Canad. Math. Bull. Vol. 59 (4), 2016 pp. 776–793
http://dx.doi.org/10.4153/CMB-2016-051-1

The Carathéodory Reflection Principle and Osgood–Carathéodory Theorem on Riemann Surfaces

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Abstract. The Osgood–Carathéodory theorem asserts that conformal mappings between Jordan domains extend to homeomorphisms between their closures. For multiply-connected domains on Riemann surfaces, similar results can be reduced to the simply-connected case, but we find it simpler to deduce such results using a direct analogue of the Carathéodory reflection principle.

1 Introduction

The reflection principles of Schwarz and Carathéodory give conditions under which holomorphic functions extend holomorphically to the boundary and the theorem of Osgood–Carathéodory states that a one-to-one conformal mapping from the unit disc to a Jordan domain extends to a homeomorphism of the closed disc onto the closed Jordan domain. In this note, we study similar questions for holomorphic mappings on Riemann surfaces. We give a Carathéodory type reflection principle for bordered Riemann surfaces that are arbitrary. That is, we do not assume that they are compact nor do we assume that they are of finite genus. From this follows a Schwarz type reflection principle, as well as an Osgood–Carathéodory type theorem.

The Osgood–Carathéodory theorem was extended by Osgood and Taylor [14] to domains bounded by finitely many disjoint Jordan curves, where the proof was reduced to the simply-connected case. Using a certain amount of cleverness, a similar approach can be employed for bordered Riemann surfaces. We prefer to deduce the Osgood–Carathéodory theorem for bordered Riemann surfaces immediately from the Carathéodory reflection principle for Riemann surfaces, since a holomorphic extension is obviously a continuous extension. In this manner (to paraphrase Larry Zalcman), cleverness is rendered superfluous.

When we speak of a conformal mapping f from a domain Ω_1 of one Riemann surface R_1 to a domain Ω_2 in another Riemann surface R_2, we always mean an orientation preserving conformal mapping that is one-to-one, but not necessarily onto. The expressions "one-to-one conformal mapping onto" and "biholomorphic mapping" will be used interchangeably. For an overview of conformal mappings in the plane, see [5, 8, 13, 15, 16].

Received by the editors February 27, 2016; revised July 30, 2016.
Published electronically September 16, 2016.
Research of author F. S. supported by NSERC (Canada).
AMS subject classification: **30C25**, 30F99.
Keywords: bordered Riemann surface, reflection principle, Osgood-Carathéodory.

A Riemann surface is said to be *planar* if it is homeomorphic to a subset of the complex plane $\mathbb{C}$. In extending results from the complex plane to Riemann surfaces, the following general uniformization theorem of Koebe is extremely helpful.

Theorem 1.1 *Every planar Riemann surface is conformally equivalent to a plane domain.*

It will also be helpful to recall that meromorphic functions on Riemann surfaces are the same as holomorphic mappings to the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

For a domain $G \subset \mathbb{C}$, a function $f: G \to \overline{\mathbb{C}}$ and a boundary point $\zeta \in \partial G$, Carathéodory defines a point $\alpha \in \overline{\mathbb{C}}$ to be a boundary value of f at ζ if there is a sequence z_ν in G for which $\lim_{\nu\to\infty} z_\nu = \zeta$, $\lim_{\nu\to\infty} f(z_\nu) = \alpha$. The set of boundary values of f at ζ is precisely the cluster set $C(f,\zeta)$. Also, for a subset $E \subset \partial G$, we denote

$$C(f,E) \equiv \bigcup_{\zeta\in E} C(f,\zeta).$$

For a set $E \subset \overline{\mathbb{C}}$, let us set $E^* = \{\overline{z} : z \in E\}$, where $\infty^* = \infty$.

The first part of the following theorem is the Carathéodory reflection principle [4].

Theorem 1.2 *Let V be a domain in the open upper half-plane $\{\Im z > 0\}$. Let I be the interior of $\{z \in \partial V : \Im z = 0\}$ in the topology of $\mathbb{R}$. Set $\widehat{V} = V \cup I \cup V^*$. Let f be meromorphic in V and suppose all boundary values of f on I are real or ∞. Then f extends to a surjective meromorphic function $\widehat{f}: \widehat{V} \to f(V) \cup C(f,I) \cup f(V)^*$ and $\widehat{f}(p^*) = \overline{f}(p)$.*

Suppose, moreover, that $f(V)$ is contained in the open upper half-plane

$$H^+ = \{w \in \mathbb{C} : \Im w > 0\}.$$

If f is respectively locally conformal, or conformal, then so is $\widehat{f}$.

A similar version of the reflection principle can be found in [15, p. 4].

Proof The final two sentences are not in Carathéodory's formulation of the theorem, but, as we shall see, this final portion follows from the first part.

Suppose then that $f(V)$ is contained in the open upper half-plane. Then combining the first part of the theorem with the Schwarz reflection principle, we conclude that if f is locally conformal (respectively conformal) in V, then $\widehat{f}$ is locally conformal (respectively conformal) in V^*. Suppose for some value $p \in I$ that $\widehat{f}(p)$ were assumed to have multiplicity greater than 1. Then at p all angles would be multiplied by the multiplicity of p, which contradicts the assumption that the image by f of any upper half-disc "centred" at p is contained in the open upper half-plane. Thus, $\widehat{f}$ is locally conformal at each point of I. Now suppose f is conformal. We have already verified that $\widehat{f}$ is injective on $V \cup V^*$. Since $\widehat{f}(V), \widehat{f}(V^*)$, and $\widehat{f}(I)$ are disjoint, it is sufficient to show that $\widehat{f}$ is injective on I. Then $\widehat{f}$ will be injective and hence conformal. Suppose, to obtain a contradiction, that $\widehat{f}(p) = \widehat{f}(q)$, for $p \neq q$ in I. Let U_p and U_q be disjoint neighbourhoods of p and q, respectively, sufficiently small that $\widehat{f}$ is conformal, hence injective, in U_p and U_q. Since $\widehat{f}$ is an open mapping and $\widehat{f}(I)$ is of

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measure zero, it follows that there are points a and b in $U_p \setminus I$ and $U_q \setminus I$, respectively, such that $\widehat{f}(a) = \widehat{f}(b)$. This contradicts the fact that $\widehat{f}$ is injective on $V \cup V^*$. Thus, $\widehat{f}$ is injective on I. ∎

The example $f(z) = z^2$ shows that if we omit the assumption that $f(V)$ is contained in the open upper half-plane, it does not always follow that $\widehat{f}$ is locally conformal when f is locally conformal, in fact, not even when if f is conformal.

2 A Few Facts on Conformal Mappings in the Plane

An open (respectively compact) Jordan arc is defined as the homeomorphic image of the interval $(0,1)$ (respectively the interval [0,1]). A Jordan curve is the homeomorphic image of a circle and a Jordan domain in $\overline{\mathbb{C}}$ is a domain whose boundary is a Jordan curve. By the Jordan curve theorem, if J is a Jordan curve in $\overline{\mathbb{C}}$, then its complement $\overline{\mathbb{C}} \setminus J$ consists of two disjoint Jordan domains, both having J as boundary. A closed Jordan domain is the closure of a Jordan domain. By the Schoenflies theorem [15, p. 25], a closed Jordan domain is the homeomorphic image of the closed unit disc. The Schoenflies theorem could be phrased as follows. A homeomorphism from the boundary of the disc to the boundary of a Jordan domain extends to a homeomorphism of the interiors. The Osgood–Carathéodory theorem goes in the opposite direction, and has as a consequence that a conformal mapping of the unit disc onto a Jordan domain (which, of course, is a homeomorphism) extends to a homeomorphism of the boundaries.

More precisely, the Osgood–Carathéodory theorem states that a conformal mapping from the open unit disc onto a Jordan domain in the Riemann sphere $\overline{\mathbb{C}}$ extends to a homeomorphism of the closed disc onto the closed Jordan domain. If we think of a Jordan domain U as the complement of a closed Jordan domain $\overline{V}$, then a natural generalization would be to replace $\overline{V}$ by a compact Jordan arc J (thinking of a Jordan arc as a "compressed" Jordan domain). In this spirit, we shall consider to what extent we can obtain an analogue of the Osgood–Carathéodory theorem if we are mapping the unit disc to the complement of a compact Jordan arc. The following discussion describes the situation.

A topological space is said to be locally connected if every point has a fundamental system of connected neighbourhoods. The continuous image of a locally connected space need not be locally connected. For example, the closure of the image of the curve

$$\gamma(t) = \left|\sin\left(\frac{2\pi}{t}\right)\right| e^{it}, \quad 0 < t \le 1$$

is not locally connected.

Theorem 2.1 (Continuity theorem [15]) *Let f be a conformal mapping of the open unit disc Δ onto a domain $G \subset \overline{\mathbb{C}}$. The function f has a continuous extension to $\overline{\Delta}$ if and only if ∂G is locally connected.*

Lemma 2.2 *Let $f\colon \Delta \to G$ be conformal with ∂G locally connected. Then the continuous extension to $\overline{\Delta}$ maps the circle $\mathbb{T}$ onto ∂G.*

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Proof If $w \in \partial G$, there is a sequence $z_n \in \Delta$ such that $f(z_n) \to w$. By choosing a subsequence, we may assume that z_n converges to a point ζ of the unit circle. Then $f(\zeta) = w$, so $f(\mathbb{T}) \supset \partial G$. Conversely, if $\zeta \in \mathbb{T}$ and $z_n \in D$ converges to ζ, then $f(z_n)$ is eventually outside of every compact subset of $f(\Delta) = G$, so $f(\zeta) \in \partial G$. Thus, $f(\mathbb{T}) \subset \partial G$. ∎

Lemma 2.3 *Let ϕ be a conformal mapping of the open unit disc Δ onto the complement J^c of a compact Jordan arc J in $\overline{\mathbb{C}}$. Then ϕ extends to a continuous mapping of $\overline{\Delta}$ onto $\overline{\mathbb{C}}$, which maps the unit circle $\mathbb{T}$ onto J.*

Proof Since J is locally connected, the lemma follows from the previous theorem and lemma. ∎

Let E be a locally connected continuum. We say that $a \in E$ is a cut point of E if $E \setminus \{a\}$ is no longer connected. For a Jordan arc all points except end points are cut-points.

Lemma 2.4 *Let ϕ be as in the previous lemma. Then for $a \in J$, the set $\phi^{-1}(a)$ is a singleton if and only if a is an end point of J.*

Proof By [15, Proposition 2.5], if ϕ is a conformal mapping of Δ onto a bounded domain G, where ∂G is locally connected, then for each $a \in \partial G$, the set $f^{-1}(a)$ is a singleton if and only if a is not a cut-point of ∂G. In our situation, J^c is not a bounded domain in $\mathbb{C}$, but the proof can be easily modified to apply to our case. Since $a \in J$ is not a cut-point of J if and only if a is an end point, the lemma follows. ∎

Lemma 2.5 *Let ϕ be as in the previous lemma. Let p and q be the ends of J and J^0 be the inner points of J. There are points a and b on the unit circle, such that $\phi(a) = p$, $\phi(b) = q$ and ϕ maps each of the two arcs comprising $\mathbb{T} \setminus \{a, b\}$(homeomorphically) onto J^0.*

Proof From the previous lemma, $\phi^{-1}(p)$ is a singleton $\{a\}$ and $\phi^{-1}(q)$ is a singleton $\{b\}$. Let A be one of the two arcs comprising $\mathbb{T} \setminus \{a, b\}$. Since $\phi(A)$ is a connected subset of the (open) Jordan arc J^0, it is a point or an arc. It cannot be a point, for then ϕ would be constant on the arc A and hence constant by uniqueness theorems. Hence, $\phi(A)$ is a sub-arc of J^0. Since p and q are in the closure of $\phi(A)$, the arc $\phi(A)$ must be all of J^0. ∎

A cross-cut C of an open set G is an open Jordan arc in G such that $\overline{C} = C \cup \{a, b\}$ with $a, b \in \partial G$. We allow that $a = b$ (see [1]).

Lemma 2.6 *Let J be a compact Jordan arc in $\overline{\mathbb{C}}$. Then for every neighbourhood G of J, there is a Jordan domain $W \subset \overline{\mathbb{C}}$, such that $J^0 \subset W \subset \overline{W} \subset G$ and J is a cross-cut of W. That is, J is contained in W, except for the end points, which (of course) lie on ∂W.*

Proof It follows from the Jordan arc separation theorem that $J^c = \overline{\mathbb{C}} \setminus J$ is connected. For a proof, see [2, Lemma 4]. Let $\phi\colon \Delta \to J^c$ be a conformal map. By Lemma 2.3, ϕ extends to a continuous mapping (which we continue to denote by ϕ) of $\overline{\Delta}$ onto $\overline{\mathbb{C}}$

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that maps $\mathbb{T}$ onto J. There are two points $a, b \in \mathbb{T}$ which are mapped to the end points of J and the two arcs of $\mathbb{T} \setminus \{a, b\}$ are mapped onto J^0. We can assume that $\{a, b\} = \{-1, +1\}$. Let G be a neighbourhood of J. The neighbourhood $\phi^{-1}(G)$ of $\mathbb{T}$ contains an annulus $A_r = \{z : r \le |z| \le 1\}$, for some $r \in (0, 1)$. Let L be a "lens domain" in Δ such that $\overline{L} \cap \overline{\mathbb{T}} = \{-1, +1\}$ and the disc $\overline{D}_r = \{z : |z| \le r\}$ is contained in L. Then $\Gamma = \phi(\partial L)$ is a Jordan curve in $\overline{\mathbb{C}}$, which separates $\overline{\mathbb{C}}$ into two Jordan domains with boundary Γ. One of these domains $\phi(L)$ contains $\phi(\overline{D}_r)$, so the other Jordan domain, call it W, is contained in $\phi(A_r) \subset G$. Since $\partial L \subset A_r$, we also have $\phi(\partial L) = \Gamma \subset G$. Hence $\overline{W} = W \cup \Gamma \subset G$. Since ϕ maps the two semicircles $\mathbb{T} \setminus \{-1, +1\}$ onto J^0 and these semicircles are disjoint from ∂L, it follows that $J^0 \subset W$. Since $\phi(\pm 1)$ are the end points of J and they lie on $\Gamma = \partial W$, it follows that J is a cross-cut of W. ∎

A domain $W \subset \overline{\mathbb{C}}$ is called a *circular domain* if ∂W consists of finitely many disjoint spherical circles. A domain is non-degenerate if no component of its complement is a single point. The following theorem of Koebe states that circular domains are conformally canonical for the class of non-degenerate n-connected domains.

Theorem 2.7 *Every non-degenerate n-connected domain in $\overline{\mathbb{C}}$ is conformally equivalent to a circular domain.*

We define a (finitely connected) Jordan region Ω in $\overline{\mathbb{C}}$ to be a domain bounded by finitely many disjoint Jordan curves and, if Ω is a Jordan region, we say that $\overline{\Omega}$ is a closed Jordan region. If there is only one boundary curve, then we call the Jordan region a Jordan domain.

Occasionally, the Osgood–Carathéodory theorem is invoked not only for Jordan domains, but also (implicitly) for Jordan regions (for example in [10]). The following extension of the Osgood–Carathéodory theorem for Jordan regions in $\overline{\mathbb{C}}$ was proved in [14] (see also [5, Chapter 15]) and can be deduced from the simply-connected case.

Theorem 2.8 *If G and Ω are two Jordan regions and $f : \Omega \to G$ is a conformal equivalence, then f extends to a homeomorphism of $\overline{\Omega}$ onto $\overline{G}$.*

3 Bordered Riemann Surfaces

Let us denote a bordered Riemann surface with interior Ω and border $b\Omega$ by $\widetilde{\Omega} = \Omega \cup b\Omega$. A bordered Riemann surface is not necessarily compact. Every bordered Riemann surface is a bordered surface, so there is an open cover $\{U_\alpha\}$ of $\widetilde{\Omega}$ and corresponding homeomorphisms $h_\alpha : \overline{U}_\alpha \to \overline{\Delta}_\alpha$ that we call *closed charts*, where each Δ_α is either a disc whose closure is contained in the open upper half-plane of $\mathbb{C}$ or an upper half-disc $\{w : |w - t| < r, \Im w \ge 0\}$ for some real "center" t and positive radius r. Points of $\widetilde{\Omega}$ that correspond to points on the real line form the border $b\Omega$ and the remaining points, which correspond to points of the open upper half-plane, form the "interior" Ω of $\widetilde{\Omega}$. The changes of charts $h_\beta^{-1} \circ h_\alpha$, when defined, preserve interior points and border points, and are clearly homeomorphisms. If $\widetilde{\Omega}$ is not only a bordered surface, but also a bordered Riemann surface, then we additionally require these changes of charts to be conformal. At interior points the meaning of conformal is obvious and at border points we ask that $h_\beta^{-1} \circ h_\alpha$ be the restriction of a conformal mapping in an open subset of $\mathbb{C}$. The closed upper half-plane is an example of a bordered Riemann surface. A good introduction to bordered Riemann surfaces can be found in [1, §II. 3A].

In a Riemann surface (bordered or not), a (Jordan) arc J is a homeomorphic image of an interval. If the interval is open, we say that J is an open Jordan arc and if the interval is closed, we say that J is a compact Jordan arc. A Jordan curve is a homeomorphic image of a circle.

Lemma 3.1 *If $\widetilde{\Omega} = \Omega \cup b\Omega$ is a bordered Riemann surface, then each border point $p \in b\Omega$ has a neighbourhood system given by closed border charts $h_r : \widetilde{U}_r \to \overline{\Delta}_r^+, 0 < r < 1$, where Δ_r^+ is the open upper half-disc $\{z : |z| < r, \Im z > 0\}$. Set $U_r = h_r^{-1}(\Delta_r^+)$. Then $\widetilde{U}_r = \overline{U}_r$. Each closed neighbourhood $\overline{U}_r$ is thus a closed Jordan domain, where the Jordan curve $\overline{U}_r \setminus U_r$ consists of an open border arc $\beta_r \subset b\Omega$ and a cross-cut C_r of $\widetilde{\Omega}$ having the same end points as β_r.*

Proof Fix $p \in b\Omega$. Let $h : \overline{U} \to \overline{\Delta}^+$ be a closed chart at p where

$$\Delta^+ = \{z : |z| < 1, \Im z > 0\}$$

and h sends p to zero. Denote by Δ_r^+ the open upper half-disc $\{z : |z| < r, \Im z > 0\}$ and U_r the inverse image $h^{-1}(\Delta_r^+)$. Since $\overline{\Delta}_r^+, 0 < r < 1$, is a neighbourhood system of 0 in the closed upper half-plane and h is a homeomorphism, it follows that the $\overline{U}_r, 0 < r < 1$ are closed Jordan domains and form a neighbourhood system of p. The Jordan curve ∂U_r consists of the open border arc $\beta_r = h^{-1}\{(-r, r)\}$ and the cross-cut $h^{-1}(c_r)$, where c_r is the closed semi-circle $\{z : |z| = r, \Im z \geq 0\}$. If we denote by h_r the restriction of h to $\overline{U}_r$, then $h_r : \overline{U}_r \to \overline{\Delta}_r^+, 0 < r < 1$ are closed border charts at p. ∎

Given a bordered Riemann surface $\widetilde{\Omega} = \Omega \cup b\Omega$, we construct a bordered Riemann surface $\widetilde{\Omega}^*$ called the conjugate of $\widetilde{\Omega}$ (see [1]). The conjugate $\widetilde{\Omega}^*$ of $\widetilde{\Omega}$ is a topological copy of $\widetilde{\Omega}$. For each α, denote by U_α^* the corresponding topological copy of the U_α and for each $p \in \widetilde{\Omega}$ by p^* the corresponding point in $\widetilde{\Omega}^*$. The space $\widetilde{\Omega}^*$ is endowed with the complex structure obtained by replacing the closed charts $h_\alpha : \overline{U}_\alpha \to \overline{\Delta}_\alpha$ of $\widetilde{\Omega}$ by the charts $h_\alpha^* : \overline{U}_\alpha^* \to \overline{\Delta}_\alpha^*$, where $h_\alpha^*(p^*) = -\overline{h}_\alpha(p)$ and $\overline{\Delta}_\alpha^* = h_\alpha^*(U_\alpha^*)$.

We now form the double $\widehat{\Omega}$ of the bordered Riemann surface $\widetilde{\Omega}$ by welding $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ together by the identity mapping on $b\Omega$. The double of a bordered Riemann surface is a Riemann surface (not a bordered Riemann surface). The complex structure of the double $\widehat{\Omega}$ is given by charts $\widehat{h}_\alpha : \widehat{U}_\alpha \to \widehat{\Delta}_\alpha$, which we now describe. If U_α is contained in the interior of Ω, then we set $\widehat{U}_\alpha = U_\alpha$ and $\widehat{h}_\alpha = h_\alpha$. Similarly, if U_α^* is contained in the interior of $\widetilde{\Omega}^*$, we set $\widehat{h}_\alpha = h_\alpha^*$. There remains to define charts at points of $b\widetilde{\Omega} = b\widetilde{\Omega}^*$. If U_α corresponds to a half-disc, then we denote by $\widehat{U}_\alpha$ the set obtained by welding together U_α and U_α^*. We define the function $\widehat{h}_\alpha$ on the closure of $\widehat{U}_\alpha$ by setting $\widehat{h}_\alpha = h_\alpha$ on $\overline{U}_\alpha$ and on $\overline{U}_\alpha^*$, $\widehat{h}_\alpha(w) = -h_\alpha^*(w) = -(-\overline{h}_\alpha(p)) = \overline{h}_\alpha(p)$, where $w = p^* \in \overline{U}_\alpha^*$.

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A manifold need not be second countable (consider the long line), but it is a profound property of Riemann surfaces that they are second countable (Radó's theorem). They are therefore σ-compact, *i.e.*, they can be represented as a countable union of compacta. Similar properties hold for bordered Riemann surfaces but, since non-compact bordered Riemann surfaces are less familiar, we state the following result, which makes it easier to see these properties (and many others) for bordered Riemann surfaces.

Theorem 3.2 *Every bordered Riemann surface is homeomorphic to a closed subset of* $\mathbb{R}^3$.

Proof Let $\widetilde{\Omega}$ be a bordered Riemann surface. The remarkable result of Rüedy [17] states that every Riemann surface admits a smooth proper conformal embedding into $\mathbb{R}^3$. Let $h: \widehat{\Omega} \to \mathbb{R}^3$ be such an embedding. Since $\widetilde{\Omega}$ is closed in $\widehat{\Omega}$, it follows that $h(\widetilde{\Omega})$ is closed in $h(\widehat{\Omega})$ and, since $h(\widehat{\Omega})$ is closed in $\mathbb{R}^3$, and closed subsets of closed subsets are closed, it follows that $h(\widetilde{\Omega})$ is also closed in $\mathbb{R}^3$. ■

A subset of a Riemann surface or bordered Riemann surface is said to be *bounded* if its closure is compact.

Corollary 3.3 *In a bordered Riemann surface* $\widetilde{\Omega}$*, a subset is compact if and only if it is closed and bounded. Hence, a closed subset is non-compact if and only if it contains a sequence which tends to infinity (the Alexandroff point of* $\widetilde{\Omega}$*).*

4 A Reflection Principle for Bordered Riemann Surfaces

Various reflection principles for Riemann surfaces are known. For example, see [12, §6]. In this section we present a reflection principle for holomorphic maps between bordered Riemann surfaces.

Theorem 4.1 *Let* $\widetilde{\Omega} = \Omega \cup b\Omega$ *be a bordered Riemann surface. Let* f *be meromorphic in* Ω *and suppose all boundary values of* f *on* $b\Omega$ *are real or* ∞*. Then* f *extends to a meromorphic function* $\widehat{f}$ *on* $\widehat{\Omega}$*. Suppose* $f(\Omega)$ *is contained in the open upper half-plane. Then if* f *is locally conformal, so is* $\widehat{f}$ *and, if* f *is conformal, so is* $\widehat{f}$.

Proof First, we shall extend f to a point p of the border $b\Omega$. At p considered as a point of $\widehat{\Omega}$, there is a chart $\widehat{h}: \widehat{U} \to \Delta$, where Δ is the open unit disc. Set

$$\Delta^+ = \{w : |w| < 1, \Im w \geq 0\} \quad \text{and} \quad \Delta^- = \{w : |w| < 1, \Im w \leq 0\}.$$

Setting $U^+ = \widehat{h}^{-1}(\Delta^+)$ and $U^- = \widehat{h}^{-1}(\Delta^-)$, we have $\widehat{U} = U^+ \cup U^-$. Moreover, $\widehat{h}|_{U^+} = h : U^+ \to \Delta^+$ and $\widehat{h}|_{U^-} = -h^* : U^- \to -\Delta^-$ are border charts of p in $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ respectively.

Denote $\Delta_0^+ = \{w : |w| < 1, \Im w > 0\}$. The meromorphic function $f \circ \widehat{h}^{-1}: \Delta_0^+ \to \mathbb{C}$ satisfies the hypotheses of Theorem 1.2 and so extends meromorphically to the open disc Δ. Consequently, f extends meromorphically to the neighbourhood $\widehat{U}$ of p.

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If p and q are two border points and $\widehat{U}_p$ and $\widehat{U}_q$ are corresponding neighbourhoods as above that intersect, then the corresponding meromorphic extensions agree, since they agree on $\widehat{U}_p^+ \cap \widehat{U}_q^+$. Setting $\widehat{U}_b = \bigcup\{\widehat{U}_p : p \in b\widetilde{\Omega}\}$, we obtain a meromorphic extension $\widehat{f}$, defined on the neighbourhood $\widehat{U}_b$ of $b\widetilde{\Omega}$.

Since this extension to the neighbourhood $\widehat{U}_b$ of the common border $b\widetilde{\Omega}$ is defined explicitly on $\widetilde{\Omega}^* \cap \widehat{U}_b$ by the formula $\widehat{f}(p^*) = \overline{f}(p)$, we may extend $\widehat{f}$ to all of $\widetilde{\Omega}^*$ by the same formula. Namely, we set $\widehat{f}(p^*) = \overline{f}(p)$ for all $p^* \in \widetilde{\Omega}^*$. From this formula, we see that if f is locally conformal on Ω, then $\widehat{f}$ is locally conformal on $\Omega \cup \Omega^*$.

Now suppose $f(\Omega)$ is contained in the open upper half-plane. The proof that $\widehat{f}$ is locally conformal or conformal if f is respectively locally conformal or conformal is the same as the proof of the corresponding portion of Theorem 1.2. ■

We have defined the cluster set earlier for mappings to the Riemann sphere. More generally, let G be a subset of a metric space X, let $f: G \to Y$ be a mapping from G to a topological space Y, and $p \in \partial G \setminus G$. Denote the cluster set of f at p by $C(f,p)$. That is, $C(f,p) = \{q \in Y : \exists p_n \in G, p_n \to p, f(p_n) \to q\}$. For $B \subset \partial G \setminus G$, we define the cluster set at B as

$$C(f,B) = \{q \in Y : \exists p \in B, \exists p_n \in G, p_n \to p, f(p_n) \to q\} = \bigcup_{p \in B} C(f,p).$$

In the sequel, we shall consider the cluster sets $C(f,p)$ and $C(f,B)$ for mappings $f: \Omega \to Y$, where Y is a bordered Riemann surface or a Riemann surface, Ω is the interior of a bordered Riemann surface $\widetilde{\Omega} = \Omega \cup b\Omega$, and both p and B are contained in the border $b\Omega$. In particular, suppose $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are bordered Riemann surfaces, $f: \Omega_1 \to \widetilde{\Omega}_2$ is a continuous mapping, and $p \in b\Omega_1$. If $\widetilde{\Omega}_2$ is compact, then $C(f,p)$ is not empty, but if $\widetilde{\Omega}_2$ is not compact, $C(f,p)$ may be empty. For example, this is the case for $C(f,0)$, when $\widetilde{\Omega}_1 = \widetilde{\Omega}_2$ is the closed upper half-plane and $f(z) = 1/z$. We shall say that the mapping f sends the border $b\Omega_1$ to the border $b\Omega_2$, if for every sequence $p_j \in \Omega_1$ converging to a point of $b\Omega_1$, the sequence $f(p_j)$ has a limit point in $b\Omega_2$. If f sends the border to the border, then $C(f,p)$ is a non-empty subset of $b\Omega_2$. Moreover, $C(f,p)$ is compact and connected, since

$$C(f,p) = \bigcap_{0<r<1} \overline{f(U_r^+)},$$

where $h: \widetilde{U} \to \{z : |z| < 1, \Im z \geq 0\}$ is a border chart at p and

$$U_r^+ = h^{-1}\{z : |z| < 1, r > \Im z > 0\}.$$

For B a closed subset of $b\Omega_1$, the cluster set $C(f,B)$ may not be closed, even if f is continuous and sends the border to the border. For example, let $\widetilde{\Omega}_2$ be the closed upper half-plane, $\widetilde{\Omega}_1$ the closed upper half-plane except the point 0, and $f(z) = z$. Then for $B = b\Omega_1$, the set B is closed in $b\Omega_1$, but $C(f,B)$ is not closed in $b\Omega_2$.

If f sends the border to the border, then $C(f,p)$ is a non-empty subset of $b\Omega_2$ and since, as we have seen, $C(f,p)$ is connected, it lies in a single component of $b\Omega_2$. Similarly, we shall say that f sends a border component B_1 of $b\Omega_1$ to the border $b\Omega_2$ if for every sequence $p_j \in \Omega_1$ converging to a point of B_1, the sequence $f(p_j)$ has a limit point in $b\Omega_2$. Also, we shall say that f sends a border component B_1 of $b\Omega_1$ to a border component B_2 of $b\Omega_2$ if for every sequence $p_j \in \Omega_1$ converging to a point of B_1, the sequence $f(p_j)$ has a limit point in B_2.

Let S be a topological space, and B a subset of S. Following Brown [3], we say that B is *collared* in S if there exists a homeomorphism h from $B \times [0,1)$ onto a neighbourhood of B such that $h(b,0) = b$ for all $b \in B$. Moreover, we say that the image $h(B \times [0,1))$ is a collar of B. If B can be covered by a collection of subsets relatively open in B each of which is collared in S, then B is said to be *locally collared* in S.

A bordered n-manifold is a connected metrizable topological space such that each point has a closed neighbourhood homeomorphic to the closed n-ball.

Theorem 4.2 (Brown [3]) *The border of a bordered n-manifold M is collared in M.*

Lemma 4.3 *Let $\widetilde{\Omega}_j = \Omega_j \cup b\Omega_j$, $j = 1,2$ be bordered Riemann surfaces. Let $f: \Omega_1 \to \widehat{\Omega}_2$ be a continuous mapping that sends the border to the border and let $B \subset b\Omega_1$. If B is compact or connected, then $C(f,B)$ is compact or connected respectively.*

Proof Suppose B is compact. Since each component of $b\Omega_1$ is closed and open in $b\Omega_1$, it follows that B is contained in the union $b_1 \cup \cdots \cup b_n$ of finitely many components of $b\Omega_1$ and that each $B_j = B \cap b_j$ is compact. Since $C(f,B) = \bigcup_{k=1}^n C(f,B_k)$, we may assume that B is contained in a single component b of $b\Omega_1$.

Let $h: b \times [0,1) \to H_b$ be a collar of b in $\widetilde{\Omega}_1$. Let I_n be a nested sequence of open subsets of b such that each $\overline{I}_n$ is compact and $B \subset I_n \subset \overline{I}_n \subset b$, $B = \bigcap_{n=1}^\infty I_n$ and put $U_n = h(I_n \times (0,1/n])$. Then $C(f,B) = \bigcap_{n=1}^\infty \overline{f(U_n)}$, and we see that $C(f,B)$ is closed.

To see that $C(f,B)$ is compact, it is sufficient to show that $\overline{f(U_1)}$ is compact. Suppose $\overline{f(U_1)}$ is not compact. Then since $\overline{f(U_1)}$ is closed, $\widehat{\Omega}_2$ is surely not compact and there is a sequence $q_n \in \overline{f(U_1)}$, such that $q_n \to *_2$, where $*_2$ is the ideal (Alexandroff) point of $\widehat{\Omega}_2$. By a diagonal process, we can construct a sequence $p_n \in U_1$, such that $f(p_n) \to *_2$. By choosing a subsequence, if necessary, we can assume that p_n converges to a point $p \in \overline{I}_1$. This contradicts the assumption that f sends the border to the border. Thus, $\overline{f(U_1)}$ is compact. Since $C(f,B)$ is a closed subset of the compact set $\overline{f(U_1)}$, it follows that $C(f,B)$ is also compact.

Suppose that B is not only compact, but also connected. Then we can take the I_n to be connected. Recall that, since f sends the border to the border, $C(f,B) \neq \varnothing$. The sets $\overline{f(U_n)}$ are connected subsets of the compact Hausdorff space $\overline{f(U_1)}$ and

$$\liminf \overline{f(U_n)} = \limsup \overline{f(U_n)} = C(f,B) \neq \varnothing.$$

It follows [9, Theorem 2-101] that $C(f,B)$ is connected.

We have shown that if B is compact and if B is moreover connected, then $C(f,B)$ is also connected.

Now we show that if B is connected, then $C(f,B)$ is connected, even if B is not compact. Since B is connected, it is contained in a single border component b. The only connected subsets of b are Jordan arcs and Jordan curves. Jordan curves are compact, so we may assume that B is a Jordan arc, possibly containing one or both end points. In any case, we may write B as the union of an increasing sequence of compact Jordan arcs $\overline{I}_n$. Since the $\overline{I}_n$ are compact and connected, we have shown that the sets $C(f,\bar{I}_n)$ are connected. Now $C(f,B)=\bigcup_{n=1}^{\infty} C(f,\bar{I}_n)$ and the $C(f,\bar{I}_n)$ are increasing, so $C(f,B)$ is connected. ∎

Lemma 4.4 *Let $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ be two bordered Riemann surfaces and $f:\Omega_1\to\widehat{\Omega}_2$ a holomorphic map. Then f sends a border component B_1 to the border $b\Omega_2$ if and only if it sends B_1 to some border component B_2 of $b\Omega_2$.*

Proof By the definition, the direction "if" is obvious. Now suppose f sends a border component B_1 of $b\Omega_1$ to the border $b\Omega_2$. It suffices to show that $C(f,B_1)$ is connected, but since B_1 is a border component, it is connected, and so by the previous lemma, $C(f,B_1)$ is connected. ∎

The following result extends the Carathéodory reflection principle to bordered Riemann surfaces.

Theorem 4.5 *For $j=1,2$, let $\widetilde{\Omega}_j=\Omega_j\cup b\Omega_j$ be bordered Riemann surfaces with respective interiors Ω_j, respective borders $b\Omega_j$, and respective doubles $\widehat{\Omega}_j$. Let $f:\Omega_1\to\widehat{\Omega}_2$ be a holomorphic mapping which sends the border $b\Omega_1$ to the border $b\Omega_2$. Then there is a holomorphic surjective extension*

$$\widehat{f}:\widehat{\Omega}_1\longrightarrow f(\Omega_1)\cup C(f,b\Omega_1)\cup f(\Omega_1)^*\subset\widehat{\Omega}_2,$$

such that $\widehat{f}(b\Omega_1)=C(f,b\Omega_1)$.

Proof Fix $p\in b\Omega_1$. By Lemma 4.4, $C(f,p)$ is contained in a single component B_2 of the border $b\Omega_2$. We consider two cases, depending on whether B_2 is an open Jordan arc or a Jordan curve. Throughout this proof, when we speak of a compact Jordan arc $[\alpha,\beta]$, we mean a compact Jordan arc whose end points are α and β. Similarly by an "open" Jordan arc (α,β), we mean the image of the open unit interval by a homeomorphism h such that $h(t)$ tends to the distinct points α and β, as t tends to 0 and 1, respectively.

Suppose first that B_2 is a Jordan arc. By the proof of Lemma 4.3, there is some compact Jordan arc $[\alpha,\beta]\subset B_2$ such that $C(f,p)$ is contained in the open Jordan arc (α,β). We may choose a closed arc $[a,b]$ about p in $b\Omega_1$, such that $C(f,q)\subset(\alpha,\beta)$, for each $q\in[a,b]$.

Construct a closed Jordan domain $\overline{G}_2$ in $\widetilde{\Omega}_2$, such that the Jordan curve $\overline{G}_2\smallsetminus G_2$, consists of the closed arc $[\alpha,\beta]$ and a cross-cut γ_2 of $\widetilde{\Omega}_2$. To see that this is possible, use a collar of B_2. Similarly, (see also Lemma 3.1) we may construct a closed border chart $\overline{G}_1$ for p, which is a closed Jordan domain in $\widetilde{\Omega}_1$, such that the Jordan curve $\overline{G}_1\smallsetminus G_1$ consists of a closed arc $[a,b]$ in $b\Omega_1$ and a cross-cut γ_1 of $\widetilde{\Omega}_1$. Let ϕ be the restriction of f to G_1. Denote by $\widetilde{G}_2$ the bordered Riemann surface whose interior is G_2 and whose border is (α,β). By Lemma 3.1, we may further assume that G_1 is so small that $\phi(G_1)\subset\widehat{G}_2$ and all boundary values of ϕ on (a,b) lie in (α,β).

Let h be a conformal mapping of G_2 onto the upper half-plane H^+. By Theorem 4.1, h extends to a conformal mapping $\widehat{h}:\widehat{G}_2\to\widehat{h(G_2)}\subset\overline{\mathbb{C}}$. The function $h\circ\phi$ also satisfies the hypotheses of Theorem 4.1, so $h\circ\phi$ extends to a meromorphic function $\widehat{h\circ\phi}:\widehat{G}_1\to\overline{\mathbb{C}}$. Since meromorphic functions on Riemann surfaces are the same as

https://doi.org/10.4153/CMB-2016-051-1 Published online by Cambridge University Press

holomorphic maps to the Riemann sphere, this extension can be considered as a holomorphic mapping $\widehat{G}_1 \to \overline{\mathbb{C}}$. On G_1 we have

$$\phi = h^{-1} \circ h \circ \phi = (\widehat{h})^{-1} \circ \widehat{h} \circ \phi = (\widehat{h})^{-1} \circ \widehat{(h \circ \phi)}.$$

Hence, ϕ extends to a holomorphic mapping $\widehat{\phi}: \widehat{G}_1 \to \widehat{G}_2$. Since ϕ is the restriction of f to G_1, this gives a holomorphic extension of f which we denote by f_p and $f_p: \widehat{G}_1 \to \widehat{G}_2$. Moreover, the value $f_p(p)$ lies on B_2, since $C(f,p) \subset B_2$.

Now we need to consider the case that B_2 is a Jordan curve. Let $\widetilde{C}_2 = C_2 \cup B_2$ be a collar about B_2 in $\widetilde{\Omega}_2$. The interior C_2 of $\widetilde{C}_2$ is planar and so, by Theorem 1.1 and the Koebe theorem on circular domains (see Theorem 2.7), there is a conformal mapping h of C_2 onto a domain $A = H^+ \setminus K$, where K is a closed disc in H^+. By Lemma 4.4, we can assume that h sends B_2 to $\mathbb{R} \cup \{\infty\}$. By Theorem 4.1, we can extend h to a meromorphic function $\widehat{h}: \widehat{C}_2 \to \overline{\mathbb{C}}$. Let B_1 be the border component containing p. Then by Lemma 4.4, $C(f,q) \subset B_2$, for each $q \in B_1$. Hence, if we fix a sufficiently small open arc α in B_1 which contains p and which is pre-compact in B_1, then we can construct a collar $\widetilde{C}_1 = C_1 \cup \alpha$ of α in $\widetilde{\Omega}_1$, such that $f(C_1) \subset \widehat{C}_2$. Let ϕ be the restriction of f to C_1. As for the case that B_2 was not compact, the function h extends meromorphically to $\widehat{C}_2$ and $h \circ \phi$ extends meromorphically to $\widehat{C}_1$. Consequently f extends to a holomorphic mapping $f_p: \widehat{C}_1 \to \widehat{C}_2$ and $f_p(p) \in B_2$. By the construction, $\widehat{C}_2 \subset \widehat{\Omega}_2$.

From the preceding, it follows that for every $p \in b\Omega_1$, there is a closed Jordan domain, $\overline{U}_p \subset \widetilde{\Omega}_1$, such that the Jordan curve $\overline{U}_p \setminus U_p$ consists of an open border arc α_p containing p and a cross-cut σ_p of Ω_1. Furthermore, there is a closed Jordan domain $\overline{V}_p \subset \widetilde{\Omega}_2$, such that the Jordan curve $\overline{V}_p \setminus V_p$ consists of an open border arc β_p and a cross-cut τ_p of Ω_2, such that, denoting $\widetilde{U}_p = U_p \cup \alpha_p$ and $\widetilde{V}_p = V_p \cup \beta_p$, f restricted to U_p extends to a holomorphic mapping $f_p: \widehat{U}_p \to \widehat{V}_p$, where $\widehat{U}_p$ is the double of $\widetilde{U}_p$ and $\widehat{V}_p$ is the double of $\widetilde{V}_p$. Moreover, $f_p(\alpha_p) \subset \beta_p$. We can assume that we have a closed border chart $h_p: \widetilde{V}_p \to \overline{\Delta}^+$.

These various holomorphic extensions $f_p, p \in b\Omega_1$ are compatible. That is, suppose p and q are two arbitrary points in the border $b\Omega_1$ of Ω_1, with corresponding holomorphic extensions $f_p: \widehat{U}_p \to \widehat{V}_p$ and $f_q: \widehat{U}_q \to \widehat{V}_q$. Suppose $\alpha_p \cap \alpha_q \neq \emptyset$. Then $f_p = f_q$ on $\widehat{U}_p \cap \widehat{U}_q$ by the uniqueness of holomorphic continuation.

It follows that there is an open neighbourhood of $b\Omega_1$ in $\widetilde{\Omega}_1$, which is a bordered surface of the form $\widetilde{U} = U \cup bU$, with interior $U \subset \Omega_1$ and border $bU = b\Omega_1$ and there is a holomorphic extension $\widehat{f}: \Omega_1 \cup \widehat{U} \to \widehat{\Omega}_2$, such that $\widehat{f}(b\Omega_1) \subset b\Omega_2$. Since, for $p^* \in U^*$, this extension is given by $\widehat{f}(p^*) = f(p)^*$, we may define the extension on all of Ω_1^* by the same formula. We now have a holomorphic extension $\widehat{f}: \widehat{\Omega}_1 \to \widehat{\Omega}_2$. ∎

As we already mentioned, for maps $f: \Omega_1 \to \widehat{\Omega}_2$ Theorem 4.5 can be considered as an extension of the Carathéodory reflection principle to Riemann surfaces. In the following, we consider the particular case that $f(\Omega_1) \subset \Omega_2$ and obtain a generalization of the Schwarz reflection principle.

https://doi.org/10.4153/CMB-2016-051-1 Published online by Cambridge University Press

Theorem 4.6 *For $j = 1,2$, let $\widetilde{\Omega}_j = \Omega_j \cup b\Omega_j$ be bordered Riemann surfaces; let $f: \Omega_1 \to \Omega_2$ be a holomorphic map which sends the border $b\Omega_1$ to the border $b\Omega_2$, and let $\widehat{f}: \widehat{\Omega}_1 \to \widehat{\Omega}_2$ be the holomorphic extension given by Theorem 4.5. If f is locally conformal (respectively conformal), then so is $\widehat{f}$.*

If f is conformal and onto and $C(f, b\Omega_1) = b\Omega_2$, then $\widehat{f}$ is a biholomorphic mapping of $\widehat{\Omega}_1$ onto $\widehat{\Omega}_2$.

Proof Let $\widehat{U}, \widehat{U}_p, h_p$, and f_p be the same as in the proof of Theorem 4.5. From the formula $\widehat{f}(p^*) = f(p)^*$, it clear that if f is locally conformal on Ω_1, then $\widehat{f}$ is locally conformal on $\Omega_1 \cup \Omega_1^*$, and we claim that it is also locally conformal on $\widehat{U}$. It is sufficient to show that it is locally conformal on each $\widehat{U}_p$. Clearly, $(h_p \circ f_p)(U_p)$ is contained in the open upper half-plane and so, by Theorem 4.1, $\widehat{h_p \circ f_p}$ is locally conformal on $\widehat{U}_p$. Consequently f_p is also locally conformal on $\widehat{U}_p$. It follows that $\widehat{f}$ is locally conformal.

The proof that if f is conformal, then $\widehat{f}$ is also conformal, is similar to that of the analogous statement in Theorems 1.2 and 4.1. If f is conformal and onto and $C(f, b\Omega_1) = b\Omega_2$, then $\widehat{f}$ is conformal and onto and hence is a biholomorphic mapping of $\widehat{\Omega}_1$ onto $\widehat{\Omega}_2$. ∎

5 Bordered Regions in Riemann Surfaces

We wish to show the equivalence between bordered Riemann surfaces and certain domains in Riemann surfaces together with a portion of their boundary. We shall call these *bordered domains*, and they include Jordan domains as the prime example.

Let Ω be a domain in a Riemann surface. An open Jordan arc $A \subset \partial\Omega$ is called a *free boundary arc* of the domain Ω if for each point $p \in A$ there is an open neighborhood U of p in Ω and a homeomorphism $h_p: \overline{U} \to \overline{\Delta}^+$, where Δ^+ is the open upper half-disc $\{|z| < 1, \Im z > 0\}$, $h(\overline{U} \cap A) = [-1, +1]$, and $h_p(p) = 0$. The maps h_p are similar to border charts in a bordered Riemann surface, where the h_p were additionally required to have a certain analyticity property.

An open arc $A \subset \partial\Omega$ is called a *doubly free boundary arc* of the domain Ω if, for each point $p \in A$, there is an open set $U \subset R$ and a homeomorphism $h: \overline{U} \to \overline{\Delta}$, where Δ is the open unit disc, $h(U \cap A) = (-1, +1)$, $h(U \cap \Omega) = \Delta^+$, and $h(p) = 0$.

As an example, if Ω is a Jordan domain in $\overline{\mathbb{C}}$, then it follows from the Schoenflies theorem that $\partial\Omega$ is doubly free.

Remark If A is a doubly free boundary arc of a domain Ω in a Riemann surface, then A is a free boundary arc of Ω.

Proof Fix $p \in A$. By the definition, there exists an open set $N \subset R$, and a homeomorphism $g: \overline{N} \to \overline{\Delta}$ such that $g(N \cap A) = (-1, 1)$, $g(N \cap \Omega) = \Delta^+$, and $g(p) = 0$. Take $U := N \cap \Omega$ and $h := g|_{\overline{U}}$ in the definition of free boundary arc. ∎

Let us say that a subset $E \subset \partial\Omega$ is a *doubly free boundary set* of Ω if each point of E is contained in a doubly free boundary arc of E.

https://doi.org/10.4153/CMB-2016-051-1 Published online by Cambridge University Press

If Ω is a domain (open connected set) in a Riemann surface R and B is a (non empty) doubly-free boundary set of Ω, then we shall say that $\widetilde{\Omega} = \Omega \cup B$ is a bordered region in R. We note that a bordered region $\widetilde{\Omega} = \Omega \cup B$ is compact if and only if $\widetilde{\Omega} = \overline{\Omega}$ and Ω is bounded. In this case, $B = \partial\Omega$ and B consists of finitely many disjoint Jordan curves. For this reason, we call a compact bordered region a closed Jordan region. A closed Jordan region of genus zero whose boundary is a single Jordan curve is a closed Jordan domain. A Jordan region is the interior of a closed Jordan region and a Jordan domain is the interior of a closed Jordan domain. The following theorem asserts that every bordered region can be considered to be a bordered Riemann surface, thus giving us a multitude of bordered Riemann surfaces. It is similar to a result in [1], where there is the further hypothesis that the border B is a locally analytic arc.

Theorem 5.1 *Suppose $\widetilde{\Omega} = \Omega \cup B$ is a bordered region in a Riemann surface R. Then $\widetilde{\Omega}$ admits the structure of a bordered Riemann surface with interior Ω and border B. The complex structures on Ω as interior of the bordered Riemann surface and as domain in R are the same. In the other direction, if $\widetilde{\Omega}$ is a bordered Riemann surface, then $\widetilde{\Omega}$ may be considered as a bordered region in the double $\widehat{\Omega}$.*

Proof Fix a point $p \in B$. Since B is a doubly free boundary set of Ω, there is an open set $U \subset R$ and a homeomorphism $h: \overline{U} \to \overline{\Delta}$, where Δ is the open unit disc, $h(U \cap B) = (-1, +1)$, $h(U \cap \Omega) = \Delta^+$, and $h(p) = 0$. Since U is planar, it follows from Theorem 1.1 that there is a biholomorphic mapping ϕ_p of U onto a plane domain G_p. We may assume that $\phi_p(p) = 0$. Let $A \subset U \cap B$ be a compact Jordan arc containing p, not as an end point. Then $J = \phi_p(A)$ is a compact Jordan arc in G_p, containing 0, not as an end point. By Lemma 2.6, there is a closed Jordan domain $\overline{W}_p$ in G_p, such that $\overline{W}_p \cap \phi_p(U \cap B) = J$, and J is a cross-cut of W_p. That is, J is contained in W_p, except for its end points that lie on the Jordan curve ∂W_p. By the Jordan curve theorem, J separates $\overline{W}_p$ into two closed Jordan domains, whose intersection is J. By construction, ϕ_p^{-1} maps one of these, call it $\overline{W}_p^+$, homeomorphically to a closed Jordan domain $\overline{V}_p \subset \widetilde{\Omega}$. We note that $\overline{V}_p$ is a closed neighbourhood of p in $\widetilde{\Omega}$, $\phi: \overline{V}_p \to \overline{W}_p^+$ is a homeomorphism, $\phi: V_p \to W_p^+$ is biholomorphic, ϕ_p maps $\overline{V}_p \cap B$ onto J, and $\phi(p) = 0$. By the Riemann mapping theorem and the Osgood–Carathéodory theorem, there is a conformal mapping $\sigma_p: W_p^+ \to \Delta^+$, which extends to a homeomorphism $\overline{W}_p^+ \to \overline{\Delta}^+$, such that $\sigma_p(J) = [-1, +1]$ and $\sigma_p(0) = 0$.

Set $\eta_p = \sigma_p \circ \phi_p$. We may consider the family of maps $\eta_p: \overline{V}_p \to \overline{\Delta}^+$, $p \in B$, as closed border charts and if, for every $p \in \Omega$, we add to this family a chart $\eta_p: V_p \to \Delta^+$ at p for the Riemann surface Ω, then these combined charts give $\widetilde{\Omega} = \Omega \cup B$ the desired structure of a bordered Riemann surface. Although the subset B of $\partial\Omega$ is locally an arc, these arcs may be non-analytic. Nevertheless, the change of border charts $\eta_q \circ \eta_p^{-1}$ is analytic on $\eta_p(B \cap (\overline{V}_p \cap \overline{V}_q))$, by the Schwarz reflection principle. This completes the proof of the first part of the theorem.

In the other direction it is not hard to see that if $\widetilde{\Omega} = \Omega \cup b\Omega$ is a bordered Riemann surface, then $\widetilde{\Omega}$ is a bordered region in the Riemann surface $\widehat{\Omega} = \Omega \cup b\Omega \cup \Omega^*$. ∎

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A particular consequence of the preceding theorem is that every bordered region $\widetilde{\Omega} = \Omega \cup B$ in $\mathbb{C}$ can be endowed with the structure of a bordered Riemann surface and the restriction of this structure to Ω is compatible with the given holomorphic structure on Ω. This is striking, considering that the curves which comprise the border of Ω need not be analytic. Nevertheless, the change of border charts

$$\phi_q \circ \phi_p^{-1}, \quad p, q \in B \subset \partial\Omega,$$

which map the real interval $\phi_p(\partial\Omega \cap (V_p \cap V_q))$ to the real interval $\phi_q(\partial\Omega \cap (V_p \cap V_q))$ is analytic. Of course, an illustration of this is the Riemann mapping theorem (with the Osgood–Carathéodory theorem), which sends an arbitrary closed Jordan domain $\overline{\Omega}$ to the closed unit disc. If the Jordan curve $\partial\Omega$ is not analytic, the structure of a bordered Riemann surface we give to $\overline{\Omega}$ is definitely *not* the restriction to $\overline{\Omega}$ of the complex structure of $\mathbb{C}$, although the restrictions to Ω of both structures are the same.

Theorem 5.2 *If $\widetilde{\Omega} = \Omega \cup B$ is a bordered region in a Riemann surface R where the region Ω is planar, then $\widetilde{\Omega}$ has a planar neighbourhood.*

Proof If A is a component of the exterior $R \setminus \overline{\Omega}$ whose boundary meets B, denote by B_A the intersection $B \cap \partial A$. Then $\widetilde{A} = A \cup B_A$ is also a bordered region.

Since the border of every bordered manifold is collared [3], each set B_A is collared in both $\Omega \cup B_A$ and $A \cup B_A$. Hence, there is an open neighbourhood W of B and a homeomorphism $h: B \times (-1, +1) \to W$, with

$$h(B \times (-1, 0]) = \widetilde{\Omega} \cap W, \quad h(p, 0) = p, \quad h(B \times [0, +1)) = W \setminus \Omega.$$

The function $\phi(t) = -1/2 + 3(t + 1/2)$ defines a homeomorphism

$$\phi: [-1/2, 0) \to [-1/2, +1),$$

which induces a homeomorphism $\Phi: B \times [-1/2, 0) \to B \times [-1/2, +1)$, given by $\Phi(p, t) = (p, \phi(t))$. Set

$$C = h(B \times \{-1/2\}), \quad V = h(B \times [-1/2, 0]), \quad U = h(B \times [-1/2, +1)).$$

The function $G = h \circ \Phi \circ h^{-1}$ defines a homeomorphism of V onto U, which fixes points of C. Denoting $N = \Omega \cup U$, we have a homeomorphism $H: \Omega \to N$, defined by setting $H(p) = p$ for $p \in \Omega \setminus V$ and $H(p) = G(p)$ for $p \in V$. Since Ω is of genus zero, it is planar, and since N is homeomorphic to Ω, the neighbourhood N is also planar, which completes the proof. ■

The following theorem may be considered as a generalization of the Osgood–Carathéodory theorem to bordered regions in Riemann surfaces.

Theorem 5.3 *For $j = 1, 2$, let $\widetilde{\Omega}_j = \Omega_j \cup B_j$ be bordered regions in Riemann surfaces R_j with respective interiors Ω_j and respective borders B_j. Let $f: \Omega_1 \to \Omega_2$ be a holomorphic mapping, which sends B_1 to B_2. Then f extends to a (unique) continuous surjective mapping $\widetilde{f}: \widetilde{\Omega}_1 \to f(\Omega_1) \cup C(f, B_1) \subset \widetilde{\Omega}_2$. If f is locally conformal or conformal, then $\widetilde{f}$ is respectively locally injective or injective. If f is conformal and onto and $C(f, B_1) = B_2$, then $\widetilde{f}$ is a homeomorphism of $\widetilde{\Omega}_1$ onto $\widetilde{\Omega}_2$.*

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Proof By Theorem 5.1, each $\widetilde{\Omega}_j$ can be endowed with the structure of a bordered Riemann surface, with interior Ω_j and border B_j, and such that on Ω_j this structure is compatible with the given holomorphic structure. This implies that f is also holomorphic, when considered as a mapping between the interiors of the bordered Riemann surfaces. By the Carathéodory reflection principle for bordered Riemann surfaces (Theorem 4.5), the mapping f extends to a holomorphic mapping $\widehat{f}: \widehat{\Omega}_1 \to \widehat{\Omega}_2$.

We claim that the restriction $\widetilde{f}$ of $\widehat{f}$ to $\widetilde{\Omega}_1$ has the desired properties. First of all, since $\widehat{f}$ is continuous, the restriction $\widetilde{f}$ is certainly a continuous extension of f. Since Ω_1 is dense in $\widetilde{\Omega}_1$, the continuous extension of f is unique. By Theorem 4.5, $\widehat{f}(B_1) = C(f, B_1)$, so $\widetilde{f}$ is surjective onto $f(\Omega_1) \cup C(f, B_1)$. Since f sends B_1 to B_2, this image is certainly contained in $\widetilde{\Omega}_2$.

If f is locally conformal or conformal, then by Theorem 4.6, the mapping $\widehat{f}$ is locally conformal or conformal, respectively, and hence $\widetilde{f}$ is locally injective or injective, respectively.

It follows that if f is conformal onto and $C(f, B_1) = B_2$, then by Theorem 4.6 $\widehat{f}: \widehat{\Omega}_1 \to \widehat{\Omega}_2$ is a biholomorphism and hence a homeomorphism. In particular, $\widehat{f}|_{\widetilde{\Omega}_1} = \widetilde{f}$ is a homeomorphism. ∎

We remark that in the above theorem, if f is conformal and onto but $C(f, B_1)$ is strictly included in B_2 (instead of equal to B_2), then f does not in general extend to a homeomorphism, even of $\widetilde{\Omega}_1$ onto $\widetilde{\Omega}_2$. For example, let Ω_1 and Ω_2 be the open unit disc. Let B_2 be the circle $\mathbb{T}$ and $B_1 = \mathbb{T} \setminus \{1\}$. Let $f(z) = z$. Then f is conformal and onto, and f sends B_1 strictly into B_2, but f does not extend to a homeomorphism of $\widetilde{B}_1$ onto $\widetilde{B}_2$.

6 Some Comments and Applications

If two plane domains are conformally equivalent, then their automorphism groups are isomorphic. Thus, by the Riemann mapping theorem, for simply connected plane domains, we only need to understand the automorphism groups of the disc and the plane which are well known.

If a plane domain is not simply connected, the group $\mathrm{Aut}(\Omega)$ of conformal self-maps is "in general small". However, for a given domain, there may be many conformally equivalent domains that are presented in very different ways. For example, let Ω be a Jordan region in $\mathbb{C}$ and let D be a disc containing $\overline{\Omega}$. Now let f be an arbitrary conformal mapping of D onto a simply connected domain. Then $f(\Omega)$ is conformally equivalent to Ω, but may appear quite different as a subset of $\mathbb{C}$.

For $n > 2$, an example of an n-connected Jordan region $\overline{\Omega} \subset \mathbb{C}$ for which $\mathrm{Aut}(\Omega)$ is not trivial is obtained by choosing $0 < r < 1$ and taking as Ω the unit disc Δ from which we have removed $n-1$ disjoint closed discs of the same small radius, whose centres are equidistributed on the circle $|z| = r$. Clearly, rotations of angle $j2\pi/(n-1)$, $j = 0, 1, \ldots, n-1$, are distinct elements of $\mathrm{Aut}(\Omega)$.

Let Ω be the interior of a compact bordered Riemann surface $\widetilde{\Omega}$. Let $p \in \Omega$ and S_p be the family of holomorphic functions from Ω to the unit disc which take p to zero, and which have in a fixed coordinate chart, a non-negative derivative at p. The

Ahlfors function for Ω and p is the unique function A in S_p such that

$$A'(p) = \max_{f \in H_p} \operatorname{Re} f'(p).$$

It is a non-trivial fact that every Ahlfors function is a proper mapping of Ω onto the unit disc Δ. The Ahlfors function for a Jordan region in $\mathbb{C}$ is presented in [8, Ch. VI]. For a monumental treatment of Ahlfors functions, see [7].

Corollary 6.1 *Let $\widetilde{\Omega}$ be a compact bordered Riemann surface and let $f: \Omega \to \Delta$ be an Ahlfors function of Ω onto the open unit disc Δ. Then f extends to a meromorphic function $\widehat{f}: \widehat{\Omega} \to \mathbb{C} \cup \{\infty\}$.*

For Riemann surfaces Ω, Ω_1, and Ω_2, let us denote by $\operatorname{Iso}(\Omega_1, \Omega_2)$ the space of biholomorphic mappings $\Omega_1 \to \Omega_2$ and by $\operatorname{Aut}(\Omega)$, the automorphism group $\operatorname{Iso}(\Omega, \Omega)$. Similarly, for bordered Riemann surfaces $\widetilde{\Omega}, \widetilde{\Omega}_1$, and $\widetilde{\Omega}_2$, let us denote by $\operatorname{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2)$ the space of homeomorphisms $\widetilde{\Omega}_1 \to \widetilde{\Omega}_2$ whose restrictions to Ω_1 are in $\operatorname{Iso}(\Omega_1, \Omega_2)$ and by $\operatorname{Aut}(\widetilde{\Omega})$, the space $\operatorname{Iso}(\widetilde{\Omega}, \widetilde{\Omega})$.

Theorem 6.2 (Schwarz 1879) *The automorphism group of every compact Riemann surface of genus $g \geq 2$ is finite.*

A compact bordered Riemann surface is said to be of type (g, n) if it is of genus g and the number of border components is n.

Corollary 6.3 *If $\widetilde{\Omega}$ is a compact bordered Riemann surface of type (g, n) and $2g+n \geq 3$, then $\operatorname{Aut}(\widetilde{\Omega})$ is finite.*

Proof It follows from Theorem 4.6 that every $\phi \in \operatorname{Aut}(\widetilde{\Omega})$ extends to $\widehat{\phi} \in \operatorname{Aut}(\widehat{\Omega})$. The genus of the double $\widehat{\Omega}$ is $2g + n - 1$, which is greater than or equal to 2. By the Schwarz theorem, $\operatorname{Aut}(\widehat{\Omega})$ is finite. Consequently, since the mapping $\phi \mapsto \widehat{\phi}$ is injective, $\operatorname{Aut}(\widetilde{\Omega})$ is also finite. ∎

The hypothesis on the type is satisfied if the genus g is not zero or if the genus is zero and the number n of border components is at least 3.

The restriction mapping gives a natural embedding $\operatorname{Aut}(\widetilde{\Omega}) \hookrightarrow \operatorname{Aut}(\Omega)$, but this need not be surjective. For example, if $\widetilde{\Delta}$ is the bordered Riemann surface whose interior is the open unit disc Δ and whose border is an arc $e^{i\theta}, 0 < \theta < \beta$ for some $\beta \in (0, 2\pi)$, then $\operatorname{Aut}(\widetilde{\Delta})$ is the proper subgroup of $\operatorname{Aut}(\Delta)$ described as follows. Fix $\alpha \in (0, \beta)$. The group $\operatorname{Aut}(\widetilde{\Delta})$ consists of the elements $\phi_\gamma \in \operatorname{Aut}(\Delta)$ which send the points $1, e^{i\alpha}, e^{i\beta}$ to the points $1, e^{i\gamma}, e^{i\beta}$, respectively, for $0 < \gamma < \beta$. They are thus parametrized by the values $\gamma, 0 < \gamma < \beta$.

If $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are bordered Riemann surfaces and $\operatorname{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2) \neq \varnothing$, then every element f of $\operatorname{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2)$ induces bijections

$$\operatorname{Aut}(\widetilde{\Omega}_1) \longrightarrow \operatorname{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2), \quad \phi \mapsto f \circ \phi$$

and

$$\operatorname{Aut}(\widetilde{\Omega}_2) \longrightarrow \operatorname{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2), \quad \psi \mapsto \psi \circ f.$$

In this situation, the groups $\mathrm{Aut}(\widetilde{\Omega}_1)$ and $\mathrm{Aut}(\widetilde{\Omega}_2)$ are isomorphic and have the same cardinality as the family $\mathrm{Iso}(\widetilde{\Omega}_1, \widetilde{\Omega}_2)$. Of course, for "most" Riemann surfaces Ω, the group $\mathrm{Aut}(\Omega)$ is trivial. Similarly, for most bordered Riemann surfaces $\widetilde{\Omega}$, the group $\mathrm{Aut}(\Omega)$ is trivial. For such $\widetilde{\Omega}$, the subgroup $\mathrm{Aut}(\widetilde{\Omega})$ is, of course, also trivial.

Since the interior of every bordered region in a Riemann surface can be viewed as a Riemann surface, it follows that if $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are two such bordered regions, the family $\mathrm{Iso}(\Omega_1, \Omega_2)$ is usually empty and, if not, then it has the same cardinality as $\mathrm{Aut}(\widetilde{\Omega}_1)$ and $\mathrm{Aut}(\widetilde{\Omega}_2)$. For a general Riemann surface, and in particular for a general domain Ω in a Riemann surface, the group $\mathrm{Aut}(\Omega)$ is usually trivial.

There are interesting exceptional bordered regions $\widetilde{\Omega}$ of infinite genus, for which $\mathrm{Aut}(\widetilde{\Omega})$ is infinite. For example, consider the bordered region in $\mathbb{C}$:

$$\widetilde{\Omega} = \mathbb{C} \setminus \bigcup_{-\infty}^{+\infty} \Delta_j,$$

where Δ_j is the open disc of center j and radius $1/3$. Then the interior Ω is of infinite connectivity and $\mathrm{Aut}(\Omega)$ is clearly infinite. We can easily modify this example to obtain an example of infinite genus. Take two copies of $\mathbb{C}$ from which we have removed the slits $z = x + i : j < x < j + 1/2, j = 0, \pm1, \pm2, \dots$ and let R be the Riemann surface obtained by gluing these two slit domains along the slits in the usual way. Let $\widetilde{W}$ be the bordered region in R, obtained by removing the open discs Δ_j from each sheet of R. Then $\widetilde{W}$ is of infinite genus, has infinitely many border components and $\mathrm{Aut}(W)$ is again clearly infinite.

For more information regarding domains with infinite automorphism groups, see [11].

Acknowledgment We thank Dmitry Khavinson and Malik Younsi for helpful comments and references regarding the Osgood–Carathéodory theorem for multiply connected planar domains.

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