# RINGS OF MEROMORPHIC FUNCTIONS ON NON-COMPACT RIEMANN SURFACES

JAMES KELLEHER

**1. Introduction.** In this paper we shall be concerned with the algebraic structure of certain rings of functions meromorphic on a non-compact (connected) Riemann surface $\Omega$. In this setting, $A = A(\Omega)$ and $K = K(\Omega)$ denote (respectively) the ring of all complex-valued functions analytic on $\Omega$ and its field of quotients, the field of functions meromorphic on $\Omega$. The rings considered here are those subrings of $K$ containing $A$, which we term *$A$-rings* of $K$. Most of the results given here were previously announced without proof (**15**) and are contained in the author's doctoral dissertation (**16**), completed at the University of Illinois under the direction of Professor M. Heins, whose encouragement and advice are gratefully acknowledged.

The ring $A$ itself has been extensively investigated in recent years. The fundamental result here is the theorem of Helmer (**10**), which states that every finitely generated ideal of $A$ is principal in $A$, for almost all of the results obtained on the ideal theory of $A$ are based on this theorem. We shall see that many of these results extend to any $A$-ring of $K$, for we are able to identify these as the rings of quotients of $A$ with respect to its multiplicative subsets. This allows us to apply general results of commutative algebra in a natural and uniform way to the rings in question. Indeed, many results obtained on the ideal theory of $A$ by previous authors, such as Henriksen (**11**; **12**) and Banaschewski (**3**), can be derived almost trivially from the algebraic theory of rings of quotients, which is presented below in brief outline, together with the requisite valuation theory.

The basic method used is the study and exploitation of the valuation theory of $K$, previously considered by Schilling (**19**) and Alling (**2**). Helmer's theorem leads to an immediate identification of those $A$-rings of $K$ which are valuation rings as the localizations of $A$ at its prime ideals. This result then implies that every $A$-ring is an intersection of valuation rings and a ring of quotients of $A$.

Having identified the $A$-rings in this way, some applications of the algebraic theory are made to the ideal theory of $A$ itself, and it is shown that a large part of the algebraic structure of this ring, such as Helmer's theorem, extends to any $A$-ring of $K$. The ideal theory of $A$ is then applied to the problem of classification of $A$-rings, some of which can be characterized geometrically

Received August 25, 1967. This paper was partially supported by the National Science Foundation under Contract NSF GP-5455.

and satisfy very restrictive algebraic conditions. These results and methods show that the theory of rings of quotients, and valuation theory in particular, are the natural framework in which the rings in question should be considered.

We also briefly discuss the extent to which the conformal structure of $\Omega$ is determined by the algebraic structure of its $A$-rings. Problems of this type were initially considered by Bers (**4**), who showed that if $\Omega$ is a plane region, then $\Omega$ is determined to within conformal or anti-conformal equivalence by the algebraic structure of $A(\Omega)$, and this was later generalized to arbitrary $\Omega$ by Nakai (**17**). The corresponding problem with $A(\Omega)$ replaced by the field $K(\Omega)$ had long been outstanding and had been considered by Heins (**9**) and Royden (**18**), but was solved in full generality only recently by Iss'sa (**13**). Among other things, Iss'sa proved that the ring $A(\Omega)$ can be characterized algebraically in $K(\Omega)$, and hence it follows from the Bers-Nakai theorem that $\Omega$ is determined to within conformal or anti-conformal equivalence by the algebraic structure of $K(\Omega)$. Here we shall prove a theorem of this type relating to isomorphisms between rings of meromorphic functions, but we shall not require the Nakai theorem in the proof.

**2. Algebraic preliminaries.** In this section we discuss some results in commutative algebra which are particularly relevant to rings of analytic and meromorphic functions. For further details on this material, the reader is referred to Bourbaki (**5**) and Zariski and Samuel (**20**). These results, while familiar to algebraists, may not be known to many analysts, and therefore the required theory will be given in some detail, though no proofs are presented and there is no attempt at complete generality. In the following, $A$ will denote an arbitrary commutative integral domain with identity $1 \neq 0$ and $K$ its field of quotients. For the most part, we follow the notation and terminology of (**20**), though the prime and primary ideals of $A$ are always assumed to be *proper* ideals.

2.1. *Rings of quotients.* A non-empty subset $S$ of the integral domain $A$ is said to be a *multiplicative subset* of $A$ if $xy \in S$ whenever $x, y \in S$. For each such set $S$, *the ring of quotients* of $A$ with respect to $S$ is the set $S^{-1}A = \{x/y\colon x \in A, y \in S\}$, which is evidently a subring of $K$ which contains $A$. Also, $K$ is the field of quotients of $S^{-1}A$, and each $x \in S$ is a unit of $S^{-1}A$. If $B$ is a ring of quotients of $A$, then $\{x \in A\colon x^{-1} \in B\}$ is the largest of the multiplicative subsets $S$ of $A$ for which $B = S^{-1}A$.

Let $S$ be a multiplicative subset of $A$, let $A^* = S^{-1}A$, and let $\Lambda$ and $\Lambda^*$ denote the collections of all ideals of $A$ and $A^*$, respectively. Given $I \in \Lambda$, $e(I)$ will denote the ideal of $A^*$ generated by $I$ (considered as a subset of $A^*$), the *extension* of $I$ to $A^*$. Then

$$e(I) = \{x/y\colon x \in I, y \in S\} = \{xy\colon x \in I, y \in A^*\},$$

and $e(I)$ is a proper ideal of $A^*$ if and only if $I \cap S = \emptyset$. If $I^* \in \Lambda^*$, then $r(I^*) = I^* \cap A$ is an ideal of $A$, the *restriction* of $I^*$ to $A$, and $r(I^*)$ is a proper ideal of $A$ whenever $I^*$ is a proper ideal of $A^*$. Further, $e[r(I^*)] = I^*$ for all $I^* \in \Lambda^*$; thus, every ideal of $A^*$ is an extended ideal, and for $I \in \Lambda$ we have that

$$I \subset r[e(I)] = \{x \in A \colon xy \in I \text{ for some } y \in S\}.$$

We denote by $\Lambda_r$ the collection of all restricted ideals of $A$; therefore, $r[e(I)] = I$ for all $I \in \Lambda_r$. We have that $\Lambda = \Lambda_r$ if and only if $S$ contains no non-units of $A$, in which case $A = A^*$. The mappings $e\colon \Lambda_r \to \Lambda^*$ and $r\colon \Lambda^* \to \Lambda_r$ are reciprocal one-to-one inclusion-preserving correspondences between $\Lambda_r$ and $\Lambda^*$.

Every primary ideal (hence every prime ideal) of $A$ which is disjoint from $S$ is a restricted ideal of $A$, and the mappings $e$ and $r$ define reciprocal one-to-one correspondences between the primary (prime) ideals of $A^*$ and the primary (prime) ideals of $A$ which do not intersect $S$.

A proper ideal of $A$ is said to be *$S$-maximal* if it does not intersect $S$ and is not properly contained in any ideal of $A$ which does not intersect $S$. Every $S$-maximal ideal is prime, and every ideal of $A$ which does not intersect $S$ is contained in an $S$-maximal ideal of $A$. The mappings $e$ and $r$ define reciprocal one-to-one correspondences between the maximal ideals of $A^*$ and the $S$-maximal ideals of $A$.

If $I \in \Lambda$ is principal (finitely generated) in $A$, then $e(I)$ is principal (finitely generated) in $A^*$ and is generated in $A^*$ by the generator(s) of $I$ in $A$. Every principal (finitely generated) ideal $I^*$ of $A^*$ is the extension of a principal (finitely generated) ideal of $A$, though the restriction of $I^*$ to $A$ need not be finitely generated in $A$. If $A$ is a principal ideal ring (a Noetherian ring), then so is $A^*$.

If $I^* \in \Lambda^*$ has radical $J^*$ in $A^*$, then $r(I^*)$ has radical $r(J^*)$ in $A$. Also, if $I \in \Lambda$ has radical $J$ in $A$, then $e(J)$ is contained in the radical of $e(I)$ in $A^*$, and $e(J)$ is the radical of $e(I)$ if $I$ is a restricted ideal. The intersection of any set of restricted ideals of $A$ is again a restricted ideal of $A$. Finally, if $I$ is the product of the ideals $I_1, \ldots, I_n \in \Lambda$, then $e(I)$ is the product of the ideals $e(I_1), \ldots, e(I_n) \in \Lambda^*$.

2.2. *Localization.* If $P$ is a prime ideal of $A$, then $S = A - P$ is a multiplicative subset of $A$. In this case, we write $S^{-1}A = A_P$, the *localization* of $A$ at $P$. The set $\{x \in A_P \colon x^{-1} \notin A_P\}$ of all non-units of $A_P$ is the extension $e(P)$ of $P$ to $A_P$ and is the (unique) maximal ideal of $A_P$, for $P$ is evidently the unique $S$-maximal ideal of $A$.

If $B$ is a ring of quotients of $A$, let $\Sigma$ denote the collection of all restrictions to $A$ of the maximal ideals of $B$. Then each $P \in \Sigma$ is a prime ideal of $A$ and

$$\{x \in A \colon x^{-1} \in B\} = \bigcap_{P \in \Sigma} (A - P).$$

This is a multiplicative subset of $A$ with ring of quotients $B$, and thus each ring of quotients of $A$ is determined by some collection of prime ideals of $A$.

https://doi.org/10.4153/CJM-1969-030-3 Published online by Cambridge University Press

As mentioned above, if $A^* = S^{-1}A$, then the extension mapping $e: \Lambda \to \Lambda^*$ is not one-to-one (except in the trivial case, $A = A^*$) and does not distinguish all the ideals of $A$. However, it is possible to recover an ideal $I$ of $A$ from its extensions to the rings of quotients of $A$. More precisely, if $\Sigma$ denotes the collection of *all* maximal ideals of $A$, then for every ideal $I \in \Lambda$ we have that

$$I = \bigcap_{M \in \Sigma} r_M[e_M(I)],$$

where for $M \in \Sigma$, $r_M$ and $e_M$ denote the restriction and extension mappings relative to the multiplicative subset $A - M$. Obviously, for each $I \in \Lambda$, it suffices to consider (in the above intersection) only those $M \in \Sigma$ which contain $I$. In particular, if $I$ is contained in only one maximal ideal $M$ of $A$, then $I = A \cap e_M(I)$.

2.3. *Valuation theory.* In this section $K$ will denote an arbitrary field. A *valuation ring* of $K$ is a subring $R$ of $K$ such that for each $x \in K - \{0\}$, $x \in R$ or $x^{-1} \in R$. We say that $R$ is *non-trivial* if $R \neq K$. Evidently, $R$ is an integral domain with field of quotients $K$, and every subring of $K$ which contains $R$ is also a valuation ring of $K$. Note that if $x, y \in R$, then $x$ divides $y$ in $R$ or $y$ divides $x$ in $R$.

Given a valuation ring $R$ of $K$, the set $M = M(R) = \{x \in R: x^{-1} \notin R\}$ of all non-units of $R$ is the (unique) maximal ideal of $R$, and $R$ is trivial if and only if $M = \{0\}$. The set $\Lambda(R)$ of all ideals of $R$ is totally ordered under set inclusion. The intersection of any collection of prime (primary) ideals of $R$ is again a prime (primary) ideal of $R$, and every ideal of $R$ has prime radical in $R$. The *rank* of $R$ is the order type of the (totally ordered) collection of all non-zero prime ideals of $R$, and $R$ is trivial if and only if $R$ has rank zero. Every finitely generated ideal of $R$ is principal; therefore, $R$ is a principal ideal ring if $R$ is Noetherian.

THEOREM 2.3.1. *Suppose that $K$ is the field of quotients of an integral domain $A$ and that $R$ is a valuation ring of $K$ which contains $A$. Then $P = A \cap M(R)$ is a prime ideal of $A$, $R$ contains the localization $A_P$ of $A$ at $P$, and $A_P \cap M(R)$ is the maximal ideal of $A_P$. Further, if $A_P$ is a valuation ring, then $R = A_P$. Thus, if $A_P$ is a valuation ring for every prime ideal $P$ of $A$, then every valuation ring of $K$ which contains $A$ is of the form $A_P$ for some prime ideal $P$ of $A$.*

The integral domains $A$ for which each localization $A_P$ is a valuation ring are commonly called *Prüfer domains*. These have been investigated in a number of recent articles, and the reader is referred to these for further details; cf. Gilmer and Ohm (7) for such references.

A *valuation* of $K$ is a mapping $v$ from $K - \{0\} = K^*$ into the group of integers $N$ so that $v(xy) = v(x) + v(y)$ for all $x, y \in K^*$ and $v(x - y) \geqq \text{Min}\{v(x), v(y)\}$ for all $x, y \in K^*$ with $x \neq y$. The valuation $v$ is said to be non-trivial if $v(x) \neq 0$ for some $x \in K^*$. In this case, the range of $v$ is a non-zero subgroup of $N$, hence is isomorphic to $N$; we shall always suppose that a non-zero valuation of $K$ maps $K^*$ *onto* $N$. (*Note*: this notion may be generalized considerably by allowing the range of $v$ to be an arbitrary totally ordered abelian group; however, we shall not consider this here.)

We shall extend $v$ to $K$ by defining $v(0) = \infty$, where $\infty$ is an element such that $\infty > n$ and $\infty + n = \infty$ for all $n \in N$. With this convention, the defining relations above are valid for all $x, y \in K$.

The connection between valuations and valuation rings is described by the following theorem.

THEOREM 2.3.2. *If $v$ is a non-trivial valuation of $K$, then $R_v = \{x \in K\colon v(x) \geqq 0\}$ is a non-trivial valuation ring of $K$ with maximal ideal $M_v = \{x \in K\colon v(x) > 0\}$ and $R_v$ is a Noetherian ring. Conversely, if $R \neq K$ is a Noetherian valuation ring of $K$, then there exists a unique valuation $v$ of $K$ such that $R = R_v$.*

Let $R$ be a non-trivial valuation ring of $K$ for which the maximal ideal $M = M(R)$ is *principal*, and let $x \in R$ generate $M$ in $R$. Then for each $k \geqq 1$, the set $M^k = \{x_1x_2 \ldots x_k\colon x_1, x_2, \ldots, x_k \in M\}$ is the principal ideal of $R$ generated by $x^k$. Further, $P = \bigcap_{k\geqq 1} M^k$ is the largest prime ideal of $R$ which is properly contained in $M$; therefore, $R$ has rank one if and only if $P = 0$. In this case, the function $v$ defined on $R - \{0\}$ by

$$v(y) = \sup\{k \geqq 0\colon y \in M^k\}, \qquad y \in R - \{0\},$$

can be uniquely extended to a valuation $v^*$ of $K$, and $R$ is the valuation ring of $v^*$; hence, $R$ is Noetherian. Then every non-zero proper ideal of $R$ is of the form $M^k$ for some $k \geqq 1$ and is a primary ideal of $R$ with radical $M$. In particular, every non-trivial Noetherian valuation ring of $K$ has rank one. (The converse is false.)

## 3. Rings of meromorphic functions.

In this section we begin our discussion of rings of functions meromorphic on a non-compact Riemann surface $\Omega$. We first consider the ring $A(\Omega)$ of functions analytic on $\Omega$ and discuss briefly some results on the ideal theory of this ring. The basic result here is the theorem of Helmer, which states that every finitely generated ideal of $A(\Omega)$ is a principal ideal of $A(\Omega)$. This theorem is fundamental for the further study of $A(\Omega)$ and is the basis of our investigation of the rings of quotients of $A(\Omega)$, which we consider in subsequent sections.

3.1. *The ring* $A(\Omega)$. Let $\Omega$ be a non-compact Riemann surface. We denote by $A(\Omega)$ the collection of all mappings from $\Omega$ into the complex plane $C$ which are analytic on $\Omega$, and by $K(\Omega)$ the collection of all mappings from $\Omega$ into the extended complex plane which are meromorphic on $\Omega$. Given $f \in K(\Omega) - \{0\}$, we denote by $\partial_f$ the *divisor* of $f$ (see Heins (**8**, p. 10)), which is an integer-valued function on $\Omega$ supported on the discrete set $\{a \in \Omega\colon f(a) = 0, \infty\}$.

https://doi.org/10.4153/CJM-1969-030-3 Published online by Cambridge University Press

Evidently, $f$ is analytic at $a \in \Omega$ if and only if $\partial_f(a) \geqq 0$, and $f \in A(\Omega)$ if and only if $\partial_f(a) \geqq 0$ for all $a \in \Omega$. Further, if $f, g \in K(\Omega) - \{0\}$, then

$$\partial_{fg} = \partial_f + \partial_g,$$
$$\partial_{f-g} \geqq \operatorname{Min}\{\partial_f, \partial_g\}, \qquad f \neq g,$$
$$Z(f) = \{a \in \Omega: f(a) = 0\} = \{a \in \Omega: \partial_f(a) > 0\}.$$

The theorem of Weierstrass for non-compact Riemann surfaces (Florack (**6**)) then states that for every integer-valued function $\partial$ on $\Omega$ which is supported on a discrete subset of $\Omega$, there exists $f \in K(\Omega) - \{0\}$ such that $\partial = \partial_f$.

Now, under the operations of pointwise addition and multiplication of functions, $A(\Omega)$ is a commutative integral domain with identity (indeed, $A(\Omega)$ is an algebra over the complex field $C$, since it contains the constant functions, which we identify with $C$), and it follows from the Weierstrass theorem that $K(\Omega)$ is the field of quotients of $A(\Omega)$. A function $f \in A(\Omega)$ is a unit of $A(\Omega)$ if and only if $\partial_f \equiv 0$. Given $f, g \in A(\Omega)$, $f$ divides $g$ in $A(\Omega)$ if and only if $\partial_f \leqq \partial_g$, and $f$ and $g$ are associates in $A(\Omega)$ if and only if $\partial_f = \partial_g$. The irreducible elements of $A(\Omega)$ are exactly those functions $f \in A(\Omega)$ for which there exists $a \in \Omega$ such that $Z(f) = \{a\}$ and $\partial_f(a) = 1$. Finally, if $f \in K(\Omega)$, then there exist $g, h \in A(\Omega)$ such that $Z(g) \cap Z(h) = \emptyset$, $f = g/h$.

Given $a \in \Omega$, we define a map $v_a: K(\Omega) - \{0\} \to N$ by $v_a(f) = \partial_f(a)$. It is evident from the above that $v_a$ is a valuation of $K(\Omega)$. The valuation ring of $v_a$ is

$$R_a = \{f \in K(\Omega): v_a(f) \geqq 0\} = \{f \in K(\Omega): f(a) \neq \infty\},$$
$$M_a = M(R_a) = \{f \in K(\Omega): v_a(f) > 0\} = \{f \in K(\Omega): f(a) = 0\}.$$

Thus (Theorem 2.3.2), for each $a \in \Omega$, $R_a$ is a non-trivial Noetherian valuation ring of $K(\Omega)$ which contains the ring $A(\Omega)$.

The basic theorem regarding the ideal theory of $A(\Omega)$ is the theorem of Helmer (cf. Helmer (**10**); for a proof in the general case, see Royden (**18**)). More precisely, we have the following theorem.

HELMER'S THEOREM 3.1.1. *Let $f_1, \ldots, f_n \in A(\Omega)$, Then the ideal of $A(\Omega)$ generated by $f_1, \ldots, f_n$ is the principal ideal of $A(\Omega)$ generated by $f$, where $f$ is any element of $A(\Omega)$ for which $\partial_f = \operatorname{Min}\{\partial_{f_1}, \ldots, \partial_{f_n}\}$. In particular, if $Z(f_1) \cap \ldots \cap Z(f_n) = \emptyset$, then the ideal generated by $f_1, \ldots, f_n$ is all of $A(\Omega)$.*

First note that if $I$ is any ideal of $A(\Omega)$ and if $f_1, \ldots, f_n \in I$, then $f \in I$ for each $f \in A(\Omega)$ such that $\partial_f = \operatorname{Min}\{\partial_{f_1}, \ldots, \partial_{f_n}\}$. Hence, if $I$ is a proper ideal of $A(\Omega)$, so that $f$ cannot be a unit of the ring, we must have that $Z(f_1) \cap \ldots \cap Z(f_n) \neq \emptyset$. This leads to the following definition, first introduced by Kakutani (**14**).

*Definition* 3.1.2. A non-empty collection $\Delta$ of non-empty discrete subsets of $\Omega$ is a *$\delta$-filter*

(i) If $D_1, D_2 \in \Delta$, then $D_1 \cap D_2 \in \Delta$;

(ii) If $D_1 \in \Delta$ and if $D_2$ is a discrete subset of $\Omega$ containing $D_1$, then $D_2 \in \Delta$. We say that $\Delta$ is *fixed* if there exists $a \in \Omega$ such that $a \in D$ for all $D \in \Delta$; otherwise, $\Delta$ is *free*. Finally, $\Delta$ is said to be *maximal* if $\Delta$ is not properly contained in any $\delta$-filter of $\Omega$.

It follows from Zorn's lemma that every $\delta$-filter of $\Omega$ is contained in a maximal $\delta$-filter. Further, $\Delta$ is maximal if and only if: for each non-empty discrete subset $D_1$ of $\Omega$ which is not an element of $\Delta$, there exists $D_2 \in \Delta$ so that $D_1 \cap D_2 = \emptyset$. The fixed maximal $\delta$-filters are exactly those $\Delta$ for which there exists $a \in \Omega$ so that $\Delta$ is the set of all discrete subsets of $\Omega$ which contain $a$. If $\Delta$ is maximal and if $D \in \Delta$ is such that $D = D_1 \cup D_2$ with $D_1$ and $D_2$ disjoint, then either $D_1 \in \Delta$ or $D_2 \in \Delta$. If $\Delta$ is not maximal and if $\Delta^*$ is maximal with $\Delta \subset \Delta^*$, then for each $D \in \Delta$ there exists $D^* \in \Delta^* - \Delta$ so that $D^* \subset D$. Finally, if $\Delta$ is free, then $\Delta$ contains no finite subsets of $\Omega$.

*Definition* 3.1.3. An ideal $I$ of $A(\Omega)$ is *fixed* if there exists $a \in \Omega$ so that $f(a) = 0$ for all $f \in I$; otherwise, $I$ is *free*. (Evidently, every proper, principal ideal of $A(\Omega)$ is fixed, but the converse is not true.)

The connection between the ideals of $A(\Omega)$ and the $\delta$-filters of $\Omega$ is given by the following theorem.

THEOREM 3.1.4. *If $I$ is a non-zero proper ideal of $A(\Omega)$, then $\Delta(I) = \{Z(f): f \in I, f \neq 0\}$ is a $\delta$-filter of $\Omega$, and $I$ is fixed if and only if $\Delta(I)$ is fixed. Conversely, if $\Delta$ is a $\delta$-filter of $\Omega$, then $I(\Delta) = \{f \in A(\Omega): Z(f) \in \Delta\} \cup \{0\}$ is a non-zero proper ideal of $A(\Omega)$, and $I(\Delta)$ is fixed if and only if $\Delta$ is fixed. Furthermore, the mappings $I \to \Delta(I)$ and $\Delta \to I(\Delta)$ define reciprocal, one-to-one correspondences between the set of all maximal ideals of $A(\Omega)$ and the set of all maximal $\delta$-filters of $\Omega$.*

If $\Delta$ is a $\delta$-filter of $\Omega$, then $\Delta = \Delta[I(\Delta)]$, and if $I$ is a non-zero proper ideal of $A(\Omega)$, then

$$I \subset I[\Delta(I)] = \{f \in A(\Omega): Z(f) = Z(g) \text{ for some } g \in I\},$$

equality holding if $I$ is a maximal ideal. If $I$ is a proper, free ideal of $A(\Omega)$, then each $f \in I$ has infinitely many zeros in $\Omega$, and the fixed maximal ideals are the ideals $I_a = \{f \in A(\Omega): f(a) = 0\}$, where $a \in \Omega$, the ideals generated by an irreducible element of $A(\Omega)$. To show that free (maximal) ideals of $A(\Omega)$ exist, it suffices to exhibit a free $\delta$-filter of $\Omega$; to this purpose, simply let $D$ be any infinite discrete subset of $\Omega$ and define $\Delta$ to be the collection of all discrete subsets of $\Omega$ which contain all but finitely many points of $D$. It is evident that the elements of $A(\Omega)$ belonging to free maximal ideals of $A(\Omega)$ are exactly those functions $f$ for which $Z(f)$ is an infinite set.

Most of the results obtained on the ideal theory of $A(\Omega)$ are based on Theorem 3.1.4; cf. Henriksen (**11**; **12**) and Banaschewski (**3**). In the following, we shall use several of these results. First, if $I_1, \ldots, I_n$ are ideals of $A(\Omega)$,

https://doi.org/10.4153/CJM-1969-030-3 Published online by Cambridge University Press

then the set $I_1 \ldots I_n = \{x_1 \ldots x_n: x_k \in I_k, 1 \leqq k \leqq n\}$ is an ideal of $A(\Omega)$, this following easily from Helmer's theorem. Second, if $P$ is a non-maximal prime ideal of $A(\Omega)$, then $P^2 = P$. Finally, if $P$ is a prime ideal of $A(\Omega)$, then $\Delta(P)$ is a *maximal* $\delta$-filter of $\Omega$.

3.2. *Rings of quotients of* $A(\Omega)$. We now consider the rings of quotients of $A = A(\Omega)$, the basic result being Theorem 3.2.5 where it is stated that every subring of $K = K(\Omega)$ which contains $A$ is the ring of quotients of $A$ with respect to some multiplicative subset of $A$. The general theory outlined in the second section will apply in a natural way to these rings and yield further information on the ideal theory of $A$ and the valuation theory of $K$. We first give some applications of the general theory to the study of the non-maximal prime ideals of $A$; cf. Henriksen (**12**). Throughout this section, $A$ will denote the ring $A(\Omega)$ and $K$ its field of quotients, where $\Omega$ is a non-compact Riemann surface.

THEOREM 3.2.1. *Every non-zero, non-maximal prime ideal of* $A$ *is a free ideal of* $A$.

*Proof*. Suppose that $P$ is a non-maximal fixed prime ideal of $A$ and let $a$ be the (unique) point of $\Omega$ for which $P \subset I_a$. Letting $S = A - I_a$, $e(P)$ is a non-maximal prime ideal of $R_a = S^{-1}A$. However, $R_a$ is a valuation ring of $K$ with rank one; whence, $e(P) = 0$ and $P = r[e(P)] = 0$.

*Definition* 3.2.2. Given $f \in K - \{0\}$ we define $\pi(f) = \sup\{\partial_f(a): a \in \Omega\}$; thus, $\pi(f)$ is either $+\infty$ or a non-negative integer, and we define $\pi(0) = +\infty$.

THEOREM 3.2.3. *The elements of* $A$ *which belong to non-maximal prime ideals of* $A$ *are exactly those* $f \in A$ *for which* $\pi(f) = +\infty$.

*Proof*. Define $S = \{f \in A: \pi(f) < +\infty\}$. Since $\pi(fg) \leqq \pi(f) + \pi(g)$ for all $g \in K$, $S$ is a multiplicative subset of $A$. Further, if $f \in A - S$, then $fg \in A - S$ for all $g \in A$; thus, the ideal generated by $f$ in $A$ does not intersect $S$ and is therefore contained in an $S$-maximal ideal of $A$. However, no such ideal of $A$ is a maximal ideal of $A$, for if $J$ is a maximal ideal of $A$, then

$$J = I[\Delta(J)] = \{g \in A: Z(g) = Z(h) \text{ for some } h \in J\},$$

and each such $J$ contains elements of $S$. Thus, non-maximal prime ideals exist and each $f \in A - S$ belongs to such an ideal. Finally, suppose that $P$ is a non-maximal prime ideal and let $f \in P$. Given $n \geqq 1$, $P^n = P$ and there exist $f_1, \ldots, f_n \in P$ so that $f = f_1 \ldots f_n$. Since $P$ is proper, $f_1, \ldots, f_n$ have a common zero in $\Omega$; whence, $\partial_f(a_n) \geqq n$ for some $a_n \in \Omega$. Since $n \geqq 1$ is arbitrary, $\pi(f) = +\infty$.

To begin our study of the rings of quotients of $A$ we first consider the localizations of $A$. Now, if $f \in K$, then there exist $g, h \in A$ with $f = g/h$ and $Z(g) \cap Z(h) = \emptyset$. By Helmer's theorem, $f$ and $g$ generate $A$ and cannot both belong to any proper ideal of $A$, and thus to no prime ideal of $A$. Therefore, $A_P$ is a valuation ring of $K$ for each prime ideal $P$ of $A$, and hence (Theorem 2.3.1) *the valuation rings of $K$ which contain $A$ are exactly the localizations $A_P$ of $A$ at its prime ideals.* Equivalently, $A$ is a Prüfer domain. The theory of these rings could now be applied; however, for the ring $A = A(\Omega)$, the appropriate results are easily obtained directly.

*Definition* 3.2.4. An *$A$-ring* of $K$ is a subring $B$ of $K$ which contains the ring $A$.

We first note that each $A$-ring is a Prüfer domain; for, if $B$ is an $A$-ring of $K$ and $Q$ a prime ideal of $B$, then $P = Q \cap A$ is a prime ideal of $A$ and $A_P \subset B_Q$; therefore, $B_Q$ is a valuation ring. Indeed, $A_P = B_Q$, by Theorem 2.3.1.

THEOREM 3.2.5. *Every $A$-ring of $K$ is the ring of quotients of $A$ with respect to some multiplicative subset of $A$.*

*Proof.* Let $B$ be an $A$-ring of $K$ and let $\Sigma$ denote the collection of all maximal ideals of $B$; therefore, $B^* = \bigcap_{Q\in\Sigma} B_Q$ is an $A$-ring of $K$ which contains $B$. Further, if $f \in B^*$ and $f = g/h$ with $g$, $h \in A$ and $Z(g) \cap Z(h) = \emptyset$, then, obviously, $h \notin Q$ for all $Q \in \Sigma$, thus $h$ is a unit of $B$ and $f \in B$; whence, $B = B^*$. It follows from the remarks above that there is some collection $\Sigma'$ of prime ideals of $A$ for which $B = \bigcap_{P\in\Sigma'} A_P$, and we claim that $B = S^{-1}A$ with $S = \bigcap_{P\in\Sigma'} (A - P)$. Since $S^{-1}A \subset A_P$ for all $P \in \Sigma'$, $S^{-1}A \subset B$. Conversely, suppose that $f \in B$ and take $g$, $h \in A$ so that $f = g/h$ and $Z(g) \cap Z(h) = \emptyset$. Then $h \notin S$, for otherwise we would have that $h \in P$ for some $P \in \Sigma'$ and $f \notin A_P \supset B$, a contradiction.

COROLLARY 3.2.6. *Let $B$ be an $A$-ring of $K$ and define $S = \{f \in A: 1/f \in B\}$. Then $B = S^{-1}A = \bigcap_{P\in\Sigma} A_P$, where $\Sigma$ is the set of all $S$-maximal ideals of $A$.*

From the fact that $A$ is a Prüfer domain we easily have a number of results previously obtained for the ring $A$ and these generalize to the $A$-rings of $K$. First, note that if $M$ is maximal in $A$, then the set of prime ideals of $A$ which are contained in $M$ is totally ordered by set inclusion, since this set is in one-to-one correspondence with the collection of all prime ideals of the valuation ring $A_M$. We now introduce the following definition; cf. (**20**, p. 340).

*Definition* 3.2.7. Let $B$ be an $A$-ring of $K$. A proper ideal $I$ of $B$ is a *valuation ideal* if there is a valuation ring $R$ of $K$ containing $B$ so that $I = J \cap B$ for some ideal $J$ of $R$.

THEOREM 3.2.8. *Let $I \neq \{0\}$ be a proper ideal of the $A$-ring $B$ of $K$. Then the following are equivalent:*

(i) *$I$ is a valuation ideal of $B$;*
(ii) *$I$ has prime radical in $B$;*
(iii) *$I$ is contained in only one maximal ideal of $B$.*

*Proof.* Let $I \neq \{0\}$ be a proper ideal of $B = S^{-1}A$. If (iii) holds and $M \supset I$ is maximal in $B$, then (i) follows immediately from the concluding remark of § 2.2, since $I = e(I) \cap B$, where $e(I)$ is the extension of $I$ to the valuation ring $B_M$. Second, if (i) holds, then there is a prime ideal $Q$ of $B$ so that $I = J \cap B$ for some ideal $J$ of $B_Q$. Therefore, $J$ has prime radical in $B_Q$, since $B_Q$ is a valuation ring, and thus $I$ has prime radical in $B$ since $I$ is a restricted ideal of $B$ (with respect to the ring of quotients $B_Q$ of $B$). Finally, if (ii) holds and $Q$ is the (prime) radical of $I$ in $B$, then $I \cap A$ has prime radical $P = Q \cap A$ in $A$. Now, $P$ is contained in a unique maximal ideal $M$ of $A$, since $\Delta(P)$ is a maximal $\delta$-filter of $\Omega$, and the set of all prime ideals of $A$ contained in $M$ is totally ordered by inclusion. Further, each prime ideal of $A$ containing $I \cap A$ contains $P$, hence, lies in $M$. Therefore, the union of all the prime ideals of $A$ which contain $I \cap A$ and do not intersect $S$ is the unique $S$-maximal ideal of $A$ which contains $I \cap A$; thus, (iii) follows from (ii).

The above proof shows that if $B$ is an $A$-ring and $M$ a maximal ideal of $B$, then the valuation ideals of $B$ contained in $M$ are exactly the ideals of $B$ which are restricted with respect to the ring of quotients $B_M$ of $B$. Consequently, this set of ideals is totally ordered, for it is in one-to-one correspondence with the collection of all ideals of the valuation ring $B_M$. Further, the intersection of any collection of valuation (primary, prime) ideals of $B$ contained in $M$ is again a valuation (primary, prime) ideal of $B$, since the corresponding statement is true in the valuation ring $B_M$. Moreover, from the remarks of § 2.2, it follows that *every* ideal of $B$ is an intersection of valuation ideals of $B$. Finally, from the results of § 2.1 on finitely generated ideals in rings of quotients, it is obvious that Helmer's theorem can be extended, so that in any $A$-ring of $K$ every finitely generated ideal is principal.

These remarks generalize to arbitrary $A$-rings results previously obtained for the ring $A$ itself (cf. **3**; **11**; **12**), and many other such results could also be easily handled by these methods, which are considerably more algebraic in character than were the original proofs. We shall return to the algebraic structure of $A$-rings in the next section, where some examples will be given and a classification of these rings will be made. We conclude this section with a brief discussion of the valuation rings $A_P$ and apply the above results to the ring $A$.

First note that for $P$ maximal in $A$, $P = I[\Delta(P)]$ and

$$A_P = \{f \in K\colon Z(1/f) \notin \Delta(P)\}, \qquad P \cdot A_P = \{f \in K\colon Z(f) \in \Delta(P)\},$$

as obtained by Alling (**2**). Further, Alling also showed the following result.

THEOREM 3.2.9. *If $P$ is a non-zero prime ideal of $A$, then $P \cdot A_P = M(A_P)$ is a principal ideal of $A_P$ if and only if $P$ is maximal.*

*Proof.* In order that $f \in P$ generate $P \cdot A_P$ in $A_P$, it is evidently necessary and sufficient that $f$ divide each $g \in P$ in $A_P$. For $P$ non-maximal, however, there exist $f_1, f_2 \in P$ so that $f = f_1 f_2$, and $f$ cannot divide both $f_1$ and $f_2$ in $P$, since then $1/f = (f_1f_2)/f^2$ would be a unit of $A_P$, which is impossible. Conversely, suppose that $P$ is maximal; thus, $P = I[\Delta(P)]$, and there exists $f \in P$ so that $\pi(f) = 1$. We claim that $f$ generates $P \cdot A_P$ in $A_P$. If not, then $g/f \notin A_P$ for some $g \in P$; whence, $f/g \in P \cdot A_P$. Let $h_1, h_2 \in A$ so that $f/g = h_1/h_2$ and $Z(h_1) \cap Z(h_2) = \emptyset$. Then $h_1 \in P$ and there exists $a \in \Omega$ so that $h_1(a) = g(a) = 0$; whence, $h_2(a) \neq 0$. We then have that

$$v_a(fh_2) = v_a(f) = v_a(gh_1) \geqq 2,$$

a contradiction to the fact that $\pi(f) = 1$.

*Note.* It can be shown without difficulty that if $P$ is maximal, then $f \in PA_P$ generates $P \cdot A_P$ in $A_P$ if and only if $\{a \in \Omega: \partial_f(a) > 1\} \notin \Delta(P)$. Theorem 3.2.9 has been applied by Alling to the quotient field $A/P$; cf. (**2**) for further details.

As an application of the above theorem, it follows from the concluding remarks of § 2.3 that for any maximal ideal $M$ of $A$, $\bigcap_{k \geqq 1} M^k$ is the largest prime ideal of $A$ contained in $M$, a result previously obtained by Henriksen (**12**). As a further application of these ideas, we prove the following theorem.

THEOREM 3.2.10. *Let $P$ be a non-zero free prime ideal of $A$ and let $f_0 \in P$, $f_0 \neq 0$. Let $\{a_k\}_{k \geqq 1}$ be a (univalent) enumeration of $Z(f_0)$ and suppose that $f \in A$ with $Z(f) \supset Z(f_0)$ so that $\partial_f(a_k)/\partial_{f_0}(a_k) \rightarrow +\infty$ as $k \rightarrow +\infty$. Then there exists a prime ideal $Q$ of $A$ so that $f \in Q \subset P$, $f_0 \notin Q$.*

*Proof.* Define $S = \{f_0^n g: n \geqq 0, g \in A - P\}$; therefore $S$ is a multiplicative subset of $A$ containing $A - P$. Then $A_P \subset S^{-1}A$, $S^{-1}A$ is a valuation ring, and $S^{-1}A = A_Q$ for some prime ideal $Q$ of $A$, where obviously $Q \subset P$ and $f_0 \notin Q$. Now, evidently $f \notin Q$ if and only if $1/f \in A_Q$. Suppose this is the case; whence, $1/f = h/f_0^n g$ for some $h \in A$, $n \geqq 0$, $g \in A - P$. Then for all $k \geqq 1$ we have that $v_{a_k}(f_0^n g) \geqq v_{a_k}(f)$, or

$$v_{a_k}(g) \geqq v_{a_k}(f) - n \cdot v_{a_k}(f_0).$$

It follows from our hypotheses that there exists an integer $k_0$ so that $\partial_g(a_k) \geqq \partial_{f_0}(a_k)$ for $k \geqq k_0$. Take $g_0 \in A$ so that $Z(g_0) = \{a_1, \ldots, a_{k_0}\}$ and $\partial_{g_0}(a_k) = \partial_{f_0}(a_k)$, $1 \leqq k \leqq k_0$. Then $f_0$ divides $gg_0$ in $A$; thus, $gg_0 \in P$. However, $g_0 \notin P$, since $P$ is free and $Z(g_0)$ is a finite set, and thus $g \in P$, a contradiction; therefore $f \in Q$.

Thus, every non-zero free prime ideal $P$ of $A$ properly contains another ideal of the same type, and thus $A_P$ is a valuation ring of infinite rank in this case. In fact, more than this is true.

THEOREM 3.2.11. *Let $\{I_n\}_{n \geqq 1}$ be any countable collection of non-zero free valuation ideals of $A$. Then $I = \bigcap_{n \geqq 1} I_n$ is a non-zero free ideal of $A$.*

*Proof.* First note that $\Delta_n = \Delta(I_n)$ is a maximal $\delta$-filter of $\Omega$ containing no finite subsets of $\Omega$. Further, since $\Delta_n$ is maximal, if $D \in \Delta_n$ and if $E \subset \Omega$ is relatively compact, then $D - E \in \Delta_n$. Now, let $\{E_n\}_{n \geqq 1}$ be any countable exhaustion of $\Omega$; cf. Ahlfors and Sario (**1**). Fix $b \in \Omega$ and for each $n \geqq 1$ let $D_n \in \Delta_n$. Then for each $n \geqq 1$, $D_n^* = D_n - E_n \cup \{b\}$ belongs to $\Delta_n$ and there exists $f_n \in I_n$ so that $Z(f_n) = D_n^*$. Now, $D^* = \bigcup_{n \geqq 1} D_n^*$ is a discrete subset of $\Omega$ and $\partial = \sup\{\partial_{f_n}: n \geqq 1\}$ is a well-defined integer-valued function supported on $D^*$. Taking $f \in A$ so that $\partial_f = \partial$, we see that $f_n$ divides $f$ in $A$ for all $n \geqq 1$, thus $f \in I_n$ for all $n \geqq 1$. Further, $f(b) \neq 0$, since $b \notin D^*$. Now, the function $f$ described here can be obtained for any $b \in \Omega$; therefore, $I = \bigcap_{n \geqq 1} I_n$ is free and non-zero.

COROLLARY 3.2.12. *If $M$ is a free maximal ideal of $A$ and if $\{I_n\}_{n \geqq 1}$ is any countable collection of non-zero valuation (primary, prime) ideals of $A$, each of which is contained in $M$, then $I = \bigcap_{n \geqq 1} I_n$ is a non-zero valuation (primary, prime) ideal of $A$.*

From this we see that every non-zero free prime ideal $P$ of $A$ contains uncountably many distinct prime ideals of $A$. For if there were only countably many such ideals, say $\{P_n\}_{n \geqq 1}$, then $\bigcap_{n \geqq 1} P_n$ is a *non-zero* free prime ideal of $A$ which contains no other non-zero prime ideals of $A$, in contradiction to Theorem 3.2.10. Thus, the valuation ring $A_P$ has *uncountable* rank unless $P$ is fixed.

**4. The classification of $A$-rings.** In this section we introduce a classification of $A$-rings based on the type of valuation rings which contain them. Two types are of special interest, those contained in no Noetherian valuation rings of $K$, and those contained only in Noetherian valuation rings of $K$. The latter are easily identified geometrically and must satisfy numerous strong algebraic conditions; the former, however, are described explicitly only in two special cases, and the results on their algebraic structure are generally of a negative character. The section concludes with a theorem describing possible isomorphisms between rings of meromorphic functions by means of a result of Iss'sa (**13**).

4.1. *Fixed and free A-rings.* The distinction between free and fixed ideals of $A$, together with Corollary 3.2.6, suggests a natural classification of the $A$-rings of $K$.

*Definition* 4.1.1. If $P$ is a non-zero prime ideal of $A$, the valuation ring $A_P$ is said to be *fixed* (*free*) if $P$ is a fixed (free) ideal of $A$. An $A$-ring $B \neq K$ is called *fixed* (*free*) if $B$ is the intersection of a collection of fixed (free) valuation rings of $K$. The field $K$ itself is considered as both fixed and free, but it is excluded from consideration in what follows.

Evidently, $B \neq K$ is fixed if and only if there exists a subset $E \neq \emptyset$ of $\Omega$ such that $B = \{f \in K: f(a) \neq \infty, a \in E\}$. This ring will be denoted by $A(E)$. In this case, if $f, g \in A(E) - \{0\}$, then $f$ divides $g$ in $A(E)$ if and only if $\partial_f|E \leqq \partial_g|E$, and $f$ and $g$ are associates in $A(E)$ if and only if $\partial_f|E = \partial_g|E$.

Furthermore, $f$ is a unit of $A(E)$ if and only if $\partial_f|E = 0$, and $f$ is an irreducible element of $A(E)$ exactly when there exists $a \in E$ so that $Z(f) \cap E = \{a\}$, $\partial_f(a) = 1$.

Every $A$-ring is the intersection of a fixed $A$-ring and a free $A$-ring (one of which may be $K$), and if $B$ is a free $A$-ring, then $B$ is contained in no fixed valuation ring of $K$. We shall say that an $A$-ring $B$ is *strongly fixed* if $B$ is contained in no *free* valuation ring of $K$. These rings are, in some sense, the antithesis of the free $A$-rings and satisfy very restrictive algebraic conditions.

THEOREM 4.1.2. *For any $A$-ring $B$ of $K$ the following are equivalent:*
(i) *$B$ is strongly fixed;*
(ii) *$B = A(E)$ with $E \subset \Omega$ relatively compact;*
(iii) *Every non-zero prime ideal of $B$ is maximal.*

*Proof*. Suppose that $B \neq K$ is strongly fixed. Then obviously $B$ is fixed and $B = A(E)$ for some $E \subset \Omega$; therefore, $B = S^{-1}A$, where $S = \{f \in A: f(a) \neq 0 \;\forall\; a \in E\}$. If $E$ is not relatively compact, then $E$ contains an infinite discrete subset $D$ of $\Omega$ and the set $I$ of all elements of $A$ which vanish at all but finitely many points of $D$ is a non-zero free ideal of $A$ which is disjoint from $S$. Then for any $S$-maximal ideal $P$ of $A$ which contains $I$, $P$ is a non-zero free prime ideal for which $B \subset A_P$, and this is impossible. Thus, $E$ is relatively compact.

Second, suppose that $B = A(E)$ with $E \neq \emptyset$ relatively compact in $\Omega$, and let $Q$ be a non-zero prime ideal of $B$. Then $P = Q \cap A$ is non-zero and prime in $A$, $P \cap S = \emptyset$. Now, if $P$ is free and $f \in P - \{0\}$, then $D = Z(f) - E$ belongs to $\Delta(P)$, which is a maximal, free $\delta$-filter of $\Omega$, and there exists $g \in P$ with $Z(g) = D$; whence, $g \in P \cap S \neq \emptyset$, a contradiction. Thus, $P$ is fixed, $P = I_a$ for some $a \in E$ and is maximal; therefore, $Q$ is maximal in $B$, being the extension of $P$ to $B$. Note that this proves that the maximal ideals of $B$ are exactly the ideals $J_a = \{f \in B: f(a) = 0\}$, $a \in E$.

Finally, let $B$ be an $A$-ring with no non-zero non-maximal prime ideals, and let $P$ be a non-zero prime ideal of $A$ for which $B \subset A_P$. Then $A_P = B_Q$, where $Q$ is the extension of $P$ to $B$. Thus, $P$ cannot be free, for in that case there would exist non-zero non-maximal prime ideals in $B_Q$, implying that $Q$ properly contains a non-zero prime ideal of $B$, contrary to hypothesis. Thus, $B$ is strongly fixed, as desired.

THEOREM 4.1.3. *Let $B$ be an $A$-ring of $K$. Then $B$ is strongly fixed if and only if $B$ is a principal ideal ring.*

*Proof*. Let $B$ be strongly fixed; therefore, $B = A(E)$ with $E$ relatively compact in $\Omega$, and let $I \neq \{0\}$ be a proper ideal of $B$. By the above proof, the maximal ideals of $B$ are the ideals $J_a = \{f \in B: f(a) \neq 0\}$, $a \in E$; thus, $I$ is contained in only finitely many maximal ideals of $B$, say $J_{a_1}, \ldots, J_{a_n}$. By the remarks of § 2.2, if $I_k$ is the extension of $I$ to the valuation ring

$R_{a_k} = \{f \in K: f(a_k) \neq \infty\}$, $1 \leqq k \leqq n$, then there exist positive integers $q_1, \ldots, q_n$ so that $I_k = \{f \in K: \partial_f(a_k) \geqq q_k\}$, $1 \leqq k \leqq n$; thus,

$$I = \bigcap_{k=1}^{n} I_k \cap B = \{f \in B: \partial_f(a_k) \geqq q_k, 1 \leqq k \leqq n\}.$$

Taking $f \in A$ so that $Z(f) = \{a_1, \ldots, a_n\}$ and $\partial_f(a_k) = q_k$, $1 \leqq k \leqq n$, $I$ is evidently the ideal of $B$ generated by $f$.

Conversely, suppose that $B$ is a principal ideal ring and let $P \neq \{0\}$ be a prime ideal of $A$ for which $B \subset A_P$. Then $Q = B \cap M(A_P)$ is prime in $B$ and $A_P = B_Q$. That is, $A_P$ is a ring of quotients of $B$, and therefore $A_P$ is also a principal ideal ring and $P$ is fixed.

In view of the fact that Helmer's theorem is valid in any $A$-ring, the above shows that the strongly fixed $A$-rings are exactly those $A$-rings which are Noetherian rings, and other characterizations are possible. For example, they are also those $A$-rings which are unique factorization domains. Moreover, an $A$-ring is fixed if and only if it is the intersection of a decreasing sequence of strongly fixed $A$-rings. For the fixed $A$-rings one can also obtain results on maximal and prime ideals analogous to those for the ring $A$. For example, in the ring $B = A(E)$ the non-zero elements of $B$ belonging to non-principal maximal ideals of $B$ are those $f \in B - \{0\}$ such that $Z(f) \cap E$ is an infinite set, and those belonging to non-maximal prime ideals are those $f \in B$ such that $\sup\{\partial_f(a): a \in E\} = +\infty$. Of course, both of these classes of ideals are empty when $B$ is strongly fixed.

Obviously, one could completely describe the $A$-rings of $K$ if an explicit characterization of the free $A$-rings were available, but this is not possible since there is no such characterization of the free prime ideals of $A$. These rings are somewhat pathological and cannot be nicely described except in two instances.

First, let $\Sigma_1$ denote the collection of all free maximal ideals of $A$ and let $S_1 = \bigcap_{P\in\Sigma_1} (A - P)$. Then $B_1 = S_1^{-1}A$ is the ring of all functions in $K$ which have only finitely many poles in $\Omega$, the units of this ring being those $f \in K$ for which $\partial_f$ is supported on a finite set. This ring is evidently the smallest free $A$-ring of $K$.

Second, let $\Sigma_2$ denote the collection of all non-maximal prime ideals of $A$ and let $S_2 = \bigcap_{P\in\Sigma_2} (A - P)$. Then $B_2 = S_2^{-1}A$ is the set of all functions $f \in K$ for which $\inf\{\partial_f(a): a \in \Omega\} > -\infty$. The units of this ring are those $f \in K$ for which $\partial_f$ is bounded on $\Omega$.

In any free $A$-ring $B$ of $K$, every non-zero prime ideal of $B$ contains uncountably many non-zero prime ideals of $B$. Also, $B$ admits no irreducible elements, and no non-zero prime ideal of $B$ is principal. In fact, in the ring $B_2$ every prime ideal is idempotent, and this will be the case in every $A$-ring which contains $B_2$.

4.2. *Isomorphism theorems.* We conclude with a brief discussion of isomorphisms between $A$-rings. Here we shall use a result of Iss'sa (**13**), who has characterized the Noetherian valuation rings of $K = K(\Omega)$, namely the following theorem.

THEOREM 4.2.1. *Every Noetherian valuation ring of $K$ contains the ring $A$, and hence is of the form $\{f \in K: f(a) \neq \infty\}$ for some $a \in \Omega$, if it is non-trivial.*

Although there exist many valuation rings of $K$ which do not contain $A$, none of these rings is Noetherian. The ring $A$ is then algebraically characterized in $K$ as the intersection of the Noetherian valuation rings of $K$, and therefore the conformal structure of $\Omega$ is determined by $K$, since it is determined by $A$, as shown by Nakai (**17**). Here we shall use Theorem 4.2.1 to prove an isomorphism theorem of this type, but we shall not require Nakai's result in the proof.

THEOREM 4.2.2. *Let $\Omega_1$ and $\Omega_2$ be Riemann surfaces, $\Omega_1$ non-compact. Let $B_1$ be an $A$-ring of $K_1 = K(\Omega_1)$, and let $B_2$ be any ring of functions meromorphic on $\Omega_2$ which contains the constants. Suppose that $\theta: B_1 \to B_2$ is a ring isomorphism of $B_1$ onto $B_2$. Then either*

(i) *$\theta i = i$ and there exists a unique analytic map $\phi: \Omega_2 \to \Omega_1$ such that $\theta f = f \circ \phi$ for all $f \in B_1$, or*

(ii) *$\theta i = -i$ and there exists a unique conjugate-analytic map $\phi: \Omega_2 \to \Omega_1$ such that $\theta f = \overline{f \circ \phi}$ for all $f \in B_1$.*

*Proof.* We shall suppose that $\theta i = i$, the other case being treated in a similar way. Let $K_0$ denote the field of quotients of $B_2$; thus, $K_0$ is a subfield of $K_2 = K(\Omega_2)$ and $\theta$ may be uniquely extended to a field isomorphism $\theta: K_1 \to K_0$ of $K_1$ onto $K_0$. Now, $K_0$ contains non-constant functions, since the complex field $C$ admits no non-trivial Noetherian valuation rings, and therefore for each $b \in \Omega_2$ the ring $R_0(b) = \{g \in K_0: g(b) \neq \infty\}$ is a non-trivial Noetherian valuation ring of $K_0$ with maximal ideal $M_0(b) = \{g \in K_0: g(b) = 0\}$. Hence, $\theta^{-1}[R_0(b)]$ is a non-trivial Noetherian valuation ring of $K_1$, and therefore there exists, by Theorem 4.2.1, a unique $a \in \Omega_1$ such that $\theta^{-1}[R_0(b)] = \{f \in K_1: f(a) \neq \infty\}$. The map $\phi$ in question is defined by $\phi(b) = a$.

Evidently, $\theta\alpha = \alpha$ for constants $\alpha \in C_r$, the subfield of $C$ of all complex numbers with rational real and imaginary coordinates. Further, the constant functions in $K_i$ are algebraically characterized as those $f \in K_i$ for which $(f - \alpha)$ has roots of all orders in $K_i$ for all $\alpha \in C_r$, $i = 0, 1$. Therefore, $\theta$ maps constants into constants, and therefore $\theta|C$ is an automorphism of the complex field $C$.

Fix $b \in \Omega_2$ and let $g \in R_0(b)$; then $g - g(b) \in M_0(b)$. The function $\theta^{-1}g - \theta^{-1}[g(b)]$ then belongs to the maximal ideal of $\theta^{-1}[R_0(b)]$, and hence vanishes at $\phi(b)$. This implies that $(\theta^{-1}g)(\phi(b)) = \theta^{-1}[g(b)]$, since $\theta^{-1}[g(b)]$ is a constant, and so, taking $f \in K_1$ so that $\theta f = g$, we have that

$$(\theta f)(b) = \theta[f(\phi(b))]. \qquad (*)$$

This holds for all $f \in K_1$ with $g = \theta f \in R_0(b)$, and if we define $\theta\infty = \infty$, then $(*)$ holds for all $f \in K_1$, for if $g = \theta f \notin R_0(b)$, one simply applies the above argument to $1/g \in M_0(b)$.

Now we claim that $\phi$ maps each relatively compact subset of $\Omega_2$ onto a relatively compact subset of $\Omega_1$; cf. Nakai (**17**). For if not, then there exists an infinite subset $\{b_k: k \geqq 1\}$ of $\Omega_2$ having a cluster point in $\Omega_2$ such that the set $\{\phi(b_k): k \geqq 1\}$ is an infinite discrete subset of $\Omega_1$. Taking $f \in K_1 - \{0\}$ so that $f(\phi(b_k)) = 0$ for all $k \geqq 1$, $(*)$ implies that $(\theta f)(b_k) = 0$ for all $k \geqq 1$, and this is impossible since $\theta f$ is non-constant.

Second, we show that $\theta|C$ is continuous (whence, $\theta\alpha = \alpha$ for all $\alpha \in C$). To see this, let $f \in A(\Omega_1)$ be non-constant and let $U$ be an open, relatively compact subset of $\Omega_2$. From $(*)$ we have, since $\phi(U)$ is relatively compact in $\Omega_1$, that

$$\sup\{|\theta^{-1}(g(b))|: b \in U\} = \sup\{|f(\phi(b))|: b \in U\} < +\infty,$$

where $g = \theta f$, which implies that $g$ is analytic on $U$ and that $\theta^{-1}|C$ is bounded on $g(U)$. However, $U$ is open and $g$ non-constant; therefore, $g(U)$ is an *open* subset of $C$ on which $\theta^{-1}|C$ is bounded, and hence $\theta^{-1}|C$ is continuous.

Equation $(*)$ now states that $\theta f = f \circ \phi$ for all $f \in K_1$, and the proof that $\phi$ is analytic proceeds as usual: given $b \in \Omega_2$, it suffices to take $f \in A(\Omega_1)$ so that $f$ is univalent in a neighbourhood of $\phi(b)$; thus, the representation $\phi = f^{-1} \circ (\theta f)$ is valid in some neighbourhood of $b$. That $\phi$ is unique follows immediately from the fact that $K_1$ separates points of $\Omega_1$.

Note that the map $\phi: \Omega_2 \to \Omega_1$ of this theorem is univalent if and only if $B_2$ separates points of $\Omega_2$, and $\phi$ maps $\Omega_2$ onto $\Omega_1$ if and only if every non-trivial Noetherian valuation ring of $K_0$ is of the form $\{g \in K_0: g(b) \neq \infty\}$ for some $b \in \Omega_2$. In particular, this is the case when $\Omega_2$ is non-compact and $B_2$ is an $A$-ring of $K_2 = K(\Omega_2)$. Thus, the algebraic structure of any $A$-ring of $K(\Omega)$, $\Omega$ non-compact, determines the conformal structure of $\Omega$. (Of course, this follows immediately from the results of Iss'sa and Nakai.)

As a final point, we note that there are always many more free maximal ideals of $A = A(\Omega)$ than there are conformal and/or anti-conformal automorphisms of $\Omega$; cf. Alling (**2**). Consequently, not all of the valuation rings $A_M$, where $M$ is a free maximal ideal of $A$, are isomorphic. It would be of interest to determine how the algebraic structure of these rings may differ, for none of the theorems available for the ideal theory of $A$ make any distinction between different free maximal ideals of $A$.

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*Columbia University,*
*New York, New York*