)z - x(yz)$ so $\overline{[x, y, z]} = -[\overline{z}, \overline{y}, \overline{x}]$. The operators $L_x$ and $R_x$ are defined by $L_x(y) := xy$, $R_x(y) := yx$.

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For $\delta = \pm 1$ and $x, y \in \mathcal{A}$ define

$$ {}^{\delta}V_{x,y} := L_{L_x(\overline{y})} + \delta(R_x R_{\overline{y}} - R_y R_{\overline{x}}), \tag{2.9}$$

$$ {}^{\delta}B_{\mathcal{A}}(x, y, z) :={}^{\delta} V_{x,y}(z) = (x\overline{y})z + \delta[(z\overline{y})x - (z\overline{x})y],\ x, y, z \in \mathcal{A}. \tag{2.10}$$

Here ${}^{+}B_{\mathcal{A}}(x, y, z)$ is called the *triple system obtained from the algebra* $(\mathcal{A},^{-})$. We will call ${}^{-}B_{\mathcal{A}}(x, y, z)$ the *anti-triple system obtained from the algebra* $(\mathcal{A},^{-})$. We write for short

$$ V_{x,y} :={}^{\delta} V_{x,y}, \quad B_{\mathcal{A}} := ({}^{\delta}B_{\mathcal{A}}, \mathcal{A}). \tag{2.11}$$

A unital non-associative algebra with involution $(\mathcal{A},^{-})$ is called a *structurable algebra* if

$$ [V_{u,v}, V_{x,y}] = V_{V_{u,v}(x),y} - V_{x,V_{v,u}(y)}, \tag{2.12}$$

for $V_{u,v} ={}^{+}V_{u,v}$, $V_{x,y} ={}^{+}V_{x,y}$, $u, v, x, y \in \mathcal{A}$, and we will call $(\mathcal{A},^{-})$ an *anti-structurable algebra* if the identity (2.12) is fulfilled for $V_{u,v} ={}^{-}V_{u,v}$, $V_{x,y} ={}^{-}V_{x,y}$.

If $(\mathcal{A},^{-})$ is structurable then, in [**35**], the triple system $B_{\mathcal{A}}$ is called a *generalized Jordan triple system* (GJTS) and in [**8**], $B_{\mathcal{A}}$ is a GJTS of 2nd order, i.e. satisfies the identities (2.3) and (2.4). If $(\mathcal{A},^{-})$ is anti-structurable then we call $B_{\mathcal{A}}$ an *anti-GJTS*.

## 3. $\delta$-flexible, quasi-flexible algebras and $(-1, -1)$-FKTSs.

**3.1. $\delta$-flexible and quasi-flexible algebras.** For $\delta = \pm 1$, an algebra $\mathcal{A}$ is called *$\delta$-flexible* if

$$ -\delta[x, y, z] = [z, y, x], \quad x, y, z \in \mathcal{A}. \tag{3.13}$$

For $\delta = 1$ then $\mathcal{A}$ is flexible, while for $\delta = -1$ then $\mathcal{A}$ is called *anti-flexible*.

EXAMPLE. For $\delta = \pm 1$, if $\mathcal{A}$ is an associative algebra then it is $\delta$-flexible.

An algebra $\mathcal{A}$ is called *quasi-flexible* if

$$ [x, y, z]w + [wy, z, x] = [xy, z, w] + [w, y, z]x, \quad x, y, z, w \in \mathcal{A}. \tag{3.14}$$

REMARK. If we put $y = e$, the unit element, in (3.14) then we have $[x, z, w] = [w, z, x]$; hence, (3.14) is a generalization of the anti-flexible property.

REMARK. If we put $w = e$, the unit element, in (3.14) then we have $[x, y, z] = -[y, z, x]$; hence,

$$ [x, y, z] = -[y, z, x] = (-1)^2[z, x, y] = (-1)^3[x, y, z] = -[x, y, z]. $$

Then, $[x, y, z] = 0$ for char$\Phi \neq 2$; hence (3.14) is a generalization of the associative property (char$\Phi \neq 2$).

First, we outline the main result as follows.

THEOREM 3.1. *Let $U$ be an anti-structurable and quasi-flexible algebra. Then $U$ is a $(-1, -1)$-FKTS.*

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COROLLARY 3.1. *Let $U$ be an anti-structurable associative algebra. Then $U$ is a $(-1,-1)$-FKTS.*

EXAMPLE. Let $\mathcal{M}_{m,n}(\Phi)$ denote the vector space of $m \times n$ matrices over $\Phi$ and for $x \in \mathcal{M}_{m,n}(\Phi)$ denote by $x^\top$ the transposed matrix. Then $\mathcal{M}_{n,n}(\Phi)$ with the product

$$\{x, y, z\} := xy^\top z - zy^\top x + zx^\top y, \quad x, y, z \in \mathcal{M}_{n,n}(\Phi) \tag{3.15}$$

is a $(-1,-1)$-FKTS.

*Proof of theorem.* We remark first that for $x, y \in U$ we have $K(x, y) \equiv A(x, y) \equiv L_{x\bar{y}+y\bar{x}}$, where $K(x, y)$ and $A(x, y)$ are defined by (2.5) and (2.6), respectively. Indeed,

$$\begin{aligned} K(x, y)z = \{x, z, y\} + \{y, z, x\} &= (x\bar{z})y - (y\bar{z})x + (y\bar{x})z + (y\bar{z})x \\ &\quad - (x\bar{z})y + (x\bar{y})z = L_{x\bar{y}+y\bar{x}} \end{aligned}$$

is valid by (2.5) and (2.10), while by (2.6) and (2.10) follows

$$\begin{aligned} A(x, y)z = \{x, y, z\} + \{y, x, z\} &= (x\bar{y})z - (z\bar{y})x + (z\bar{x})y + (y\bar{x})z \\ &\quad - (z\bar{x})y + (z\bar{y})x = L_{x\bar{y}+y\bar{x}}. \end{aligned}$$

Then, by (2.4), it is enough to show $A(A(a, b)c, d) = L(d, c)A(a, b) + A(a, b)L(c, d)$. Thus, by (2.6), we must show

$$L(A(a, b)c, d) + L(d, A(a, b)c) = L(d, c)A(a, b) + A(a, b)L(c, d). \tag{3.16}$$

We note now that by (2.7),

$$A(a, b)L(c, d) = L(A(a, b)c, d) - L(c, A(a, b)d) + L(c, d)A(a, b), \tag{3.17}$$

and then replacing the last term on the right-hand side of equation (3.16) with the right-hand side of (3.17) and cancelling, (3.16) becomes

$$L(d, A(a, b)c) + L(c, A(a, b)d) = (L(d, c) + L(c, d))A(a, b) = A(c, d)A(a, b). \tag{3.18}$$

Denote $h := a\bar{b} + b\bar{a}$. Then, (3.18) can be written as

$$L(d, hc) + L(c, hd) = (L(d, c) + L(c, d))A(a, b) = A(c, d)h, \tag{3.19}$$

or equivalently, by (2.5) and (2.10),

$$(d(\overline{hc}))u - (u(\overline{hc}))d + (u\bar{d})(hc) + (c(\overline{hd}))u - (u(\overline{hd}))c + (u\bar{c})(hd) = (c\bar{d} + d\bar{c})(hu), \tag{3.20}$$

for all $a, b, c, d, u \in U$. Furthermore, since $^-$ is an involution and $\bar{h} = h$, (3.20) is equivalent to

$$(d(\bar{c}h))u - (u(\bar{c}h))d + (u\bar{d})(hc) + (c(\bar{d}h))u - (u(\bar{d}h))c + (u\bar{c})(hd) = (c\bar{d})(hu) + (d\bar{c})(hu), \tag{3.21}$$

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that is

$$(d(\overline{c}h))u - (d\overline{c})(hu) - (u(\overline{c}h))d + (u\overline{c})(hd) + (u\overline{d})(hc) \\ - (u(\overline{d}h))c + (c(\overline{d}h))u - (c\overline{d})(hu) = 0.$$

If we put now $d = x, \overline{c} = y, h = z, u = w$, in the last line, then (3.21) is equivalent to

$$(x(yz))w - (xy)(zw) - (w(yz))x + (wy)(zx) + (w\overline{x})(z\overline{y}) - (w(\overline{x}z))\overline{y} \\ + (\overline{y}(\overline{x}z))w - (\overline{y}\,\overline{x})(zw) = 0. \tag{3.22}$$

If we denote $E(x, y, z, w) := (x(yz))w - (xy)(zw) - (w(yz))x + (wy)(zx)$, then (3.22) holds if

$$E(x, y, z, w) = 0, \tag{3.23}$$

for all $x, y, z, w \in U$ since the identity (3.22) is equivalent to $E(x, y, z, w) - E(w, \overline{x}, z, \overline{y}) = 0$.

The identity (3.23) holds if and only if

$$[x, yz, w] + x((yz)w) - (xy)(zw) - [w, yz, x] - w((yz)x) + (wy)(zx) = 0,$$

that is

$$[x, yz, w] + x[y, z, w] + x(y(zw)) - (xy)(zw) - [w, yz, x] - w[y, z, x] \\ - w(y(zx)) + (wy)(zx) = 0,$$

or equivalently

$$[x, yz, w] + x[y, z, w] - [x, y, zw] - [w, yz, x] - w[y, z, x] + [w, y, zx] = 0. \tag{3.24}$$

In general, we have $[a, b, c]d + [a, bc, d] = [ab, c, d] + [a, b, cd] - a[b, c, d]$. Hence, we have

$$[x, yz, w] + x[y, z, w] - [x, y, zw] = [xy, z, w] - [x, y, z]w \tag{3.25}$$

and

$$-[w, yz, x] - w[y, z, x] + [w, y, zx] = [w, y, z]x - [wy, z, x]. \tag{3.26}$$

Then, by (3.25) and (3.26), it follows that the identity is equivalent to the quasi-flexible property (3.14). Hence, an anti-structurable and quasi-flexible algebra is a $(-1, -1)$-FKTS. □

REMARK. We note that an anti-structurable algebra which is quasi-flexible is Lie admissible. The proof will be given elsewhere.

**3.2.** $(-1, -1)$**-FKTSs with left unit element.** We outline first the main result as follows.

THEOREM 3.2. *A* $(-1, -1)$*-FKTS* $U$ *with product* ( ) *and left unit element* $e$ *can be determined in terms of the bilinear product of* $U$ *defined by*

$$x \cdot y = (exy), \quad x, y \in U. \tag{3.27}$$

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*Let $U$ be a triple system with product $(\ )$. An element $e \in U$ is called a left unit element if*

$$(eex) = x. \tag{3.28}$$

*An element $e \in U$ is called a tripotent if*

$$(eee) = e. \tag{3.29}$$

*Let us denote*

$$Q(x) := (exe), \quad R(x) := (xee), \quad x \in U. \tag{3.30}$$

LEMMA 3.1. *Let $U$ be a $(-1,-1)$-FKTS with a left unit element $e$. Then*

$$Q^2(x) = R^2(x) = x, \quad RQ(x) = QR(x), \quad x \in U.$$

*Proof.* We remark first that by (2.5) and (3.29) we have $K(e,e)e = 2e$. Moreover, by (2.4),

$$K(K(e,e)e,x)e - 2L(x,e)K(e,e)e - K(e,e)L(e,x)e = 0,$$

hence $2K(e,x)e - 2L(x,e)e - 2(eQ(x)e) = 0$. Thus, by (2.5), $(eex) + (xee) - (xee) - Q^2(x) = 0$, that is $Q^2(x) = (eex) = x$ for all $x \in U$. Furthermore, by (2.3),

$$(xe(eee)) = ((xee)ee) - (e(exe)e) + (ee(xee)),$$

or equivalently, by (3.29), $(xee) = ((xee)ee) - (e(exe)e) + (xee)$, that is $Q^2(x) = R^2(x)$.
Finally, by (2.3), it follows that $(ex(eee)) = ((exe)ee) - (e(xee)e) + (ee(exe))$, or equivalently, by (3.29), $(exe) = RQ(x) - QR(x) + (exe)$, that is $RQ(x) = QR(x)$. □

REMARK. By **[38]**, we note that for a $(-1,1)$-FKTS $U$ with product $(\ )$ and left unit element $e$ can be determined in terms of the bilinear product of $U$ defined by $x \circ y = (xey)$, $x, y \in U$, that is, by Theorem 3.3 **[38]**, $U = U_{11}^{+} \oplus U_{11}^{-} \oplus U_{13}^{+} \oplus U_{13}^{-}$ with the product

$$(xyz) = (Q^{-1}(y) \circ x) \circ z + x \circ (Q^{-1}(y) \circ z) - Q^{-1}(y) \circ (x \circ z),$$

where $Q(x) = \begin{cases} \pm x, \text{ if } x \in U_{11}^{\pm} \\ \pm 3x, \text{ if } x \in U_{13}^{\pm} \end{cases}$ and $U_{1i} := \{x \in U \mid R(x) = ix\}$.

*Proof of theorem.* By (2.3), it follows that $(xe(eey)) = ((xee)ey) - (e(exe)y) + (ee(xey))$, or equivalently, $(xey) = ((xee)ey) - (e(exe)y) + (xey)$, that is, by (3.30),

$$(eQ(x)y) = (R(x)ey), \quad x, y \in U. \tag{3.31}$$

Now, by Lemma 3.1, $Q^2(x) = x$ and by replacing $x$ by $Q(x)$ in (3.31) we have, by (3.27),

$$x \cdot y = (exy) = (RQ(x)ey), \quad x, y \in U. \tag{3.32}$$

Furthermore, by Lemma 3.1, $Q^2(x) = R^2(x) = x$ and by replacing $x$ by $QR(x)$ in (3.32), we have

$$QR(x) \cdot y = (xey). \tag{3.33}$$

On the other hand, by (2.3), it follows that $(ex(eye)) = ((exe)ye) - (e(xey)e) + (ey(exe))$, or equivalently, by (3.30), $(exQ(y)) = (Q(x)ye) - Q(xey) + (eyQ(x))$. Hence, by (3.27) and (3.33) follows $x \cdot Q(y) = (Q(x)ye) - Q(QR(x) \cdot y) + y \cdot Q(x)$ or equivalently, by Lemma 3.1, $x \cdot Q(y) = (Q(x)ye) - Q(RQ(x) \cdot y) + y \cdot Q(x)$ for all $x, y \in U$. Replacing $x$ by $Q(x)$ in the last line follows $Q(x) \cdot Q(y) = (Q^2(x)ye) - Q(RQ^2(x) \cdot y) + y \cdot Q^2(x)$; hence by Lemma 3.1,

$$(xye) = Q(x) \cdot Q(y) + Q(R(x) \cdot y) - y \cdot x. \tag{3.34}$$

Finally, by (2.3) follows $(ex(yze)) = ((exy)ze) - (y(xez)e) + (yz(exe))$, or equivalently, by (3.30), $(yzQ(x)) = (ex(yze)) - ((exy)ze) + (y(xez)e)$, for all $x, y, z \in U$. Replacing $x$ by $Q(x)$ in the last line follows, by Lemma 3.1, $(yzx) = (eQ(x)(yze)) - ((eQ(x)y)ze) + (y(Q(x)ez)e)$, or equivalently,

$$(xyz) = (eQ(z)(xye)) - ((eQ(z)x)ye) + (x(Q(z)ey)e), \quad x, y, z \in U. \tag{3.35}$$

Hence, the product $(xyz)$ can be characterized in terms of the bilinear product (3.27). $\square$

We now give a corollary which first appeared in **[7]** but without using the notion of quadratic algebra **[7]**. First, we need a remark and two lemmas.

REMARK. We shall remark that the balanced property gives

$$x + \overline{x} = T(x)e, \quad \text{where} \quad \overline{x} := R(x) = (xee), \quad T(x)e := 2 < x \,|\, e > e, \tag{3.36}$$

and

$$e \cdot x = x \cdot e = x, \quad [e, x, y] = 0 = [x, e, y], \quad x, y \in U. \tag{3.37}$$

Indeed, by (2.8), $(xyx) =< x \,|\, x > y, x, y \in U$; hence, $(xyz) + (zyx) = 2 < x \,|\, z >$, by the symmetry of the bilinear form. Then, $(xee) + (eex) = 2 < x \,|\, e > e$ so, by (3.28), it follows that (3.36) is fulfilled. Also by (2.8), $(exe) =< e \,|\, e > x, x \in U$, that is, by (3.27), $x \cdot e = x$ since $e$ is a tripotent. Then, $x \cdot e = x = e \cdot x$, by (3.28). Moreover, the last identity gives $[e, x, y] = (e \cdot x) \cdot y - e \cdot (x \cdot y) = x \cdot y - x \cdot y = 0$ as well as $[x, e, y] = (x \cdot e) \cdot y - x \cdot (e \cdot y) = x \cdot y - x \cdot y = 0$, so (3.37) is fulfilled.

LEMMA 3.2. *Let $U$ be a balanced $(-1, -1)$-FKTS with product $(\,)$ and left unit element $e$. Then the bilinear product (3.27) is flexible.*

*Proof.* Indeed, by (2.8),

$$(ex(eyx)) + (xe(eyx)) = 2 < x \,|\, e > (eyx) = 2 < x \,|\, e > y \cdot x = T(x)(y \cdot x), x, y \in U. \tag{3.38}$$

On the other hand, by (2.3), it follows that $(xe(eyx)) = ((xee)yx) - (e(exy)e) + (ey(xex))$, or equivalently, by (2.8)

$$(e(exy)e) = -(xe(eyx)) + ((xee)yx) + (ey(xex)) = -(xe(eyx)) + (\overline{x}yx) + < x \,|\, x > y \cdot e,$$

that is $(e(exy)e) = -(xe(eyx)) + ((T(x) - x)yx) + \langle x \,|\, x \rangle\, y \cdot e$; hence,

$$(e(exy)e) = -(xe(eyx)) + T(x)(y \cdot x) - (xyx) + \langle x \,|\, x \rangle\, y = -(xe(eyx)) + T(x)(y \cdot x). \tag{3.39}$$

Then by (3.38) and (3.39) follows $(ex(eyx)) = (e(exy)e)$, or equivalently $x \cdot (y \cdot x) = (x \cdot y) \cdot x$; hence, $[x, y, x] = 0$, which implies the flexible identity (3.13). □

LEMMA 3.3. *Let $U$ be a balanced $(-1, -1)$-FKTS with product $(\,)$ and left unit element $e$. Then*

$$\overline{x \cdot y} = \overline{y} \cdot \overline{x}, \quad \overline{\overline{x}} = x, \quad x, y \in U, \tag{3.40}$$

*where $\overline{x} := (xee)$ and $x \cdot y$ is the bilinear product of $U$ defined by (3.27).*

*Proof.* By definition $\overline{x \cdot y} = ((x \cdot y)ee) = ((exy)ee)$; hence, by (2.3), it follows that

$$\overline{x \cdot y} = (ex(yee)) + (y(xee)e) - (ye(exe)) = x \cdot \overline{y} + (y\overline{x}e) - \overline{y} \cdot x, \tag{3.41}$$

since by the balanced property (2.3), (3.28) and (3.30), we have $Q(x) = (exe) = (eex) = x$. Furthermore, by (3.30) we have $R(x) = (xee) = \overline{x}$; hence, by (3.34),

$$(y\overline{x}e) = y \cdot \overline{x} + \overline{y} \cdot \overline{x} - \overline{x} \cdot y. \tag{3.42}$$

Then, by (3.42) and (3.34), it follows

$$\overline{x \cdot y} = x \cdot \overline{y} + y \cdot \overline{x} + \overline{y} \cdot \overline{x} - \overline{x} \cdot y - \overline{y} \cdot x. \tag{3.43}$$

Now, by (3.36), it follows that (3.43) is equivalent to

$$\overline{x \cdot y} = (T(x)e - \overline{x}) \cdot \overline{y} + (T(y)e - \overline{y}) \cdot \overline{x} + \overline{y} \cdot \overline{x} - \overline{x} \cdot (T(y)e - \overline{y}) - \overline{y} \cdot (T(x)e - \overline{x}).$$

Then, the first identity in (3.40) follows from the last line, (3.27) and straightforward cancellations. The second identity in (3.40), that is $\overline{\overline{x}} = x$, follows from Lemma 3.1, that is from the identity $R^2(x) = x$ since we have shown above that $R(x) = \overline{x}$; hence $R^2(x) = \overline{\overline{x}}$. □

COROLLARY 3.2. *Let $U$ be a balanced $(-1, -1)$-FKTS with product $(\,)$ and left unit element $e$. Then*

$$(xyz) = (\overline{x} \cdot y) \cdot z - \overline{x} \cdot (y \cdot z) + y \cdot (\overline{x} \cdot z), \quad x, y, z \in U, \tag{3.44}$$

*where $\overline{x} := (xee)$ and $x \cdot y$ is the bilinear product of $U$ defined by (3.27).*

*Proof.* Since by the proof of Lemma 3.3 we have $Q(x) = x$, $R(x) = \overline{x}$, then by (3.35),

$$(xyz) = (ez(xye)) - ((ezx)ye) + (x(zey)e). \tag{3.45}$$

Now, since $x \cdot y = (exy)$, $\overline{x} \cdot y = (xey)$ and, by (3.42), $(xye) = x \cdot y + \overline{x} \cdot y - y \cdot x$, then $(ez(xye)) = z \cdot (x \cdot y + \overline{x} \cdot y - y \cdot x)$, $((ezx)ye) = ((z \cdot x)ye) = (z \cdot x) \cdot y + \overline{(z \cdot x)} \cdot y - y \cdot (z \cdot x)$ and $(x(zey)e) = (x(\overline{z} \cdot y)e) = x \cdot (\overline{z} \cdot y) + \overline{x} \cdot (\overline{z} \cdot y) - (\overline{z} \cdot y) \cdot x$. Hence,

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(3.45) is equivalent to

$$(xyz) = -[z, x, y] + z \cdot (\overline{x} \cdot y) + \overline{z} \cdot (y \cdot x) - T(z) \cdot (y \cdot x) - (\overline{z} \cdot y) \cdot x \\ - (\overline{x} \cdot \overline{z}) \cdot y - \overline{x} \cdot (\overline{z} \cdot y) + T(x) \cdot (\overline{z} \cdot y) + \overline{x} \cdot (\overline{z} \cdot y) + y \cdot (z \cdot x),$$

that is

$$(xyz) = -[z, x, y] + z \cdot (\overline{x} \cdot y) - [\overline{z}, y, x] - T(z) \cdot (y \cdot x) - (\overline{x} \cdot \overline{z}) \cdot y \\ + T(x) \cdot (\overline{z} \cdot y) + y \cdot (z \cdot x).$$

Since $(\overline{x} \cdot \overline{z}) \cdot y = ((T(x)e - x) \cdot \overline{z}) \cdot y = T(x)(\overline{z} \cdot y) - (x \cdot \overline{z}) \cdot y$, then the last identity is equivalent to

$$(xyz) = -[z, x, y] + z \cdot (\overline{x} \cdot y) - [\overline{z}, y, x] - T(z) \cdot (y \cdot x) + (x \cdot \overline{z}) \cdot y + y \cdot (z \cdot x). \tag{3.46}$$

We remark now that $[z, x, y] = -[\overline{z}, x, y]$ since $z = T(z)e - \overline{z}$ and $[e, x, y] = 0$, by (3.37); hence, by (3.46), it follows that

$$(xyz) = [\overline{z}, x, y] + z \cdot (\overline{x} \cdot y) - [\overline{z}, y, x] + (x \cdot \overline{z}) \cdot y - y \cdot (\overline{z} \cdot x), \tag{3.47}$$

since $-T(z) \cdot (y \cdot x) + y \cdot (z \cdot x) = -y \cdot (\overline{z} \cdot x)$. Analogously, we have $[z, x, y] = -[z, \overline{x}, y]$, by (3.37); thus $[z, x, y] = -[z, \overline{x}, y] = -[\overline{z}, x, y]$. Hence, by (3.47), it follows that

$$(xyz) = [z, \overline{x}, y] + z \cdot (\overline{x} \cdot y) - [\overline{z}, y, x] + (x \cdot \overline{z}) \cdot y - y \cdot (\overline{z} \cdot x). \tag{3.48}$$

Now, from the identity $[\overline{z}, y, x] = -[z, y, x]$ above, it follows that $[\overline{z}, y, x] = -[z, y, T(x) - \overline{x}] = [z, y, \overline{x}]$ since, by Lemma 3.2 the flexible property is satisfied for the dot product; thus $[z, y, e] = 0$ since $[e, y, z] = 0$, by (3.37). Then the identity (3.48) is equivalent to

$$\begin{aligned} (xyz) &= [z, \overline{x}, y] + z \cdot (\overline{x} \cdot y) - [z, y, \overline{x}] + (x \cdot \overline{z}) \cdot y - y \cdot (\overline{z} \cdot x), \text{ i.e.} \\ (xyz) &= (z \cdot \overline{x}) \cdot y - [z, y, \overline{x}] + (x \cdot \overline{z}) \cdot y - y \cdot (\overline{z} \cdot x) \\ &= (x \cdot \overline{z} + \overline{x \cdot \overline{z}}) \cdot y - y \cdot (\overline{z} \cdot x) + [\overline{x}, y, z]. \end{aligned} \tag{3.49}$$

Furthermore, since $x \cdot \overline{z} + \overline{x \cdot \overline{z}} = T(x \cdot \overline{z})$ by the flexible property, and

$$T(x \cdot y) = T(\overline{x \cdot y}) = T(\overline{y} \cdot \overline{x}) = T((T(y) - y) \cdot (T(x) - x)) = T(y \cdot x),$$

it follows that (3.49) is equivalent to

$$(xyz) = T(\overline{z} \cdot x)y - y \cdot (\overline{z} \cdot x) + [\overline{x}, y, z] = y \cdot (\overline{\overline{z} \cdot x}) = y \cdot (\overline{x} \cdot z) + [\overline{x}, y, z],$$

hence the identity (3.44) is fulfilled. □

## REFERENCES


**1.** B. N. Allison, A class of nonassociative algebras with involution containing the class of Jordan algebras, *Math. Ann.* **237**(2) (1978), 133–156.

**2.** B. N. Allison, Models of isotropic simple Lie algebras, *Commun. Algebra* **7**(17) (1979), 1835–1875.

**3.** H. Asano and K. Yamaguti, A construction of Lie algebras by generalized Jordan triple systems of second order, *Nederl. Akad. Wetensch. Indag. Math.* **42**(3)(1980), 249–253.

**4.** H. Asano, Classification of non-compact real simple generalized Jordan triple systems of the second kind, *Hiroshima Math. J.* **21**(1991), 463–489.

**5.** W. Bertram, Complex and quaternionic structures on symmetric spaces – correspondence with Freudenthal–Kantor triple systems, in *Theory of Lie groups and manifolds* (Miyaoka R. and Tamaru H., Editors), Sophia Kokyuroku in Math., Vol. 45 (Sophia University, 2002), pp. 61–80.

**6.** A. Elduque, N. Kamiya and S. Okubo, Simple $(-1, -1)$ balanced Freudenthal–Kantor triple systems, *Glasg. Math. J.* **11**(2)(2003), 353–372.

**7.** A. Elduque, N. Kamiya and S. Okubo, $(-1, -1)$ balanced Freudenthal–Kantor triple systems and noncommutative Jordan algebras, *J. Algebra* **294**(1) (2005), 19–40.

**8.** J. R. Faulkner, Structurable triples, Lie triples, and symmetric spaces, *Forum Math.* **6** (1994), 637–50.

**9.** L. Frappat, A. Sciarrino and P. Sorba, *Dictionary on Lie algebras and superalgebras* (Academic Press, San Diego, CA, 2000), xxii+410 pp.

**10.** H. Freudenthal, Beziehungen der $E_7$ und $E_8$ zur Oktavenebene, *I. Indag. Math.* (1954), no. 16, 218–230.

**11.** N. Jacobson, Lie and Jordan triple systems, *Amer. J. Math.* **71**(1949), 149–170.

**12.** N. Jacobson, *Structure and representations of Jordan algebras,* Amer. Math. Soc. Colloq. Publ., Vol. 39 (Amer. Math. Soc., Providence, RI, 1968), x+453 pp.

**13.** V. G. Kac, Lie superalgebras, *Adv. Math.* **26**(1) (1977), 8–96.

**14.** N. Kamiya, A structure theory of Freudenthal–Kantor triple systems, *J. Algebra* **110**(1) (1987), 108–123.

**15.** N. Kamiya, A construction of anti-Lie triple systems from a class of triple systems, *Mem. Fac. Sci. Shimane Univ.* **22**(1988), 51–62.

**16.** N. Kamiya, A structure theory of Freudenthal–Kantor triple systems. II. Comment, *Math. Univ. St. Paul.* **38**(1) (1989), 41–60.

**17.** N. Kamiya, On $(\varepsilon, \delta)$-Freudenthal–Kantor triple systems, in *Nonassociative algebras and related topics* (Hiroshima, 1990), 65–75 (World Scientific, River Edge, NJ, 1991).

**18.** N. Kamiya, The construction of all simple Lie algebras over $\mathbb{C}$ from balanced Freudenthal–Kantor triple systems, *Contributions to general algebra*, 7 (Vienna, 1990), 205–213, Hölder-Pichler-Tempsky, Vienna, 1991.

**19.** N. Kamiya, On Freudenthal–Kantor triple systems and generalized structurable algebras, *Non-associative algebra and its applications* (Oviedo, 1993), 198–203, Math. Appl., 303 (Kluwer, Dordrecht, 1994).

**20.** N. Kamiya, On the Peirce decompositions for Freudenthal–Kantor triple systems, *Commun. Algebra* **25**(6) (1997), 1833–1844.

**21.** N. Kamiya, On a realization of the exceptional simple graded Lie algebras of the second kind and Freudenthal–Kantor triple systems, *Bull. Polish Acad. Sci. Math.* **46**(1) (1998), 55–65.

**22.** N. Kamiya and S. Okubo, On $\delta$-Lie supertriple systems associated with $(\varepsilon, \delta)$-Freudenthal–Kantor supertriple systems, *Proc. Edinburgh Math. Soc.* **43**(2) (2000), 243–260.

**23.** N. Kamiya and S. Okubo, A construction of Jordan superalgebras from Jordan-Lie triple systems, in *Non-Associative Algebra and Its Applications* (Lecture Notes in Pure and Applied Mathematics), Vol. 211 (Costa R., Grishkov A., Guzzo H. and Peresi L., Editors) (Marcel Dekker, New York, 2002), pp. 171–176.

**24.** N. Kamiya and S. Okubo, A construction of simple Jordan superalgebra of $F$ type from a Jordan–Lie triple system, *Ann. Mat. Pura Appl. (4)* **181**(3) (2002), 339–348.

**25.** N. Kamiya and S. Okubo, Construction of Lie superalgebras $D(2, 1; \alpha)$, $G(3)$ and $F(4)$ from some triple systems, *Proc. Edinburgh Math. Soc.* **46**(1) (2003), 87–98.

**26.** N. Kamiya and S. Okubo, On generalized Freudenthal–Kantor triple systems and Yang–Baxter equations, *Proc. 24th Inter. Colloq. Group Theoretical Methods in Physics* (Gazeau J-P., Kerner R., Antoine J-P., Metens S. and Thibon J-Y., Editors) (IPCS, Paris, 2003) Vol. 173 (2003), 815–818.

**27.** N. Kamiya and S. Okubo, A construction of simple Lie superalgebras of certain types from triple systems, *Bull. Austral. Math. Soc.* **69**(1) (2004), 113–123.

**28.** N. Kamiya, Examples of Peirce decomposition of generalized Jordan triple system of second order-Balanced cases, in *Noncommutative geometry and representation theory in* *mathematical physics*, Contemp. Math., Vol. 391 (Fuchs J., Mickelsson J., Rosenblioum G., Stolin A. and Westerberg A., Editors) (AMS, Providence, RI, 2005), 157–165.

**29.** N. Kamiya and S. Okubo, Composition, quadratic, and some triple systems, in *Non-associative algebra and its applications*, Lecture Notes Pure Appl. Math., Vol. 246 (Sabinin L., Sbitneva L. and Shestakov I., Editors) (Chapman & Hall, Boca Raton, FL, 2006), 205–231.

**30.** N. Kamiya and D. Mondoc, A new class of nonassociative algebras with involution, *Proc. Japan Acad. Ser.* A **84**(5) (2008), 68–72.

**31.** N. Kamiya, D. Mondoc and S. Okubo, A structure theory of $(-1, -1)$-Freudenthal–Kantor triple systems, *Bull. Australian Math. Soc.* **81**(2010), 132–155.

**32.** N. Kamiya and D. Mondoc, On anti-structurable algebras and extended Dynkin diagrams, *J. Gen. Lie Theory Appl.* **3**(3) (2009), 185–192.

**33.** S. Kaneyuki and H. Asano, Graded Lie algebras and generalized Jordan triple systems, *Nagoya Math. J.* **112**(1988), 81–115.

**34.** I. L. Kantor, Graded Lie algebras, *Trudy Sem. Vect. Tens. Anal.* **15**(1970), 227–266.

**35.** I. L. Kantor, Some generalizations of Jordan algebras, *Trudy Sem. Vect. Tens. Anal.* **16**(1972), 407–499.

**36.** I. L. Kantor, Models of exceptional Lie algebras, *Soviet Math. Dokl.* **14**(1) (1973), 254–258.

**37.** I. L. Kantor, A generalization of the Jordan approach to symmetric Riemannian spaces, in *The Monster and Lie algebras* (Columbus, OH, 1996), Ohio State Univ. Math. Res. Inst. Publ., 7 (Ferrar J. and Harada K., Editors) (de Gruyter, Berlin, 1998), 221–234.

**38.** I. L. Kantor and N. Kamiya, A Peirce decomposition for generalized Jordan triple systems of second order, *Commun. Algebra* **31**(12) (2003), 5875–5913.

**39.** M. Koecher, Embedding of Jordan algebras into Lie algebras I, *Amer. J. Math* **89** (1967), 787–816.

**40.** M. Koecher, Embedding of Jordan algebras into Lie algebras II, *Amer. J. Math* **90**(1968), 476–510.

**41.** W. G. Lister, A structure theory of Lie triple systems, *Trans. Amer. Math. Soc.* **72**(1952), 217–242.

**42.** K. Meyberg, *Lectures on algebras and triple systems*, Lecture Notes (The University of Virginia, Charlottesville, VA, 1972), v+226 pp.

**43.** D. Mondoc, Models of compact simple Kantor triple systems defined on a class of structurable algebras of skew-dimension one, *Commun. Algebra* **34**(10) (2006), 3801–3815.

**44.** D. Mondoc, On compact realifications of exceptional simple Kantor triple systems, *J. Gen. Lie Theory Appl.* **1**(1) (2007), 29–40.

**45.** D. Mondoc, Compact realifications of exceptional simple Kantor triple systems defined on tensor products of composition algebras, *J. Algebra* **307**(2) (2007), 917–929.

**46.** D. Mondoc, Compact exceptional simple Kantor triple systems defined on tensor products of composition algebras, *Commun. Algebra* **35**(11) (2007), 3699–3712.

**47.** E. Neher, *Jordan triple systems by the grid approach*, Lecture Notes in Mathematics, Vol. 1280 (Springer, Berlin, 1987), xii+193 pp.

**48.** S. Okubo, *Introduction to octonion and other non-associative algebras in physics*, Montroll Memorial Lecture Series in Mathematical Physics, 2 (Cambridge University Press, Cambridge, MA, 1995), xii+136 pp.

**49.** S. Okubo and N. Kamiya, Jordan–Lie superalgebra and Jordan–Lie triple system, *J. Algebra* **198**(2) (1997), 388–411.

**50.** S. Okubo and N. Kamiya, Quasi-classical Lie superalgebras and Lie supertriple systems, *Commun. Algebra* **30**(8) (2002), 3825–3850.

**51.** S. Okubo, Symmetric triality relations and structurable algebras, *Linear Algebra Appl.* **396** (2005), 189–222.

**52.** M. Scheunert, *The theory of Lie superalgebras. An introduction*. Lecture Notes in Mathematics, Vol. 716 (Springer, Berlin, 1979), x+271 pp.

**53.** J. Tits, Une classe d'algèbres de Lie en relation avec les algèbres de Jordan, *Nederl. Acad. Wetensch. Proc. Ser.* A **65**(1962), 530–535.

**54.** K. Yamaguti and A. Ono, On representations of Freudenthal–Kantor triple systems $U(\varepsilon, \delta)$, *Bull. Fac. School Ed. Hiroshima Univ.* **7**(2) (1984), 43–51.

**55.** K. A. Zhevlakov, A. M. Slinko, I. P. Shestakov and A. I. Shirshov, *Rings that are nearly associative* (Academic Press, New York, 1982), xi+371 pp.

**56.** E. Zelmanov, Primary Jordan triple systems, *Sibirsk. Mat. Zh.* **24**(4) (1983), 23–37.