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Characterizations of Model Manifolds by Means of Certain Differential Systems

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Abstract. We prove metric rigidity for complete manifolds supporting solutions of certain second order differential systems, thus extending classical works on a characterization of space-forms. Along the way, we also discover new characterizations of space-forms. We next generalize results concerning metric rigidity via equations involving vector fields.

1 Introduction

Having fixed a smooth, even function $G\colon \mathbb{R} \to \mathbb{R}$, we let M^m_{-G} denote the m-dimensional (not necessarily complete) model manifold with radial sectional curvature $-G(r)$. More precisely, we set $M^m_{-G} = ([0, r_{-G}) \times \mathbf{S}^{m-1}, dr^2 + g(r)^2 d\theta^2)$, where $g\colon \mathbb{R} \to \mathbb{R}$ is the unique solution of the problem

$$\begin{cases} g'' = Gg, \\ g(0) = 0, \\ g'(0) = 1, \end{cases}$$

and $r_{-G} \in (0, +\infty]$ is the first zero of $g(r)$ on $(0, +\infty)$. Obviously, in case $g(r) > 0$ for every $r > 0$, we are using the convention $r_{-G} = +\infty$. In this case, the model is geodesically complete.

Examples of models come from the standard space-forms.

(i) Let $G(r) \equiv -k < 0$. Then $g(r) = k^{-1/2} \sin(k^{1/2} r)$, $r_k = \pi/k^{1/2}$ and M^m_k is isometric to the standard sphere of constant curvature k punctured at one point. Equivalently, M^m_k is isometric to the geodesic ball $B_{\pi/\sqrt{k}}(o)$ in the standard sphere of constant curvature k.

(ii) Let $G(r) \equiv k > 0$. Then $g(r) = k^{-1/2} \sinh(k^{1/2} r)$ and M^m_{-k} is isometric to the standard hyperbolic space of constant curvature $-k$.

(iii) Let $G(r) \equiv 0$. Then $g(r) = r$ and M^m_0 is isometric to the standard Euclidean space.

Characterizations of space-forms as complete manifolds supporting solutions of second order differential systems of the form $\operatorname{Hess}(u)(x) = (au(x) + b)\langle \cdot\,, \cdot \rangle_x$ have been classically investigated by M. Obata [5], Y. Tashiro [8], and M. Kanai [4]. The following theorem encloses in a single statement their results.

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Theorem 1.1 *Let $(M, \langle \cdot\,,\,\cdot \rangle)$ be a complete, connected Riemannian manifold of dimension* $\dim M = m$*. Then*

(i) *A necessary and sufficient condition for M to be isometric to the sphere of constant curvature $k > 0$ is that M supports a smooth, non trivial solution $u\colon M \to \mathbb{R}$ of the differential system*

$$\operatorname{Hess}(u)(x) = -ku(x)\langle \cdot\,,\,\cdot \rangle. \tag{1.1}$$

(ii) *A necessary and sufficient condition for M to be isometric to the hyperbolic space of constant curvature $-k < 0$ is that M supports a smooth, non trivial solution $u\colon M \to \mathbb{R}$ of the differential system*

$$\operatorname{Hess}(u)(x) = ku(x)\langle \cdot\,,\,\cdot \rangle, \tag{1.2}$$

with precisely one critical point.

(iii) *A necessary and sufficient condition for M to be isometric to the Euclidean space is that M supports a smooth, non trivial solution $u\colon M \to \mathbb{R}$ of the differential system*

$$\operatorname{Hess}(u)(x) = h\langle \cdot\,,\,\cdot \rangle,$$

for some constant $h \neq 0$.

Recently, E. García-Río, D. Kupeli and B. Ünal, [3], were able to extend the metric rigidity established in Theorem 1.1 to complete manifolds supporting vector field solutions Z of differential systems of the form $(DDZ)(X,Y) = k\langle Z, X\rangle Y$ for some constant $k \neq 0$ and for every vector fields X, Y. Here, the symbol D stands for covariant differentiation so that $(DDZ)(X,Y) = D_X D_Y Z - D_{D_X Y} Z$. Note that, in case $Z = \nabla u$ is a gradient vector field, the above equation becomes

$$D \operatorname{Hess}(u) = k du \otimes \langle \cdot\,,\,\cdot \rangle,$$

which is a third order system in the unknown function u. The following rigidity theorem summarizes the main results of [3].

Theorem 1.2 *Let $(M, \langle \cdot\,,\,\cdot \rangle)$ be a complete, connected Riemannian manifold of dimension* $\dim M = m$*. Then*

(i) *A necessary and sufficient condition for M to be isometric to the sphere of constant curvature $k > 0$ is that M supports a smooth, non trivial solution Z of the differential system $(DDZ)(X,Y) = -k\langle Z, X\rangle Y$, $\forall X, Y$.*

(ii) *A necessary and sufficient condition for M to be isometric to the hyperbolic space of constant curvature $-k < 0$ is that M supports a smooth, non trivial solution Z of the differential system $(DDZ)(X,Y) = k\langle Z, X\rangle Y$, $\forall X, Y$ satisfying $Z_o = 0$, for some $o \in M$.*

Since space-forms are very special cases of model manifolds, a natural question is whether a general model manifold M^m_{-G} can be characterized in the same perspective of Theorem 1.1 and Theorem 1.2. This note aims to answer the question in the affirmative. During our investigation, we will also give new characterizations of space-forms.

2 Second Order Systems

Quite naturally, one expects that a characterization of the model M^m_{-G}, in the spirit of Theorem 1.1 must involve more general differential systems of the form

$$\text{Hess}(u)(x) = H(r(x))u(x)\langle\cdot,\cdot\rangle, \tag{2.1}$$

where $r(x)$ denotes the geodesic distance from a fixed origin o. First of all, we need to find the right form of the radial coefficient H. Let $u(x) = \alpha(r(x))$ be a radial solution of (2.1). We assume u has been normalized in such a way that $u(0) = 1$, and we require u to have a critical point at 0. Then, recalling that

$$\text{Hess}(r) = \frac{g'}{g}\{\langle\cdot,\cdot\rangle - dr\otimes dr\} = gg'd\theta^2, \tag{2.2}$$

we have $\text{Hess}(u) = \alpha'' dr\otimes dr + \alpha' gg' d\theta^2$. On the other hand,

$$\text{Hess}(u) = H\alpha\langle\cdot,\cdot\rangle = H\alpha dr\otimes dr + H\alpha g^2 d\theta^2.$$

Comparing these two equations gives the ordinary differential system

$$\begin{cases}\alpha'' = H\alpha,\\ \alpha' gg' = H\alpha g^2,\end{cases}$$

that is,

$$\begin{cases}\alpha'' = \alpha' g'/g,\\ H = \alpha' g'/\alpha g,\end{cases}$$

where, we recall, $\alpha(0) = 1$, $\alpha'(0) = 0$. Integrating the first equation gives

$$\alpha(r) = A\int_0^r g(s)ds + 1,$$

with $A \neq 0$ any constant. Inserting this expression into the second equation we finally deduce

$$H(r) = \frac{Ag'(r)}{A\int_0^r g(s)ds + 1}.$$

In order that H is defined on all of $[0, r_{-G})$, we need to impose that

$$\inf\Big\{ t > 0 : A\int_0^t g(s)ds + 1 \leq 0\Big\} \geq r_{-G}.$$

We have thus obtained the following lemma.

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Lemma 2.1 *A necessary and sufficient condition for equation* (2.1) *to possess a radial solution u on M^m_{-G} is that*

$$H(r) = \frac{Ag'(r)}{A\int_0^r g(s)ds + 1}.$$

for any constant $A \neq 0$ such that

$$\inf\Big\{ t > 0 : A\int_0^t g(s)ds + 1 \leq 0 \Big\} \geq r_{-G}.$$

Note, in particular, the following.

- On the punctured standard sphere $M^m_1 = \mathbf{S}^m \backslash \{\text{point}\} = B_\pi(0)$, for every $A \in \mathbb{R}\backslash\{0\}$ such that $A > -1/2$, $A = -1$, there is a smooth function u_A with exactly one critical point at 0 and satisfying the equation

$$\text{(2.3)} \qquad \operatorname{Hess}(u_A)(x) = \frac{A\cos r(x)}{-A\cos r(x) + 1 + A} u_A(x)\langle\,\cdot\,,\,\cdot\,\rangle.$$

  As a matter of fact, the function $u(x) = -A\cos r(x) + 1 + A$ is well defined and solves the equation on all of $\mathbf{S}^m$. Note finally that in the special case $A = -1$, (2.3) reduces to (1.1).
- On the standard hyperbolic model $M^m_{-1} = \mathbb{H}^m_{-1}$, for every $A > 0$ there exists a smooth function u_A with exactly one critical point at 0 and satisfying the equation

$$\text{(2.4)} \qquad \operatorname{Hess}(u_A)(x) = \frac{A\cosh r(x)}{A\cosh r(x) + 1 - A} u_A(x)\langle\,\cdot\,,\,\cdot\,\rangle.$$

  In the special case $A = 1$, (2.4) reduces to (1.2).
- On the standard Euclidean space $M^m_0 = \mathbb{R}^m$, for every $A > 0$, there exists a function u_A with exactly one critical point at 0 and satisfying the equation

$$\text{(2.5)} \qquad \operatorname{Hess}(u_A)(x) = \frac{2A}{Ar(x)^2 + 2} u_A(x)\langle\,\cdot\,,\,\cdot\,\rangle.$$

We shall prove the following result. Recall that a twisted sphere of dimension n is a differentiable manifold N, homeomorphic to the standard sphere $\mathbf{S}^n$, which is obtained by gluing two n-dimensional closed, unit disks $D^n \subset \mathbb{R}^n$ via a boundary diffeomorphism.

Theorem 2.2 *Let $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a complete Riemannian manifold of dimension m, and let $o \in M$ be a reference origin. Then a necessary and sufficient condition for the existence of an isometric imbedding $\Phi\colon M^m_{-G} \to M$ is that there exists a smooth solution $u\colon B_{r_{-G}}(o) \to \mathbb{R}$ of the problem*

$$\text{(2.6)} \qquad \begin{cases} \operatorname{Hess}(u)(x) = H(r(x))u(x)\langle\,\cdot\,,\,\cdot\,\rangle, \\ u(o) = 1, \\ |\nabla u|(o) = 0, \end{cases}$$

where $r(x) = \mathrm{dist}_{(M,\langle\,\cdot\,,\,\cdot\,\rangle)}(x, o)$, $H\colon [0, R^*] \to \mathbb{R}$ *is the smooth function*

$$H(t) = \frac{Ag'(t)}{A\int_0^t g(s)ds + 1}$$

for some real number $A \neq 0$, *and*

$$R^* = \sup\{T > 0 : H(t) \textit{ well defined on } [0, T]\} > r_{-G}.$$

Furthermore, if u *is a solution of* (2.6) *on all of* M, *then the following holds*

(i) *If* $r_{-G} = +\infty$, *then* M *is isometric to the model* M^m_{-G}.
(ii) *If* $r_{-G} < +\infty$ *and* $H(r_{-G}) \neq 0$, *then* $\mathrm{cut}(o) = \{O\}$ *for some* $O \in M$, *and* $\Phi(M^m_{-G}) = M\backslash\{O\}$. *Furthermore,* M *is diffeomorphically a twisted sphere.*

As a direct consequence of Theorem 2.2 we point out the following result that generalizes, in some directions, Theorem 1.1.

Corollary 2.3 *Let* $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ *be a complete Riemannian manifold,* $o \in M$ *a reference origin, and* $r(x) = \mathrm{dist}_{(M,\langle\,\cdot\,,\,\cdot\,\rangle)}(x, o)$.

(i) *M is isometric to the standard sphere* $\mathbf{S}^m$ *if and only if* M *supports a real valued function* $u \not\equiv 0$ *with a critical point at* o *and satisfying the differential system* (2.3) *for some* $A \neq 0$ *such that either* $A > -1/2$ *or* $A = -1$.
(ii) *M is isometric to the standard hyperbolic space if and only if* M *supports a real valued function* $u \not\equiv 0$ *with a critical point at* o *and satisfying the differential system* (2.4) *for some* $A > 0$.
(iii) *M is isometric to the standard Euclidean space if and only if* M *supports a real valued function* $u \not\equiv 0$ *with a critical point at* o *and satisfying the differential system* (2.5) *for some* $A > 0$.

Before proving Theorem 2.2 we make some observations on case (i) of Corollary 2.3.

(i) First of all, to deduce that M is a standard sphere one simply observes that, as established in Theorem 2.2(ii), M is simply connected and $M\backslash\{O\}$ is isometric to a standard punctured sphere. Therefore, by continuity M itself has positive constant curvature and we can apply the Hopf classification theorem. Alternatively, we can recall that a necessary and sufficient condition for the model metric $dr\otimes dr + g(r)^2 d\theta^2$ of M^m_{-G} to smoothly extend to all of $[0, r_{-G}] \times \mathbf{S}^{m-1}$ is that $g^{(2k)}(r_{-G}) = 0$ and $g'(r_{-G}) = -1$; see [6]. In the present situation we have $g(r) = \sin(r)$ and therefore we deduce that the isometry Φ extends to cover the removed point O.

(ii) Comparing with Theorem 1.1(i), we see that, on the one hand, we enlarge the class of differential systems characterizing the sphere but, on the other hand, we make the additional assumption that u has a critical point at o. As first noted by Obata, the existence of a critical point is automatically guaranteed if $H(r) \equiv -k < 0$. To see this, one can argue as follows. By contradiction, suppose u has no critical point at all. Then the vector field $X = \nabla u/|\nabla u|$ is defined on all of M. Using the differential system $\mathrm{Hess}(u) = -ku\langle\,\cdot\,,\,\cdot\,\rangle$, it is readily seen that the integral curves $\gamma(t)\colon \mathbb{R} \to M$

of X are unit speed, but not necessarily minimizing, geodesics. Indeed,

$$\begin{aligned} D_{\dot\gamma}\dot\gamma = D_{\dot\gamma}X_\gamma &= |\nabla u|^{-1}\,\mathrm{Hess}(u)(\dot\gamma,\,\cdot\,)^\# - |\nabla u|^{-1}\,\mathrm{Hess}(u)(\dot\gamma, X)X \\ &= -ku|\nabla u|^{-1}X + ku|\nabla u|^{-1}X \\ &= 0. \end{aligned}$$

Note that the same argument works if u solves the more general equation $\mathrm{Hess}(u) = f\langle\,\cdot\,,\,\cdot\,\rangle$ for any real-valued function f. Now consider $y(t) = u\circ\gamma(t)$. Then y satisfies the oscillatory o.d.e. $y'' = -ky$. Let $t_0 > 0$ be a critical point of y. Since

$$\begin{aligned} 0 = \frac{dy}{dt}(t_0) = \langle \nabla u(\gamma(t_0)), \dot\gamma(t_0)\rangle = \Big\langle \nabla u(\gamma(t_0)), \frac{\nabla u}{|\nabla u|}(\gamma(t_0))\Big\rangle \\ = |\nabla u|(\gamma(t_0)), \end{aligned}$$

we have that $\gamma(t_0)$ is a critical point of u, which is a contradiction. Thus, u has a critical point p and we can always take $p = o$ as the reference origin in our Theorem 2.2. In case the coefficient H in the differential equation depends on the distance function $r(x)$, if we try to adapt the previous argument to the present situation, we encounter two obvious difficulties.

(a) As observed above, an integral curve $\gamma(t)\colon \mathbb{R} \to M$ of the vector field X is a geodesic, but it can be non-minimizing. Therefore, for large values of $|t|$, $H(r(\gamma(t))) \neq H(t)$. It follows that the reduction procedure of the P.D.E. to an o.d.e., via composition with γ, cannot be carried over for large values of $|t|$.

(b) Even if we were able to prove that u has a critical point at some $p \in M$, since the coefficient H depends on the distance from the reference origin o, we could not take $p = o$.

The rest of the section is entirely devoted to a proof of Theorem 2.2. The necessity part has been already discussed above. Therefore we may concentrate on the sufficiency part.

The following density result due to R. Bishop [1] will play a key role in our argument. For a nice and simplified proof, see F. Wolter [9]. Following Bishop, recall that given a complete manifold $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ and a reference point $o \in M$, then $p \in \mathrm{cut}(o)$ is an *ordinary cut point* if there are at least two distinct minimizing geodesics from o to p. Using the infinitesimal Euclidean law of cosines, it is not difficult to show that at an ordinary cut point p the distance function $r(x) = \mathrm{dist}_{(M,\langle\,\cdot\,,\,\cdot\,\rangle)}(x, o)$ is not differentiable [9].

Theorem 2.4 *Let $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a complete Riemannian manifold and let $o \in M$ be a reference point. Then the ordinary cut-points of o are dense in* $\mathrm{cut}(o)$. *In particular, if the distance function $r(x)$ from o is differentiable on the (punctured) open ball $B_R(o)\backslash\{o\}$, then $B_R(o) \cap \mathrm{cut}(o) = \varnothing$.*

Proof of Theorem 2.2 To simplify the exposition we will proceed by steps.

Step 1: First of all, we note that the function $u\colon B_{r_{-G}}(o) \to \mathbb{R}$ must be radial and, more precisely, $u(x) = \alpha(r(x))$, where

$$\alpha(t) = A \int_0^t g(s)ds + 1.$$

Indeed, fix x and choose a unit speed, minimizing geodesic $\gamma\colon [0, r(x)] \to B_{r_{-G}}(o)$ from o to x. Then composing with γ we deduce that $y(t) = u \circ \gamma(t)$ is the solution of the Cauchy problem

$$\begin{cases} y''(t) = \frac{Ag'(t)}{A\int_0^t g(s)ds+1} y(t), \\ y(0) = 1, \\ y'(0) = \langle \nabla u(o), \dot{\gamma}(0) \rangle = 0. \end{cases}$$

It follows that

$$y(t) = A \int_0^t g(s)ds + 1,$$

and, taking $t = r(x)$, we get

$$u(x) = y(r(x)) = A \int_0^{r(x)} g(s)ds + 1.$$

Step 2: The open ball $B_{r_{-G}}(o)$ is inside the cut-locus of o. Indeed, recall that $u(x) = \alpha(r(x))$ and note that α is a diffeomorphism on $(0, r_{-G})$ because $\alpha'(t) = Ag(t) \neq 0$ on that interval. Therefore, $r(x) = \alpha^{-1} \circ u(x)$ is smooth on $B_{r_{-G}}(o) \backslash \{o\}$ as a composition of smooth functions. By Theorem 2.4, it follows that $B_{r_{-G}}(o) \cap \mathrm{cut}(o) = \varnothing$.

Step 3: According to Step 2, we can introduce geodesic polar coordinates on $B_{r_{-G}}(o)$. We claim that the corresponding map

$$\Phi(r,\theta) = \exp_o(r\theta)\colon M^m_{-G} \approx \mathbf{B}^m_{r_{-G}}(0) \subseteq T_oM \to B_{r_{-G}}(o) \subseteq M$$

is a Riemannian isometry. To see this, let v be the function

$$v(x) = \frac{u(x)-1}{A} = \int_0^{r(x)} g(s)ds$$

on $B_{r_{-G}}(o)$ and note that

$$\begin{cases} \mathrm{Hess}(v) = A^{-1}Hu\langle \cdot\, , \cdot \rangle, \\ v(o) = 0, \\ |\nabla v|(o) = 0. \end{cases} \tag{2.7}$$

Furthermore,

$$\nabla r = \frac{\nabla v}{|\nabla v|}. \tag{2.8}$$

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Using geodesic polar coordinates $(r,\theta) \in (0, r_{-G}) \times \mathbf{S}^{m-1} \approx \mathbf{B}^m_{r_{-G}}(0)\setminus\{0\} \subseteq T_oM$, keeping a local orthonormal frame $\{\theta^\alpha\}$ on $\mathbf{S}^{m-1} \subset T_oM$, and recalling Gauss' lemma, we now express $\exp_o^*\langle\,\cdot\,,\,\cdot\,\rangle = dr \otimes dr + \sigma_{\alpha\beta}(r,\theta)\theta^\alpha \otimes \theta^\beta$, where $d\theta^2 = \sum \theta^\alpha \otimes \theta^\alpha$ denotes the standard metric on $\mathbf{S}^{m-1}$ and the coefficient matrix $(\sigma_{\alpha\beta})$ satisfies the asymptotic condition

$$\sigma_{\alpha\beta}(r,\theta) = r^2\delta_{\alpha\beta} + o(r^2), \text{ as } r \to 0. \tag{2.9}$$

By the fundamental equations of Riemannian geometry, we know that within the cut locus of o $L_{\nabla r}\langle\,\cdot\,,\,\cdot\,\rangle = 2\,\mathrm{Hess}(r)$, where, furthermore, $\nabla r = \partial_r$, the radial vector field. Therefore, on $B_{r_{-G}}(o)$, we have

$$\partial_r\sigma_{\alpha\beta}(r,\theta) = 2\,\mathrm{Hess}(r)_{\alpha\beta}. \tag{2.10}$$

But, according to (2.7) and (2.8), we have for every $X, Y \in (\nabla r)^\perp$,

$$\begin{aligned}\mathrm{Hess}(r)(X,Y) &= \left\langle D_X \frac{\nabla v}{|\nabla v|}, Y \right\rangle = \frac{1}{|\nabla v|}\,\mathrm{Hess}(v)(X,Y) = \frac{1}{|\nabla v|} A^{-1}Hu\,\langle X,Y\rangle \\ &= \frac{g'}{g}\,\langle X,Y\rangle\,.\end{aligned}$$

Using this information into (2.10) and recalling (2.9), we deduce that

$$\begin{cases} \partial_r\sigma_{\alpha\beta}(r,\theta) = 2\dfrac{g'}{g}(r)\sigma_{\alpha\beta}(r,\theta), \\ \sigma_{\alpha\beta}(r,\theta) = r^2\delta_{\alpha\beta} + o(r^2), \text{ as } r \to 0, \end{cases} \tag{2.11}$$

which integrated gives $\sigma_{\alpha\beta}(r,\theta) = g(r)^2\delta_{\alpha\beta}$. We have thus shown that

$$\exp_o^*\langle\,\cdot\,,\,\cdot\,\rangle = dr \otimes dr + g(r)^2 d\theta^2,$$

proving that $\exp_o\colon M^m_{-G}\setminus\{0\} \to B_R(o)\setminus\{o\}$ is a Riemannian isometry. To conclude, note that by the assumptions on g, this isometry smoothly extends even to the origin 0.

Step 4: We now assume that u is a solution of (2.6) on all of M. In case $r_{-G} = +\infty$, then it follows directly from Step 3 that $\Phi\colon M^m_{-G} \to M$ is a Riemannian isometry. Accordingly, in what follows, we assume $r_{-G} < +\infty$.

Step 5: We show that $\partial B_{r_{-G}}(o)$ is discrete, hence a finite set. Indeed, for every $x \in \partial B_{r_{-G}}(o)$, let γ be a unit speed, minimizing geodesic from o to x. Then $|\nabla u| \circ \gamma(t) = Ag(t) \to 0$ as $t \to r_{-G}$. Therefore, $\partial B_{r_{-G}}(o)$ is made up of critical points of u. Since u satisfies the differential equation $\mathrm{Hess}(u)(x) = H(r(x))u(x)\langle\,\cdot\,,\,\cdot\,\rangle$ and, by assumption, $H(r_{-G}) \neq 0$ and $u \neq 0$ on $\partial B_{r_{-G}}(o)$, we deduce that such critical points are non-degenerate, *i.e.,* the quadratic form $\mathrm{Hess}(u)$ has no zero eigenvalues. Hence, by Morse's Lemma, they are isolated. Accordingly, $\partial B_{r_{-G}} = \{p_1, \ldots, p_k\}$, as claimed.

Step 6: We prove that $\mathrm{cut}(o) = \{O\} = \partial B_{r_{-G}}(o)$, for some $O \in M$. Indeed, by Step 2, the standard m-dimensional ball $\mathbf{B}^m_{r_{-G}}(0) \subset T_oM$ of radius r_{-G} lies in the domain $D_o \subset T_oM$ of the normal coordinates at o. Therefore, it suffices to show that

$$\exp_o(\partial \mathbf{B}^m_{r_{-G}}(0)) = \partial B_{r_{-G}}(o) = \{O\}. \tag{2.12}$$

If this occurs, then $\partial \mathbf{B}^m_{r_{-G}}(0)$ is precisely the tangential cut-locus of o and, hence, $\mathrm{cut}(o) = \{O\}$. Note that, in particular, all the geodesics issuing from o will meet at O (and cannot minimize distances past r_{-G}).

Now for the proof of (2.12). Let us observe that $\exp_o(\partial \mathbf{B}^m_{r_{-G}}(0)) \subseteq \overline{B_{r_{-G}}(o)}$ and $\exp_o(\partial \mathbf{B}^m_{r_{-G}}(0) \cap D_o) = \partial B_{r_{-G}}(o) \cap (M \setminus \mathrm{cut}(o))$. Since $B_{r_{-G}}(o)$ does not contain any cut-point of o, it follows that also the tangential cut points in $\partial \mathbf{B}^m_{r_{-G}}(0)$ are mapped on $\partial B_{r_{-G}}(o)$ by $\exp_o$. Thus, $\exp_o(\partial \mathbf{B}^m_{r_{-G}}(0)) = \partial B_{r_{-G}}(o)$. Now recall from Step 5 that $\partial B_{r_{-G}}(o)$ is a finite set. Since $\partial \mathbf{B}^m_{r_{-G}}(0)$ is connected and $\exp_o$ is a continuous map, we conclude the validity of (2.12).

Step 7: We note that $\Phi(M^m_{-G}) = M \setminus \{O\} = B_{r_{-G}}(o)$. Indeed, this follows directly from Steps 3 and 6.

Step 8: Finally we deduce that M is, diffeomorphically, a twisted sphere. To this end, recall that by Step 6, M is compact. Moreover, u is a smooth function on M with precisely two critical points, o and O. According to (2.6) and Step 5, these critical points are non-degenerate. Therefore, to conclude, we can apply the (differentiable version of) the classical result by G. Reeb.

This completes the proof of the theorem. ■

3 Third Order Systems: from Functions to (Gradient) Vector Fields

Recently, much work has been done to characterize space-forms, and also complex Kähler and quaternionic manifolds, via differential equations involving vector fields instead of functions. We refer to [2] for a survey of such kind of results. Let us focus attention on space-forms. It is a nice observation by García-Río, Kupeli, and Ünal, [3], that if the vector field Z on M satisfies

$$(DDZ)(X,Y) = k\langle Z, X\rangle Y, \tag{3.1}$$

for every vector fields X, Y, and for some constant $k \neq 0$, where D denotes the covariant differentiation, then (a) Z has the special form

$$Z = \frac{\nabla \operatorname{div} Z}{mk}, \tag{3.2}$$

and, (b) the smooth function $u = \operatorname{div} Z$ satisfies

$$\mathrm{Hess}(u) = ku\langle\,\cdot\,,\,\cdot\,\rangle, \text{ on } M.$$

Using this latter fact, the authors are able to reduce their characterizations of space-forms to the *scalar* cases collected in Theorem 1.1. Note that once we have chosen a reference origin o and used polar coordinates with respect to o, the function u turns out to be radial and hence, by (3.2) Z is a radial gradient vector field.

One may therefore ask whether similar characterizations hold for a generic model up to considering solutions of

$$(DDZ)(X, Y) = K(r(x)) \langle Z, X \rangle Y \tag{3.3}$$

for a suitable smooth, real valued function $K(t)$, thus extending Theorem 2.2 to vector field equations. Inspection of what happens on a generic model suggests that this is the case. Indeed, suppose we are given a model M^m_{-G} with corresponding warping function g. In view of what we observed above, it is quite natural to consider the radial, gradient vector field

$$Z_x = \nabla\Big(\int_0^{r(x)} y(s)ds + B\Big) = y(r(x))\nabla r,$$

where $B \in \mathbb{R}$ is an arbitrary constant. Straightforward calculations show that

$$\begin{aligned}\langle (DDZ)(X, Y), W\rangle = {}& y'' dr(X)dr(Y)dr(W)\\ &+ y' \operatorname{Hess}(r)(X, Y)dr(W) + y' \operatorname{Hess}(r)(X, W)dr(Y)\\ &+ y' \operatorname{Hess}(r)(Y, W)dr(X) + y(D_X \operatorname{Hess}(r))(Y, W).\end{aligned}$$

On the other hand, using (2.2), we see that

$$\begin{aligned}(D_X \operatorname{Hess}(r))(Y, W) = {}& \Big\{ \frac{(gg')'}{gg'} - 2\frac{g'}{g} \Big\} \operatorname{Hess}(r)(Y, W)dr(X)\\ &- \frac{g'}{g} \operatorname{Hess}(r)(X, Y)dr(W) - \frac{g'}{g} \operatorname{Hess}(r)(X, W)dr(Y),\end{aligned}$$

holds for every vector fields X, Y, W. Whence, we deduce

$$\begin{aligned}\langle (DDZ)(X, Y), W\rangle = {}& y'' dr(X)dr(Y)dr(W)\\ &+ \Big(y' - y\frac{g'}{g}\Big) \operatorname{Hess}(r)(X, Y)dr(W)\\ &+ \Big(y' - y\frac{g'}{g}\Big) \operatorname{Hess}(r)(X, W)dr(Y)\\ &+ (y' + y\frac{(gg')'}{gg'} - 2y\frac{g'}{g}) \operatorname{Hess}(r)(Y, W)dr(X).\end{aligned}$$

Since

$$K(r)\langle Z, X\rangle\langle Y, W\rangle = K(r) y dr(X)dr(Y)dr(W) + K(r) y \frac{g}{g'} \operatorname{Hess}(r)(Y, W)dr(X),$$

it follows that (3.3) is verified for the chosen vector fields X and Y if and only if the equation

$$\begin{aligned}0 &= (y'' - Ky)dr(X)dr(Y)dr(W)\\ &\quad + \Big(y' - y\frac{g'}{g}\Big)\operatorname{Hess}(r)(X,Y)dr(W)\\ &\quad + \Big(y' - y\frac{g'}{g}\Big)\operatorname{Hess}(r)(X,W)dr(Y)\\ &\quad + \Big(y' + y\frac{(gg')'}{gg'} - 2y\frac{g'}{g} - Ky\frac{g}{g'}\Big)\operatorname{Hess}(r)(Y,W)dr(X)\end{aligned}$$

is satisfied for every W. Using appropriate choices of X, Y, W, we immediately see that equation (3.3) is equivalent to

$$\begin{cases} y'' - Ky = 0\\ y' - yg'/g = 0\\ y' + y(gg')'/gg' - 2yg'/g - Kyg/g' = 0.\end{cases}$$

Whence, up to imposing $y(0) = 0$ (which is a natural assumption in order to extend the above computations to the pole of M^m_{-G}), we conclude that these conditions imply

$$K(r) = G(r), \quad y(r) = Ag(r),$$

for any constant $A \neq 0$. We have thus obtained the following result.

Lemma 3.1 *A necessary and sufficient condition for equation* (3.3) *on M^m_{-G} to possess a (non-trivial) radial gradient vector field solution Z is that $K(r) = G(r)$. In this case,*

$$Z_x = \nabla(A\int_0^{r(x)} g(s)ds + B),$$

where $A \neq 0$ and $B \in \mathbb{R}$ are arbitrary constants.

Observe that Z is the gradient vector field associated with the radial solution $u(x) = \alpha(r(x))$ of the *scalar* equation (2.1). Also, as we already remarked at the beginning of the section, if $G(r) \equiv k$ a non-zero constant then, according to (3.2), any solution Z of (3.1) must be of the form $Z = \nabla u$ where $u = \operatorname{div} Z/mk$, and equation (3.1) becomes $(D\operatorname{Hess}(u))(X;Y,W) = k\langle\nabla u, X\rangle\,\langle Y,W\rangle$. According to these considerations, we are naturally led to state the next rigidity result which represents a genuine extension of Theorem 1.2 stated in the Introduction. Our approach is rather different from that presented in [3]. Indeed, the reduction procedure outlined above cannot be carried over to this more general situation.

Theorem 3.2 *Let $(M, \langle\cdot,\cdot\rangle)$ be an m-dimensional, complete Riemannian manifold, let $o \in M$ be a reference origin, and set $r(x) = \operatorname{dist}_{(M,\langle\cdot,\cdot\rangle)}(x,o)$. A necessary and*

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sufficient condition for the existence of an isometric imbedding $\Phi\colon M^m_{-G} \to M$ *is that there exists a non-trivial, smooth solution* $u\colon B_{r_{-G}}(o) \to \mathbb{R}$ *of the problem*

$$(3.4)\qquad \begin{cases} (D\,\mathrm{Hess}(u))(X;Y,W) = G(r(x))\langle \nabla u, X\rangle \langle Y, W\rangle, \\ \mathrm{Hess}(u)(o) = A\langle\,\cdot\,,\,\cdot\,\rangle, \\ |\nabla u|(o) = 0, \end{cases}$$

for some $A \neq 0$. *Furthermore, if u is a solution of* (3.4) *on all of* M, *then the following holds:*

(i) *If* $r_{-G} = +\infty$, *then* M *is isometric to the model* M^m_{-G}.
(ii) *In case* $r_{-G} < +\infty$, *and* $g'(r_{-G}) \neq 0$, *then* $\mathrm{cut}(o) = \{O\}$ *for some* $O \in M$, *and* $\Phi(M^m_{-G}) = M\backslash\{O\}$. *Moreover,* M *is diffeomorphically a twisted sphere.*

Remark 3.3 In case $G(s) \equiv k$, it can be shown that assumption $\mathrm{Hess}(u)(o) = A\langle , \rangle$ is unessential. Furthermore, if $k < 0$, then even the request $|\nabla u|(o) = 0$ can be omitted.

Comparing Theorems 2.2 and 3.2 we see that the characterization of a model M^m_{-G} via a third order system seems to be more natural. Indeed, the system involves directly the radial sectional curvature $-G(r)$ of the model. On the other hand, in the situation of second order systems, we are able to characterize the same space via a one-parameter family of differential systems as remarked, *e.g.*, in Corollary 2.3. These further characterizations are invisible from the third order point of view.

Proof Let us begin by showing that $u(x) = \alpha(r(x))$ where

$$\alpha(t) = A\int_0^t g(s)ds + B,$$

for some constant $B \in \mathbb{R}$. To this end, fix $x \in B_{r_{-G}}(o)$ and let $\gamma(s)\colon [0, r(x)] \to B_{r_{-G}}(o)$ be a unit speed, minimizing geodesic from $\gamma(0) = o$ to $\gamma(r(x)) = x$. Then evaluating (3.4) along γ, we readily deduce that $y(s) = u \circ \gamma(s)$ solves the Cauchy problem

$$(3.5)\qquad \begin{cases} y''' = G(s)y', \\ y(0) = B, \\ y'(0) = 0, \\ y''(0) = A, \end{cases}$$

where $B = u(o)$. Since $G = g''/g$, integrating (3.5), we deduce that $y(s) = \alpha(s)$. Evaluating the latter at $s = r(x)$, we conclude that $u(x) = \alpha(r(x))$ as desired.

As in Step 2 of the proof of Theorem 2.2, it follows from the Bishop density result that $\mathrm{cut}(o) \cap B_{r_{-G}}(o) = \varnothing$. On the other hand, using equation (3.4), we have that the Riemann curvature tensor of M satisfies

$$\begin{aligned} \mathrm{Riem}\,(W, X, \nabla u, Y) &= (D\,\mathrm{Hess}(u))(W; X, Y) - (D\,\mathrm{Hess}(u))(X; W, Y) \\ &= G(r(x))\{\langle \nabla u, W\rangle\langle X, Y\rangle - \langle \nabla u, X\rangle\langle W, Y\rangle\}, \end{aligned}$$

for every vector fields X, Y, W. Since $\nabla u = Ag(r)\nabla r$, choosing $X = \nabla r$ and $W = Y \in (\nabla r)^{\perp}$, we deduce that the radial sectional curvature of M is given by

$$\mathrm{Sec}_{\mathrm{rad}}(x) = -G(r(x)).$$

Therefore, by Hessian comparisons [7],

$$\mathrm{Hess}(r) = \frac{g'}{g}\{\langle\cdot\,,\,\cdot\rangle - dr \otimes dr\}, \text{ on } B_{r_{-G}}(o)\backslash\{o\}. \tag{3.6}$$

Now the proof can be easily completed following the arguments of Theorem 2.2. Indeed, setting

$$\exp_o^*\langle\cdot\,,\,\cdot\rangle = dr \otimes dr + \sigma_{\alpha\beta}(r,\theta)\theta^{\alpha} \otimes \theta^{\beta},$$

and using (3.6) into (2.10) yields the validity of (2.11) which, once integrated, gives

$$\sigma_{\alpha\beta}(r,\theta) = g(r)^2\delta_{\alpha\beta}.$$

We have thus established that $B_{r_{-G}}(o)$ is isometric to M^m_{-G}. In particular, u satisfies

$$\mathrm{Hess}(u)(x) = Ag'(r(x))\langle\cdot\,,\,\cdot\rangle, \text{ on } B_{r_{-G}}(o). \tag{3.7}$$

Suppose now that u is defined on all of M. In case $r_{-G} = +\infty$ we immediately conclude that M is isometric to M^m_{-G}, as stated in (i). On the other hand, assume that $r_{-G} < +\infty$, hence $g(r_{-G}) = 0$, and $g'(r_{-G}) \neq 0$. Having fixed $x \in \partial B_{r_{-G}}(o)$ and a unit vector $v \in T_xM$, let $\gamma\colon [0, r_{-G}] \to M$ be a minimizing geodesic from $\gamma(0) = o$ to $\gamma(r_{-G}) = x$. Obviously, $\gamma(t) \in B_{r_{-G}}(o)$ for every $t < r_{-G}$. Next, consider $v(t)$ the vector field obtained by parallel transport of v along γ. Then according to (3.7),

$$\mathrm{Hess}(u)(x)(v,v) = \lim_{t\to r_{-G}} \mathrm{Hess}(u)(\gamma(t))(v(t),v(t)) = Ag'(r_{-G}) \neq 0,$$

proving that $\partial B_{r_{-G}}$ is made up entirely by non-degenerate critical points. Therefore, following exactly Steps 5–8 in the proof of Theorem 2.2, we conclude the validity of the global properties of M collected in (ii). ∎

We conclude the section by stating the following problem.

Problem *Suppose that the equation $(DDZ)(X,Y) = G(r(x))\langle Z,X\rangle Y$, $G \neq 0$, has a non-trivial solution Z with $Z_o = 0$. Is M isometric to M^m_{-G}? Is it necessary to impose some further assumption on M?*

Observe that even in this more general situation,

$$Z = \frac{\nabla\,\mathrm{div}\,Z}{mG(r)}.$$

However, this time, it does not follow from this expression that Z is gradient. Needless to say, the reduction procedure of [3] cannot be applied directly in the present situation.

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