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DEGREE OF POINTEDNESS OF A CONVEX FUNCTION

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A convex function f is said to be pointed if its epigraph has a recession cone which is pointed. Partial pointedness of f refers to the case in which such a recession cone is only partially pointed. In this note we show that the degree of pointedness of f is related to the "thickness" of the effective domain of the conjugate function f^*.

1. INTRODUCTION

Let K be a closed convex cone in some finite dimensional linear space X. It is easy to see that

$$\ell(K) := K \cap -K$$

is the largest subspace of X which is contained in K. The dimension of such a subspace can be used to measure the degree of pointedness of K.

DEFINITION 1: The *degree of pointedness* of K is defined as the integer

$$p[K] := \dim X - \dim\ \ell(K). \tag{1}$$

If $p[K] = \dim\ X$, then one says that K is pointed. If $0 < p[K] < \dim\ X$, then one says that K is partially pointed.

Consider a function $f : X \to R \cup \{+\infty\}$ whose effective domain

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$$

is nonempty, and whose epigraph

$$\operatorname{epi} f := \{(x, \lambda) \in X \times R : f(x) \leqslant \lambda\}$$

is convex and closed. The class of such functions is usually denoted by $\Gamma_0(X)$. The recession cone of the set epi f is defined by

$$(\operatorname{epi} f)_\infty := \{(u, \alpha) \in X \times R : \quad (u, \alpha) + \operatorname{epi} f \subseteq \operatorname{epi} f\}.$$

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The recession function f_∞ of f is given by

$$f_\infty(u) := \sup\{f(x+u) - f(x) : x \in \operatorname{dom} f\} \quad \text{for all} \quad u \in X.$$

Both notions are standard in the context of convex analysis and can be consulted, for instance, in the book by Rockafellar [**3**]. In this note we study the following new concept.

DEFINITION 2: The *degree of pointedness* of the function $f \in \Gamma_0(X)$ is the integer

$$p[f] := \dim X - \dim \ell((\operatorname{epi} f)_\infty). \tag{2}$$

If $p[f] = \dim X$, then f is said to be pointed. If $0 < p[f] < \dim X$, then f is called partially pointed.

REMARK. According to the above definition, f is pointed if and only if $(\operatorname{epi} f)_\infty$ is a pointed cone. This case has already been considered by Benoist and Hiriart-Urruty [**1**, Definition 2.3]. However, these authors do not address the question of the dimension of $\ell((\operatorname{epi} f)_\infty)$.

2. POINTEDNESS AND CONJUGACY.

In this section we derive a simple formula for computing the degree of pointedness of a function $f \in \Gamma_0(X)$. Recall that the Legendre–Fenchel conjugate of f is the function $f^* \in \Gamma_0(X)$ defined by

$$f^*(y) := \sup_{x \in X}\{\langle y, x\rangle - f(x)\} \quad \text{for all } y \in X,$$

where $\langle \cdot, \cdot \rangle$ is a given inner product in the space X. The next theorem says that the degree of pointedness of f is equal to the dimension of the effective domain of f^*. The dimension of a nonempty convex set $A \subseteq X$ is defined as the dimension of the affine hull of A (see [**3**, p.12]).

THEOREM 1. *Let X be a finite dimensional linear space, and let $f \in \Gamma_0(X)$. Then,*

$$\dim(\operatorname{dom} f^*) + \dim \ell((\operatorname{epi} f)_\infty) = \dim X. \tag{3}$$

PROOF: We start by proving the equality

$$\ell((\operatorname{epi} f)_\infty) = \langle \operatorname{dom} f^* \times \{-1\}\rangle^\perp. \tag{4}$$

The notation $\langle C \rangle$ refers to the linear space spanned by the set C, and $\langle C\rangle^\perp$ stands for the orthogonal complement of $\langle C \rangle$. By definition one has

$$(u, \alpha) \in \ell((\operatorname{epi} f)_\infty) \Leftrightarrow (u, \alpha) \in (\operatorname{epi} f)_\infty \cap -(\operatorname{epi} f)_\infty.$$

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Since $(\operatorname{epi} f)_\infty = \operatorname{epi} f_\infty$, one can also write

$$\begin{aligned}(u,\alpha) \in \ell((\operatorname{epi} f)_\infty) &\Leftrightarrow f_\infty(u) \leqslant \alpha \quad \text{and} \quad f_\infty(-u) \leqslant -\alpha \\ &\Leftrightarrow f_\infty(u) \leqslant \alpha \leqslant -f_\infty(-u).\end{aligned}$$

Now we use the fact that f_∞ is the support function of the set $\operatorname{dom} f^*$ (see [**2**]), that is,

$$f_\infty(u) = \sup\{\langle y,u\rangle : y \in \operatorname{dom} f^*\}.$$

One has also

$$-f_\infty(-u) = \inf\{\langle y,u\rangle : y \in \operatorname{dom} f^*\}.$$

Hence

$$\begin{aligned}(u,\alpha) \in \ell((\operatorname{epi} f)_\infty) &\Leftrightarrow \langle y,u\rangle = \alpha \quad \text{for all} \quad y \in \operatorname{dom} f^* \\ &\Leftrightarrow \langle (y,\beta),(u,\alpha)\rangle = 0 \quad \text{for all} \quad (y,\beta) \in \operatorname{dom} f^* \times \{-1\}.\end{aligned}$$

Equality (4) is proven in this way. To complete the proof of the theorem, it suffices to observe that

$$\dim A = \dim \langle A \times \{-1\}\rangle - 1, \tag{5}$$

whenever A is a nonempty convex set in X. The proof of (5) is essentially an exercise in linear algebra. □

REMARK. By combining Theorem 1 and [**3**, Theorem 13.4], one sees that $\ell((\operatorname{epi} f)_\infty)$ has the same dimension as the set $\{u \in X : f_\infty(u) = -f_\infty(-u)\}$. The latter set is known as the lineality space of f.

We end this section by illustrating Theorem 1 with two examples.

EXAMPLE 1. Consider the space $X = S_n$ of symmetric matrices of order $n \times n$ equipped with the usual inner product $\langle x,y\rangle := \operatorname{trace}(xy)$. The variance of a matrix $x \in S_n$ is defined by

$$(\mathrm{var})(x) := \frac{1}{n}\langle x,x\rangle - \left(\frac{\operatorname{trace} x}{n}\right)^2.$$

The function $\mathrm{var} : S_n \to R$ is convex, and its conjugate $(\mathrm{var})^* : S_n \to R \cup \{+\infty\}$ is given by

$$(\mathrm{var})^*(y) = \begin{cases} \dfrac{n}{4}\langle y,y\rangle & \text{if } \operatorname{trace} y = 0, \\ +\infty & \text{otherwise.}\end{cases}$$

The dimension of $\operatorname{dom}(\mathrm{var})^* = \{y \in S_n : \operatorname{trace} y = 0\}$ is equal to $\dim S_n - 1$. Hence, the function var is partially pointed and its degree of pointedness is

$$p[\mathrm{var}] = \dim S_n - 1 = (n^2 + n - 2)/2.$$

EXAMPLE 2. Let $X = S_n$ be as in Example 1. Let $\lambda_{\max}(x)$ denote the largest eigenvalue of the matrix $x \in S_n$. It is known that $\lambda_{\max} : S_n \to R$ is equal to the support function of the set

$$A = \{x \in S_n : \quad x \text{ is positive semidefinite, } \operatorname{trace} x = 1\}.$$

Hence $\operatorname{dom}(\lambda_{\max})^* = A$ is a set of dimension equal to $\dim S_n - 1$. Thus, $\lambda_{\max}$ is partially pointed and $p[\lambda_{\max}] = (n^2 + n - 2)/2$. In this example one can also compute $p[\lambda_{\max}]$ by using the very definition of this number. The set $\operatorname{epi}(\lambda_{\max})$ is a closed convex cone, so it coincides with its recession cone. One has

$$\begin{aligned}(x, \lambda) \in \ell(\operatorname{epi}(\lambda_{\max})) &\Leftrightarrow \lambda_{\max}(x) \leqslant \lambda \text{ and } \lambda_{\max}(-x) \leqslant -\lambda \\ &\Leftrightarrow \text{ all the eigenvalues of } x \text{ are equal to } \lambda.\end{aligned}$$

If $i \in S_n$ denotes the identity matrix, then

$$\ell(\operatorname{epi}(\lambda_{\max})) = \{(\lambda i, \lambda) : \lambda \in R\} = R\{(i, 1)\} \subset S_n \times R$$

is a space of dimension 1.

3. Pointedness and Growth Condition

Recall that each $f \in \Gamma_0(X)$ can be minorised by some affine function, that is, one can find $y \in X$ and $b \in R$ such that

$$f(x) \geqslant \langle y, x\rangle + b \quad \text{for all } x \in X. \tag{6}$$

Is it possible to obtain a more precise information on the growth of f? In other words, is it possible to write a stronger growth condition for f? As already observed by Benoist and Hiriart-Urruty [1, Theorem 2.4], a growth condition of the type

$$f(x) \geqslant r\,\|x\| + \langle y, x\rangle + b \quad \text{for all } x \in X, \tag{7}$$

with $r > 0$, characterises the class of functions $f \in \Gamma_0(X)$ which are pointed. This observation leads us to think that a partially pointed function $f \in \Gamma_0(X)$ satisfies a growth condition which is intermediate between (6) and (7). The purpose of the next theorem is to display in a clear-cut manner the relationship between growth conditions and degree of pointedness. To start with, observe that the conditions (6) and (7) can be written in a common format, namely:

$$f(x) \geqslant \psi_A^*(x) + b \quad \text{for all } x \in X,$$

where ψ_A^* denotes the support function of $A \subseteq X$. In the former case A corresponds to the zero-dimensional set $\{y\}$, and in the latter case A is the full dimensional ball $B(y, r) := \{z \in X : \|z - y\| \leqslant r\}$.

Theorem 2. *The degree of pointedness of the function* $f \in \Gamma_0(X)$ *is given by*

$$p[f] = \max_{A \in C(X)} \{\dim\ A : f - \psi_A^* \ \textit{ is minorised}\}, \tag{8}$$

where $C(X)$ *denotes the class of nonempty convex sets in* X. *One can also write*

$$p[f] = \max_{A \in C(X)} \{\dim\ A : f_\infty \geqslant \psi_A^*\}, \tag{9}$$

where the maximum in (9) is attained at $A = \operatorname{dom} f^*$ *(or, more generally, at any set* A *which is contained in the closure of* $\operatorname{dom} f^*$ *and which has the same dimension as* $\operatorname{dom} f^*$*).*

Proof: The inequality $f_\infty \geqslant \psi_A^*$ is equivalent to the inclusion

$$\overline{A} \subseteq \overline{\operatorname{dom} f^*},$$

where the upper bar denotes the closure operation in X. Since the closure operation does not affect the dimension of a convex set (see [**3**, Theorem 6.2]), the maximum in (9) is attained at any $A \subseteq \overline{\operatorname{dom} f^*}$ which has the same dimension as $\operatorname{dom} f^*$. Now we prove that $p[f] = m[f]$, where $m[f]$ denotes the term on the right–hand side of (8). It is fairly clear that

$$f - \psi_A^* \text{ is minorised } \Rightarrow f_\infty \geqslant \psi_A^*.$$

Thus (9) yields the inequality $p[f] \geqslant m[f]$. According to the Toland–Singer duality theorem (see [**4**, **5**]), for all $A \in C(X)$, one has:

$$\inf_{x \in X}\{f(x) - \psi_A^*(x)\} = -\sup_{y \in \overline{A}} f^*(y).$$

Hence,

$$m[f] = \max_{A \in C(X)} \{\dim\ A : f^* \text{ is majorised over } \overline{A}\}. \tag{10}$$

Take any $y \in ri(\operatorname{dom} f^*)$, where "$ri$" stands for the relative interior (see [**3**, p.44]). Then, for some $r > 0$ sufficiently small, one has

$$A_r := B(y, r) \cap \overline{\operatorname{dom} f^*} \subset ri(\operatorname{dom} f^*).$$

According to Rockafellar [**3**, Theorem 10.1], the function f^* is continuous relative to $ri(\operatorname{dom} f^*)$. Hence, f^* is majorised over the compact set A_r, and

$$m[f] \geqslant \dim\ A_r.$$

But

$$\dim\ A_r = \dim\left(\overline{\operatorname{dom} f^*}\right) = \dim\left(\operatorname{dom} f^*\right) = p[f].$$

This proves the reverse inequality $m[f] \geqslant p[f]$, and completes the proof of the theorem. ∎

REMARK. Since f and ψ_A^* may take the value $+\infty$ at the same time, we have implicitly adopted the rule $(+\infty)-(+\infty)=+\infty$. The maximum in (8) is attained at a set A of the form $A=B(y,r)\cap\overline{\operatorname{dom} f^*}$. This means that the formula (8) remains true if one defines $C(X)$ as the class of nonempty convex compact sets in X.

4. CALCULATION RULES FOR THE DEGREE OF POINTEDNESS.

Suppose $f\in\Gamma_0(X)$ is constructed from other functions, say $f_1,\dots,f_N$, whose degrees of pointedness are known, or can be easily computed. In this case it is helpful to have a formula which relates $p[f]$ to the degrees $p[f_1],\dots,p[f_N]$ of the component functions. The calculation rules recorded in the first two propositions can be proved in a fairly simple way by using Theorem 1.

PROPOSITION 1. *Let $f\in\Gamma_0(X)$. Then,*

(a) $p[f]\leqslant p[g]$ *for all* $g\in\Gamma_0(X)$ *such that* $f\leqslant g$;
(b) $p[f+\ell]=p[f]$ *for all affine functions* $\ell:X\to R$;
(c) $p[\lambda f]=p[f]$ *for all* $\lambda>0$;
(d) $p[f(\cdot/\lambda)]=p[f]$ *for all* $\lambda>0$;
(e) $p[f_c]=p[f]$, *where* $c\in X$ *and* $f_c(x)=f(x-c)$.

PROPOSITION 2. *Let $X=\prod_{k=1}^{N}X_k$ be the Cartesian product of the finite dimensional spaces $X_1,\dots,X_N$. Suppose*

$$f(x)=f_1(x_1)+\cdots+f_N(x_N)\quad \text{for all } x\in X,$$

where $f_k\in\Gamma_0(X_k)$ for all $k=1,\dots,N$. Then,

$$p[f]=\sum_{k=1}^{N}p[f_k]. \tag{11}$$

In particular, f is pointed if and only if all the f_k's are pointed.

In the next four propositions we consider some important functional operations arising in the context of convex analysis, namely, pointwise maximum, addition, closed convex hull, and infimal-convolution.

PROPOSITION 3. *Let $f_1,\dots,f_N\in\Gamma_0(X)$ be finite at some common point, and let $f=\max_{1\leqslant k\leqslant N}f_k$. Then,*

$$p[f]\geqslant\max_{1\leqslant k\leqslant N}p[f_k]. \tag{12}$$

In particular, f is pointed if at least one of the f_k's is pointed.

PROOF: Formula (12) follows from Proposition 1(a). Indeed, since each $f_k \leqslant f$, one has

$$p[f_k] \leqslant p[f] \quad \text{for all} \quad k = 1, \dots, N.$$

An alternative proof of (12) is based on Theorem 2 and runs as follows. Let $A_1, \dots, A_N \in C(X)$ and $b_1, \dots, b_N \in R$ be such that $\dim A_k = p[f_k]$, and

$$f_k(x) \geqslant \psi^*_{A_k}(x) + b_k \quad \text{for all} \quad x \in X.$$

Then,

$$f(x) \geqslant \psi^*_A(x) + b \quad \text{for all} \quad x \in X,$$

where $b = \min\limits_{1 \leqslant k \leqslant N} b_k$ and $A = \operatorname{co} \bigcup\limits_{k=1}^{N} A_k$ is the convex hull of the sets $A_1, \dots, A_N$. Since $f - \psi^*_A$ is minorised, we have

$$p[f] \geqslant \dim\, A \geqslant \max_{1 \leqslant k \leqslant N} \dim\, A_k. \qquad \square$$

PROPOSITION 4. *Let $f_1, \dots, f_N \in \Gamma_0(X)$ be finite at some common point, and let $f = \sum\limits_{k=1}^{N} f_k$. Then*

$$p[f] \geqslant \dim\left(\operatorname{dom} f_1^* + \cdots + \operatorname{dom} f_N^*\right) \geqslant \max_{1 \leqslant k \leqslant N} p[f_k]. \tag{13}$$

In particular, f is pointed if at least one of the f_k's is pointed.

PROOF: It is known that f^* is equal to the lower-semicontinuous hull of the function

$$y \in X \mapsto h(y) := \inf\{f_1^*(y_1) + \cdots + f_N^*(y_N) : y_1 + \cdots + y_N = y\}.$$

Thus,

$$\operatorname{dom} f^* \supseteq \operatorname{dom} h = \operatorname{dom} f_1^* + \cdots + \operatorname{dom} f_N^*$$

and

$$\dim\, \operatorname{dom} f^* \geqslant \dim\left(\operatorname{dom} f_1^* + \cdots + \operatorname{dom} f_N^*\right) \geqslant \max_{1 \leqslant k \leqslant N} \dim\, \operatorname{dom} f_k^*.$$

Theorem 1 completes the proof of (13). $\square$

REMARK. If $\bigcap\limits_{k=1}^{N} \operatorname{ri}(\operatorname{dom} f_k)$ is nonempty, then h is lower-semicontinuous, and the first inequality in (13) becomes $p[f] = \dim\left(\operatorname{dom} f_1^* + \cdots + \operatorname{dom} f_N^*\right)$.

Proposition 5. *Let f be the closed convex hull of the functions $f_1, \dots, f_N \in \Gamma_0(X)$. Suppose all the f_k's are minorised by a common affine function. Then*

$$p[f] = \dim \bigcap_{k=1}^{N} \operatorname{dom} f_k^* \quad \leqslant \min_{1 \leqslant k \leqslant N} p[f_k]. \tag{14}$$

Proof: Since $f^* = \max\{f_k^* : 1 \leqslant k \leqslant N\}$, one has

$$\operatorname{dom} f^* = \bigcap_{k=1}^{N} \operatorname{dom} f_k^*.$$

Formula (14) follows by applying Theorem 1. ∎

Proposition 6. *Let f be the infimal-convolution of the functions $f_1, \dots, f_N \in \Gamma_0(X)$. Suppose the sets $ri(\operatorname{dom} f_k)$, $k = 1, \dots, N$ have a point in common. Then*

$$p[f] = \dim \bigcap_{k=1}^{N} \operatorname{dom} f_k^* \quad \leqslant \min_{1 \leqslant k \leqslant N} p[f_k]. \tag{15}$$

Proof: The proof is the same as in Proposition 4. This time one starts with the equality

$$(f)^* = \sum_{k=1}^{N} f_k^*.$$

∎

An important use of the infimal-convolution operation is the regularisation of a given function $f \in \Gamma_0(X)$. The regularised version of f is defined by

$$x \in X \mapsto [f \Box \theta](x) := \inf_{u \in X} \{f(u) + \theta(x - u)\},$$

where the function $\theta \in \Gamma_0(X)$ is referred to as a "kernel".

Corollary 1. *Let $\theta \in \Gamma_0(X)$ be coercive in the sense that $\theta(x)/\|x\| \to +\infty$ as $\|x\| \to \infty$. Then*

$$p[f \Box \theta] = p[f] \quad \textit{for all} \quad f \in \Gamma_0(X).$$

Proof: That $f \Box \theta \in \Gamma_0(X)$ follows from the coercivity of θ. One also has $\operatorname{dom} \theta^* = X$. Thus, $\operatorname{dom}(f \Box \theta)^* = \operatorname{dom} f^* \cap \operatorname{dom} \theta^*$ has the same dimension as $\operatorname{dom} f^*$. ∎

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