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THE KATZ–KLEMM–VAFA CONJECTURE FOR $K3$ SURFACES

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Abstract

We prove the KKV conjecture expressing Gromov–Witten invariants of $K3$ surfaces in terms of modular forms. Our results apply in every genus and for every curve class. The proof uses the Gromov–Witten/Pairs correspondence for $K3$-fibered hypersurfaces of dimension 3 to reduce the KKV conjecture to statements about stable pairs on (thickenings of) $K3$ surfaces. Using degeneration arguments and new multiple cover results for stable pairs, we reduce the KKV conjecture further to the known primitive cases. Our results yield a new proof of the full Yau–Zaslow formula, establish new Gromov–Witten multiple cover formulas, and express the fiberwise Gromov–Witten partition functions of $K3$-fibered 3-folds in terms of explicit modular forms.

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0. Introduction

0.1. Reduced Gromov–Witten theory. Let S be a nonsingular projective $K3$ surface, and let

$$\beta \in \mathrm{Pic}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S, \mathbb{C})$$

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be a nonzero effective curve class. The moduli space $\overline{M}_g(S, \beta)$ of genus g stable maps (with no marked points) has expected dimension

$$\dim^{\text{vir}}_{\mathbb{C}} \overline{M}_g(S, \beta) = \int_\beta c_1(S) + (\dim_{\mathbb{C}}(S) - 3)(1 - g) = g - 1.$$

However, as the obstruction theory admits a 1-dimensional trivial quotient, the virtual class $[\overline{M}_g(S, \beta)]^{\text{vir}}$ vanishes. The standard Gromov–Witten theory is trivial.

Curve counting on $K3$ surfaces is captured instead by the *reduced* Gromov–Witten theory constructed first via the twistor family in [8]. An algebraic construction following [2, 3] is given in [36]. The reduced class

$$[\overline{M}_g(S, \beta)]^{\text{red}} \in A_g(\overline{M}_g(S, \beta), \mathbb{Q})$$

has dimension g. Let λ_g be the top Chern class of the rank g Hodge bundle

$$\mathbb{E}_g \to \overline{M}_g(S, \beta)$$

with fiber $H^0(C, \omega_C)$ over the moduli point

$$[f : C \to S] \in \overline{M}_g(S, \beta).$$

(The Hodge bundle is pulled back from $\overline{M}_g$ if $g \geqslant 2$. See [13, 18] for a discussion of Hodge classes in Gromov–Witten theory.) The reduced Gromov–Witten integrals of S,

$$R_{g,\beta}(S) = \int_{[\overline{M}_g(S,\beta)]^{\text{red}}} (-1)^g \lambda_g \in \mathbb{Q}, \tag{0.1}$$

are well defined. Under deformations of S for which β remains a $(1, 1)$-class, the integrals (0.1) are invariant.

Let $\epsilon : \mathcal{X} \to (B, b)$ be a fibration of $K3$ surfaces over a base B with special fiber

$$\mathcal{X}_b \cong S \quad \text{over } b \in B.$$

Let $U \subset B$ be an open set containing $b \in B$ over which the local system of second cohomology $R^2\epsilon_*(\mathbb{Z})$ is trivial. The class $\beta \in \text{Pic}(S)$ determines a *local Noether–Lefschetz locus*

$$\text{NL}(\beta) \subset U$$

defined as the subscheme where β remains a $(1, 1)$-class. (While $\text{NL}(\beta)$ is locally defined on U by a single equation, the locus may be degenerate (equal to all of U).)

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Let $(\Delta, 0)$ be a nonsingular quasiprojective curve with special point $0 \in \Delta$. The integral $R_{g,\beta}(S)$ computes the local contribution of S to the standard Gromov–Witten theory of every $K3$-fibered 3-fold

$$\epsilon : T \to (\Delta, 0) \tag{0.2}$$

with special fiber S and local Noether–Lefschetz locus $\mathrm{NL}(\beta) \subset \Delta$ equal to the reduced point $0 \in \Delta$; see [36].

0.2. Curve classes. The second cohomology of S is a rank 22 lattice with intersection form

$$H^2(S, \mathbb{Z}) \stackrel{\sim}{=} U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1), \tag{0.3}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

is the (negative) Cartan matrix. The intersection form (0.3) is even.

The *divisibility* $m(\beta)$ is the maximal positive integer dividing the lattice element $\beta \in H^2(S, \mathbb{Z})$. If the divisibility is 1, β is *primitive*. Elements with equal divisibility and norm square are equivalent up to orthogonal transformation of $H^2(S, \mathbb{Z})$; see [47]. By straightforward deformation arguments using the Torelli theorem for $K3$ surfaces, $R_{g,\beta}(S)$ depends, for effective classes, *only* on the divisibility $m(\beta)$ and the norm square

$$\langle \beta, \beta \rangle = \int_S \beta^2.$$

We will omit the argument S in the notation,

$$R_{g,\beta} = R_{g,\beta}(S).$$

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0.3. BPS counts. The KKV conjecture concerns BPS counts associated to the Hodge integrals (0.1). Throughout this paper we let

$$\alpha \in \mathrm{Pic}(S)$$

denote a nonzero class which is both *effective* and *primitive*. The Gromov–Witten potential $F_\alpha(\lambda, v)$ for classes proportional to α is

$$F_\alpha = \sum_{g \geqslant 0} \sum_{m>0} R_{g,m\alpha} \lambda^{2g-2} v^{m\alpha}. \tag{0.4}$$

The BPS counts $r_{g,m\alpha}$ are uniquely defined by the following equation:

$$F_\alpha = \sum_{g \geqslant 0} \sum_{m>0} r_{g,m\alpha} \lambda^{2g-2} \sum_{d>0} \frac{1}{d} \left(\frac{\sin(d\lambda/2)}{\lambda/2} \right)^{2g-2} v^{dm\alpha}. \tag{0.5}$$

Equation (0.5) defines BPS counts for both primitive and divisible classes.

The string theoretic calculations of Katz *et al.* [24] via heterotic duality yield two conjectures.

CONJECTURE 1. *The BPS count $r_{g,\beta}$ depends upon β only through the norm square $\langle \beta, \beta \rangle$.*

Conjecture 1 is rather surprising from the point of view of Gromov–Witten theory. From the definition, the invariants $R_{g,\beta}$ and $r_{g,\beta}$ depend upon both the divisibility m of β and the norm square $\langle \beta, \beta \rangle$. Assuming the validity of Conjecture 1, let $r_{g,h}$ denote the BPS count associated to a class β of arithmetic genus h,

$$\langle \beta, \beta \rangle = 2h - 2.$$

CONJECTURE 2. *The BPS counts $r_{g,h}$ are uniquely determined by the following equation:*

$$\sum_{g \geqslant 0} \sum_{h \geqslant 0} (-1)^g r_{g,h} (y^{1/2} - y^{-1/2})^{2g} q^h = \prod_{n \geqslant 1} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}.$$

As a consequences of Conjecture 2, $r_{g,h} \in \mathbb{Z}$, $r_{g,h}$ vanishes if $g > h$, and

$$r_{g,g} = (-1)^g (g+1).$$

The integrality of $r_{g,h}$ and the vanishing for high g (when h is fixed) fit in the framework of the Gopakumar–Vafa conjectures. The first values are tabulated below:

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$r_{g,h}$	$h=0$	1	2	3	4
$g=0$	1	24	324	3200	25650
1		-2	-54	-800	-8550
2			3	88	1401
3				-4	-126
4					5

The right side of Conjecture 2 is related to the generating series of Hodge numbers of the Hilbert schemes of points $\mathrm{Hilb}^n(S)$. The genus 0 specialization of Conjecture 2 recovers the Yau–Zaslow formula

$$\sum_{h\geqslant 0} r_{0,h} q^h = \prod_{n\geqslant 1} \frac{1}{(1-q^n)^{24}}$$

related to the Euler characteristics of $\mathrm{Hilb}^n(S)$.

The main result of the present paper is a proof of the KKV conjecture for all genera g and all $\beta \in H_2(S, \mathbb{Z})$.

THEOREM 3. *The BPS count $r_{g,\beta}$ depends upon β only through $\langle \beta, \beta\rangle = 2h-2$, and the Katz–Klemm–Vafa formula holds:*

$$\sum_{g\geqslant 0}\sum_{h\geqslant 0}(-1)^g r_{g,h}(y^{1/2}-y^{-1/2})^{2g} q^h = \prod_{n\geqslant 1} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}.$$

0.4. Past work. The enumerative geometry of curves on $K3$ surfaces has been studied since the 1995 paper of Yau and Zaslow [48]. A mathematical approach to the genus 0 Yau–Zaslow formula can be found in [4, 11, 14]. The Yau–Zaslow formula was proven for primitive classes β by Bryan and Leung [8]. The divisibility 2 case was settled by Lee and Leung in [32]. A complete proof of the Yau–Zaslow formula for all divisibilities was given in [28]. Our approach to Theorem 1 provides a completely new proof of the Yau–Zaslow formula for all divisibilities (which avoids the mirror calculation of the STU model and the Harvey–Moore identity used in [28]).

Conjecture 2 for primitive classes β is connected to Euler characteristics of the moduli spaces of stable pairs on $K3$ surfaces by the GW/P correspondence of [40, 41]. A proof of Conjecture 2 for primitive classes is given in [37] relying upon the Euler characteristic calculations of Kawai and Yoshioka [25]. For cases where $g > 0$ and β is not primitive, Theorem 1 is a new result.

The cases understood before are very special. If the genus is 0, the calculation can be moved via Noether–Lefschetz theory to the genus 0 Gromov–Witten theory of toric varieties using the hyperplane principle for $K3$-fibrations [28]. If the class β is irreducible, the moduli space of stable pairs is nonsingular [25], and the calculation can be moved to stable pairs [37]. The difficulty for positive genus imprimitive curves – which are essentially *all* curves – lies in the complexity of the moduli spaces. There is no effective hyperplane principle in higher genus, and the moduli spaces of stable maps and stable pairs are both highly singular.

Toda has undertaken a parallel study of the Euler characteristic (following Joyce) of the moduli spaces of stable pairs on $K3$ surfaces [46]. His results – together with further multiple cover conjectures which are still open – are connected to an Euler characteristic version of the KKV formula. Our methods and results essentially concern the virtual class and thus do not imply (nor are implied by) Toda's paper [46]. In fact, the motivation of [46] was the original KKV conjecture proven here.

0.5. GW/P correspondence. The Katz–Klemm–Vafa formula concerns integrals over the moduli space of stable maps. Our strategy is to transform the calculation to the theory of stable pairs. Let $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$ be the blow-up of $\mathbb{P}^2 \times \mathbb{P}^1$ in a point. Consider a nonsingular anticanonical Calabi–Yau 3-fold hypersurface,

$$\mathsf{X} \subset \widetilde{\mathbb{P}^2 \times \mathbb{P}^1} \times \mathbb{P}^1.$$

The projection onto the last factor,

$$\pi_3 : \mathsf{X} \to \mathbb{P}^1, \tag{0.6}$$

determines a 1-parameter family of anticanonical $K3$ surfaces in $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$. The interplay between the Gromov–Witten, stable pairs, and Noether–Lefschetz theories for the family π_3 will be used to transform Theorem 3 to nontrivial claims of the moduli of sheaves on $K3$-fibrations.

The KKV formula (conjecturally) evaluates the integrals $R_{g,\beta}$ occurring in the reduced Gromov–Witten theory of a $K3$ surface S. If we view S as a fiber of π_3, then

$$\beta \in \mathrm{Pic}(S) \subset H^2(S, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$$

determines a fiber class in $H_2(\mathsf{X}, \mathbb{Z})$ by push-forward. We consider both the Gromov–Witten and stable pairs invariants of X in π_3-fiber curve classes. The GW/NL correspondence of [36] precisely relates the Gromov–Witten theory of X in fiber classes with the Noether–Lefschetz numbers of the family and the integrals $R_{g,\beta}$. We prove a P/NL correspondence which establishes a parallel

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relationship between the stable pairs theory of X in fiber classes with the same Noether–Lefschetz numbers and the invariants $\widetilde{R}_{n,\beta}$ defined as follows.

0.6. Stable pairs and $K3$ surfaces. Let S be a nonsingular projective $K3$ surface with a nonzero effective curve class $\beta \in \mathrm{Pic}(S)$. We define here the stable pairs analogue $\widetilde{R}_{n,\beta}$ of the reduced Gromov–Witten invariants $R_{g,\beta}$ of S.

For Gromov–Witten invariants, we defined $R_{g,\beta}$ directly (0.1) in terms of the moduli of stable maps to S and observed the result calculated the contributions of the special fiber S to the Gromov–Witten theories of all families (0.2) appropriately transverse to the local Noether–Lefschetz locus corresponding to β. The geometry of stable pairs is more subtle. While the support of a stable pair may probe thickenings of the special fiber $S \subset T$ of (0.2), the image of a stable map does not. As a result, we will *define* $\widetilde{R}_{n,\beta}$ via the geometry of appropriately transverse families of $K3$ surfaces. Later in Section 6.11, we will see how to define $\widetilde{R}_{n,\beta}$ via the intrinsic geometry of S.

Let $\alpha \in \mathrm{Pic}(S)$ be a nonzero class which is both effective and primitive. Let T be a nonsingular 3-dimensional quasiprojective variety,

$$\epsilon : T \to (\Delta, 0),$$

fibered in $K3$ surfaces over a pointed curve $(\Delta, 0)$ satisfying:

(i) Δ is a nonsingular quasiprojective curve;

(ii) ϵ is smooth, projective, and $\epsilon^{-1}(0) \stackrel{\sim}{=} S$;

(iii) the local Noether–Lefschetz locus $\mathrm{NL}(\alpha) \subset \Delta$ corresponding to the class $\alpha \in \mathrm{Pic}(S)$ is the *reduced* point $0 \in \Delta$.

The class $\alpha \in \mathrm{Pic}(S)$ is *m-rigid* with respect to the family ϵ if the following further condition is satisfied:

($\star$) for every *effective* decomposition

$$m\alpha = \sum_{i=1}^{l} \gamma_i \in \mathrm{Pic}(S),$$

the local Noether–Lefschetz locus $\mathrm{NL}(\gamma_i) \subset \Delta$ corresponding to each class $\gamma_i \in \mathrm{Pic}(S)$ is the *reduced* point $0 \in \Delta$. (An effective decomposition requires all parts γ_i to be effective divisors.)

Let $\mathrm{Eff}(m\alpha) \subset \mathrm{Pic}(S)$ denote the subset of effective summands of $m\alpha$. Condition ($\star$) implies (iii).

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Assume α is m-rigid with respect to the family ϵ. By property ($\star$), there is a compact, open, and closed component

$$P_n^\star(T,\gamma) \subset P_n(T,\gamma)$$

parameterizing stable pairs supported set-theoretically over the point $0 \in \Delta$ for *every* effective summand $\gamma \in \mathrm{Eff}(m\alpha)$. (For any class $\gamma \in \mathrm{Pic}(S)$, we denote the push-forward to $H_2(T,\mathbb{Z})$ also by γ. Let $P_n(T,\gamma)$ be the moduli space of stable pairs of Euler characteristic n and class $\gamma \in H_2(T,\mathbb{Z})$.)

DEFINITION. Let $\alpha \in \mathrm{Pic}(S)$ be a nonzero class which is both effective and primitive. Given a family $\epsilon : T \to (\Delta, 0)$ satisfying conditions (i), (ii), and ($\star$) for $m\alpha$, let

$$\sum_{n\in\mathbb{Z}} \widetilde{R}_{n,m\alpha}(S)\, q^n = \mathrm{Coeff}_{v^{m\alpha}}\left[\log\left(1+\sum_{n\in\mathbb{Z}}\sum_{\gamma\in\mathrm{Eff}(m\alpha)} q^n v^\gamma \int_{[P_n^\star(T,\gamma)]^{\mathrm{vir}}} 1\right)\right]. \tag{0.7}$$

In Section 6.12, we will prove $\widetilde{R}_{n,m\alpha}$ depends *only* upon n, m and $\langle\alpha,\alpha\rangle$, and *not* upon S nor the family ϵ. The dependence result is nontrivial and requires new techniques to establish. The existence of m-rigid families ϵ for suitable S and α (primitive with fixed $\langle\alpha,\alpha\rangle$) then defines $\widetilde{R}_{n,m\alpha}$ for all m. (Constructions are given in Section 6.2.)

The appearance of the logarithm in (0.7) has a simple explanation. The Gromov–Witten invariants $R_{g,m\alpha}$ are defined via moduli spaces of stable maps with *connected* domains. Stable pairs invariants count sheaves with possibly *disconnected* support curves. The logarithm accounts for the difference.

The stable pairs potential $\widetilde{F}_\alpha(q,v)$ for classes proportional to the primitive class α is

$$\widetilde{F}_\alpha = \sum_{n\in\mathbb{Z}}\sum_{m>0} \widetilde{R}_{n,m\alpha} q^n v^{m\alpha}. \tag{0.8}$$

By the properties of $\widetilde{R}_{n,m\alpha}$, the potential $\widetilde{F}_\alpha$ depends only upon the norm square $\langle\alpha,\alpha\rangle$.

Via the correspondences with Noether–Lefschetz theory, we prove that the GW/P correspondence [39, 40] for suitable 3-folds fibered in $K3$ surfaces implies the following basic result for the potentials (0.4) and (0.8).

THEOREM 4. *After the variable change $-q = e^{i\lambda}$, the potentials are equal:*

$$F_\alpha(\lambda, v) = \widetilde{F}_\alpha(q,v).$$

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In order to show the variable change of Theorem 4 is well defined, a rationality result is required. In Section 7, we prove for all $m > 0$,

$$[\widetilde{F}_\alpha]_{v^{m\alpha}} = \sum_{n\in\mathbb{Z}} \widetilde{R}_{n,m\alpha} q^n$$

is the Laurent expansion of a rational function in q.

0.7. Multiple covers. While Theorem 4 transforms Theorem 3 to a statement about stable pairs, the evaluation must still be carried out.

The logarithm in definition (0.7) plays no role for the v^α coefficient,

$$[\widetilde{F}_\alpha]_{v^\alpha} = \sum_{n\in\mathbb{Z}} q^n \int_{[P_n^\star(T,\alpha)]^{\mathrm{vir}}} 1.$$

If α is irreducible (which can be assumed by deformation invariance), $P_n^\star(T,\alpha)$ is a nonsingular variety of dimension $\langle\alpha,\alpha\rangle + n + 1$. If T is taken to be Calabi–Yau, the obstruction theory on $P_n^\star(T,\alpha)$ is self-dual and

$$\sum_{n\in\mathbb{Z}} q^n \int_{[P_n(T,\alpha)]^{\mathrm{vir}}} 1 = \sum_{n\in\mathbb{Z}} q^n (-1)^{\langle\alpha,\alpha\rangle+n+1} \chi_{\mathrm{top}}\left(P_n^\star(T,\alpha)\right).$$

The Euler characteristic calculations of Kawai and Yoshioka [**25**] then imply the stable pairs KKV prediction for primitive $\alpha \in \mathrm{Pic}(S)$. A detailed discussion can be found in [**41**, Appendix C].

In order to prove the KKV conjecture for $[\widetilde{F}_\alpha]_{v^{m\alpha}}$ for all $m > 1$, we find new multiple cover formulas for stable pairs on $K3$ surfaces. In fact, the multiple cover structure implicit in the KKV formula is much more natural on the stable pairs side.

By degeneration arguments and deformation to the normal cone, we reduce the stable pairs multiple cover formula to a calculation on the trivial $K3$-fibration $S \times \mathbb{C}$, where $\mathbb{C}^*$-localization applies. A crucial point here is a vanishing result: for each k only the simplest k-fold multiple covers contribute – those stable pairs which are a trivial k-times thickening in the $\mathbb{C}$-direction of a stable pair on S. The moduli space of such trivial thickenings is isomorphic to the moduli space of stable pairs supported on S. This simple geometric relationship provides the key to the stable pairs multiple cover formula.

0.8. Guide to the proof. The main steps in our proof of the Katz–Klemm–Vafa formula are summarized as follows:

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(i) We express the Gromov–Witten invariants of the anticanonical hypersurface,

$$\mathsf{X} \subset \widetilde{\mathbb{P}^2 \times \mathbb{P}^1} \times \mathbb{P}^1,$$

in terms of the Noether–Lefschetz numbers of π_3 and the reduced invariants $R_{g,\beta}$ via the GW/NL correspondence.

(ii) We express the stable pairs invariants of X in terms of the Noether–Lefschetz numbers of π_3 and the stable pairs invariants $\widetilde{R}_{n,\beta}$ via the P/NL correspondence.

(iii) The GW/P conjecture, proved for the complete intersection X in [39], relates the Gromov–Witten and stable pairs invariants of the 3-fold X.

(iv) By inverting the relations (i) and (ii) and using the correspondence (iii), we establish the equivalence between the sets of numbers $R_{g,\beta}$ and $\widetilde{R}_{n,\beta}$ stated in Theorem 4.

(v) The invariant $\widetilde{R}_{n,\beta}(S)$ is defined via an appropriately transverse family

$$\epsilon : T \to (\Delta, 0), \quad \epsilon^{-1}(0) \stackrel{\sim}{=} S.$$

Degenerating the total space T to the normal cone of $S \subset T$, we reduce $\widetilde{R}_{n,\beta}(S)$ to a calculation of stable pairs integrals over a rubber target. After further geometric arguments, the calculation is expressed in terms of the reduced stable pairs invariants of the trivial $K3$-fibration $S \times \mathbb{P}^1$. A careful analysis of several different obstruction theories is required here.

(vi) By $\mathbb{C}^*$-localization on $S \times \mathbb{P}^1$, we reduce further to a calculation on the moduli space of $\mathbb{C}^*$-fixed stable pairs on $S \times \mathbb{C}$.

(vii) We prove a vanishing result: for each k only the simplest k-fold multiple covers contribute. We only need to calculate the contributions of stable pairs which are a trivial k-times thickening (in the $\mathbb{C}$-direction) of a stable pair scheme-theoretically supported on S.

(viii) The resulting moduli spaces are isomorphic to $P_n(S, \beta)$, the moduli space of stable pairs on S.

(ix) The resulting integral is calculated in [29, 30] in terms of universal formulas in topological constants. In particular, the result does not depend on the divisibility of β.

(x) We may therefore assume β to be primitive, and moreover, by deformation invariance, to be irreducible. The moduli space $P_n(S, \beta)$ is then nonsingular. The integrals $\widetilde{R}_{n,\beta}(S)$ can be expressed in terms of those evaluated by Kawai–Yoshioka, as explained in [37, 41].

The paper starts with a discussion of Noether–Lefschetz theory for Gromov–Witten invariants of $K3$-fibrations. The GW/NL correspondence of [36] and Borcherds' results are reviewed in Section 1. A crucial property of the family (0.6) is established in Proposition 6 of Section 2: the BPS states and the Noether–Lefschetz numbers for the family (0.6) uniquely determine *all* the integrals $R_{g,\beta}$ in the reduced Gromov–Witten theory of $K3$ surfaces. The result follows by finding a triangularity in the GW/NL correspondence.

Theorem 4 constitutes half of our proof of the KKV conjecture. In Section 3, we prove Theorem 4 *assuming* the P/NL correspondence. In fact, Theorem 4 is an easy consequence of the GW/NL correspondence, the P/NL correspondence, and the invertibility established in Proposition 6. The precise statement of the P/NL correspondence is given in Section 3.5, but the proof is presented later in Section 8.

Sections 4–8 mainly concern the geometry of the moduli of stable pairs on $K3$ surfaces and $K3$-fibrations. The first topic is a detailed study of the trivial fibration $S \times \mathbb{C}$. In Sections 4 and 5, an analysis of the perfect obstruction theory of the $\mathbb{C}^*$-fixed loci of the moduli space of stable pairs on $S \times \mathbb{C}$ is presented. We find that only the simplest $\mathbb{C}^*$-fixed stable pairs have nonvanishing contributions. Moreover, these contributions directly yield multiple cover formulas. The move from Gromov–Witten theory to stable pairs was made precisely to exploit the much simpler multiple cover structure on the sheaf theory side.

The main results of Sections 6 and 7 concern the expression of $\widetilde{R}_{n,\beta}$ in terms of the stable pair theory of $S \times \mathbb{C}$. A careful study of the obstruction theory is needed. The outcome is a multiple cover formula for $\widetilde{R}_{n,\beta}$.

After we establish the P/NL correspondence for the family X in Section 8, the proof of the Katz–Klemm–Vafa conjecture is completed in Section 9 by transforming the multiple cover formula to the Gromov–Witten invariants $R_{g,\beta}$. As a consequence of the KKV formula, the Gromov–Witten theory of $K3$-fibrations in vertical classes can be effectively computed. As an example, the classical pencil of quartic $K3$ surfaces is treated in Section 10.

A summary of our notation for the various Gromov–Witten and stable pairs invariants for $K3$ surfaces and $K3$-fibrations is given in Appendix A. Appendix B contains a discussion of degenerations of X needed for the Gromov–Witten/Pairs correspondence of [39]. Appendix C contains results about cones, the Fulton total Chern class, and virtual cycles.

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1. Noether–Lefschetz theory

1.1. Lattice polarization. Let S be a nonsingular $K3$ surface. A primitive class $L \in \mathrm{Pic}(S)$ is a *quasipolarization* if

$$\langle L, L \rangle > 0 \quad \text{and} \quad \langle L, [C] \rangle \geqslant 0$$

for every curve $C \subset S$. A sufficiently high tensor power L^n of a quasipolarization is base point free and determines a birational morphism

$$S \rightarrow \widetilde{S}$$

contracting A–D–E configurations of (-2)-curves on S. Hence, every quasipolarized $K3$ surface is algebraic.

Let Λ be a fixed rank r primitive sublattice

$$\Lambda \subset U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

with signature $(1, r-1)$, and let $v_1, \ldots, v_r \in \Lambda$ be an integral basis. (A sublattice is primitive if the quotient is torsion free.) The discriminant is

$$\Delta(\Lambda) = (-1)^{r-1} \det \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle \\ \vdots & \ddots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle \end{pmatrix}.$$

The sign is chosen so $\Delta(\Lambda) > 0$.

A *Λ-polarization* of a $K3$ surface S is a primitive embedding

$$j : \Lambda \rightarrow \mathrm{Pic}(S)$$

satisfying two properties:

(i) the lattice pairs $\Lambda \subset U^3 \oplus E_8(-1)^2$ and $\Lambda \subset H^2(S, \mathbb{Z})$ are isomorphic via an isometry which restricts to the identity on Λ;

(ii) $\mathrm{Im}(j)$ contains a quasipolarization.

By (ii), every Λ-polarized $K3$ surface is algebraic.

The period domain M of Hodge structures of type $(1, 20, 1)$ on the lattice $U^3 \oplus E_8(-1)^2$ is an analytic open set of the 20-dimensional nonsingular isotropic quadric Q,

$$M \subset Q \subset \mathbb{P}\big((U^3 \oplus E_8(-1)^2) \otimes_{\mathbb{Z}} \mathbb{C}\big).$$

Let $M_\Lambda \subset M$ be the locus of vectors orthogonal to the entire sublattice $\Lambda \subset U^3 \oplus E_8(-1)^2$.

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Let Γ be the isometry group of the lattice $U^3 \oplus E_8(-1)^2$, and let

$$\Gamma_\Lambda \subset \Gamma$$

be the subgroup restricting to the identity on Λ. By global Torelli, the moduli space $\mathcal{M}_\Lambda$ of Λ-polarized $K3$ surfaces is the quotient

$$\mathcal{M}_\Lambda = M_\Lambda / \Gamma_\Lambda.$$

We refer the reader to [12] for a detailed discussion.

Let $\widetilde{S}$ be a $K3$ surface with A–D–E singularities, and let

$$\tilde{j} : \Lambda \to \mathrm{Pic}(\widetilde{S})$$

be a primitive embedding. Via pull-back along the desingularization,

$$S \to \widetilde{S},$$

we obtain a composition $j : \Lambda \to \mathrm{Pic}(S)$. If (S, j) satisfies (i) and (ii), we define $(\widetilde{S}, \tilde{j})$ to be a Λ-polarized *singular* $K3$ surface. Then (S, j) is a Λ-polarized nonsingular $K3$ surface canonically associated to $(\widetilde{S}, \tilde{j})$.

1.2. Families. Let X be a nonsingular projective 3-fold equipped with line bundles

$$L_1, \ldots, L_r \to X$$

and a map

$$\pi : X \to C$$

to a nonsingular complete curve.

The tuple $(X, L_1, \ldots, L_r, \pi)$ is a *1-parameter family of Λ-polarized $K3$ surfaces* if

(i) the fibers $(X_\xi, L_{1,\xi}, \ldots, L_{r,\xi})$ are Λ-polarized $K3$ surfaces with at worst *a single nodal singularity* via

$$v_i \mapsto L_{i,\xi}$$

for every $\xi \in C$;

(ii) there exists a $\lambda^\pi \in \Lambda$ which is a quasipolarization of all fibers of π simultaneously.

The family π yields a morphism,

$$\iota_\pi : C \to \mathcal{M}_\Lambda,$$

to the moduli space of Λ-polarized $K3$ surfaces.

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Let $\lambda^\pi = \lambda_1^\pi v_1 + \cdots + \lambda_r^\pi v_r$. A vector $(d_1, \ldots, d_r)$ of integers is *positive* if

$$\sum_{i=1}^{r} \lambda_i^\pi d_i > 0.$$

If $\beta \in \mathrm{Pic}(X_\xi)$ has intersection numbers

$$d_i = \langle L_{i,\xi}, \beta \rangle,$$

then β has positive degree with respect to the quasipolarization if and only if $(d_1, \ldots, d_r)$ is positive.

1.2.1. Noether–Lefschetz divisors. Noether–Lefschetz numbers are defined in [36] by the intersection of $\iota_\pi(C)$ with Noether–Lefschetz divisors in $\mathcal{M}_\Lambda$. We briefly review the definition of the Noether–Lefschetz divisors.

Let $(\mathbb{L}, \iota)$ be a rank $r+1$ lattice $\mathbb{L}$ with an even symmetric bilinear form $\langle , \rangle$ and a primitive embedding

$$\iota : \Lambda \to \mathbb{L}.$$

Two data sets $(\mathbb{L}, \iota)$ and $(\mathbb{L}', \iota')$ are isomorphic if and only if there exist an isometry relating $\mathbb{L}$ and $\mathbb{L}'$ which takes ι to ι'. The first invariant of the data $(\mathbb{L}, \iota)$ is the discriminant $\Delta \in \mathbb{Z}$ of $\mathbb{L}$.

An additional invariant of $(\mathbb{L}, \iota)$ can be obtained by considering any vector $v \in \mathbb{L}$ for which

$$\mathbb{L} = \iota(\Lambda) \oplus \mathbb{Z}v. \tag{1.1}$$

(Here, $\oplus$ is used just for the additive structure, not orthogonal direct sum.) The pairing

$$\langle v, \cdot \rangle : \Lambda \to \mathbb{Z}$$

determines an element of $\delta_v \in \Lambda^*$. Let $G = \Lambda^*/\Lambda$ be the quotient defined via the injection $\Lambda \to \Lambda^*$ obtained from the pairing $\langle , \rangle$ on Λ. The group G is abelian of order given by the discriminant $|\Delta(\Lambda)|$. The image

$$\delta \in G/\pm$$

of δ_v is easily seen to be independent of v satisfying (1.1). The invariant δ is the *coset* of $(\mathbb{L}, \iota)$.

By elementary arguments, two data sets $(\mathbb{L}, \iota)$ and $(\mathbb{L}', \iota')$ of rank $r+1$ are isomorphic if and only if the discriminants and cosets are equal.

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Let $v_1, \dots, v_r$ be an integral basis of Λ as before. The pairing of $\mathbb{L}$ with respect to an extended basis $v_1, \dots, v_r, v$ is encoded in the matrix

$$\mathbb{L}_{h,d_1,\dots,d_r} = \begin{pmatrix} \langle v_1, v_1\rangle & \cdots & \langle v_1, v_r\rangle & d_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle v_r, v_1\rangle & \cdots & \langle v_r, v_r\rangle & d_r \\ d_1 & \cdots & d_r & 2h-2 \end{pmatrix}.$$

The discriminant is

$$\Delta(h, d_1, \dots, d_r) = (-1)^r \det(\mathbb{L}_{h,d_1,\dots,d_r}).$$

The coset $\delta(h, d_1, \dots, d_r)$ is represented by the functional

$$v_i \mapsto d_i.$$

The Noether–Lefschetz divisor $P_{\Delta,\delta} \subset \mathcal{M}_\Lambda$ is the closure of the locus of Λ-polarized $K3$ surfaces S for which $(\mathrm{Pic}(S), j)$ has rank $r+1$, discriminant Δ, and coset δ. By the Hodge index theorem, $P_{\Delta,\delta}$ is empty unless $\Delta > 0$. By definition, $P_{\Delta,\delta}$ is a reduced subscheme. (The intersection form on $\mathrm{Pic}(S)$ is nondegenerate for an algebraic $K3$ surface. Hence, a rank $r+1$ sublattice of $\mathrm{Pic}(S)$ which contains a quasipolarization must have signature $(1, r)$ by the Hodge index theorem.)

Let $h, d_1, \dots, d_r$ determine a positive discriminant

$$\Delta(h, d_1, \dots, d_r) > 0.$$

The Noether–Lefschetz divisor $D_{h,(d_1,\dots,d_r)} \subset \mathcal{M}_\Lambda$ is defined by the weighted sum

$$D_{h,(d_1,\dots,d_r)} = \sum_{\Delta,\delta} m(h, d_1, \dots, d_r | \Delta, \delta) \cdot [P_{\Delta,\delta}]$$

where the multiplicity $m(h, d_1, \dots, d_r|\Delta, \delta)$ is the number of elements β of the lattice $(\mathbb{L}, \iota)$ of type (Δ, δ) satisfying

$$\langle \beta, \beta\rangle = 2h-2, \quad \langle \beta, v_i\rangle = d_i. \tag{1.2}$$

If the multiplicity is nonzero, then $\Delta | \Delta(h, d_1, \dots, d_r)$ so only finitely many divisors appear in the above sum.

If $\Delta(h, d_1, \dots, d_r) = 0$, the divisor $D_{h,(d_1,\dots,d_r)}$ has a different definition. The tautological line bundle $\mathcal{O}(-1)$ is Γ-equivariant on the period domain M_Λ and descends to the *Hodge line bundle*

$$\mathcal{K} \to \mathcal{M}_\Lambda.$$

We define $D_{h,(d_1,\dots,d_r)} = \mathcal{K}^*$ if there exists $v \in \Lambda$ satisfying

$$\langle v_1, v\rangle = d_1, \quad \langle v_2, v\rangle = d_2, \dots, \langle v_r, v\rangle = d_r. \tag{1.3}$$

(If $\Delta(h, d_1, \dots, d_r) = 0$ and (1.3) holds, then $\langle v, v\rangle = 2h - 2$ is forced. Since the d_i do not simultaneously vanish, $v \neq 0$.) If v satisfies (1.3), v is unique. If no such $v \in \Lambda$ exists, then

$$D_{h,(d_1,\dots,d_r)} = 0.$$

In case Λ is a unimodular lattice, such a v always exists. See [36] for an alternate view of degenerate intersection.

If $\Delta(h, d_1, \dots, d_r) < 0$, the divisor $D_{h,(d_1,\dots,d_r)}$ on $\mathcal{M}_\Lambda$ is defined to vanish by the Hodge index theorem.

1.2.2. Noether–Lefschetz numbers. Let Λ be a lattice of discriminant $l = \Delta(\Lambda)$, and let $(X, L_1, \dots, L_r, \pi)$ be a 1-parameter family of Λ-polarized $K3$ surfaces. The Noether–Lefschetz number $NL^\pi_{h,d_1,\dots,d_r}$ is the classical intersection product

$$NL^\pi_{h,(d_1,\dots,d_r)} = \int_C \iota_\pi^*[D_{h,(d_1,\dots,d_r)}]. \tag{1.4}$$

Let $\mathrm{Mp}_2(\mathbb{Z})$ be the metaplectic double cover of $SL_2(\mathbb{Z})$. There is a canonical representation [5] associated to Λ,

$$\rho^*_\Lambda : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{End}(\mathbb{C}[G]),$$

where $G = \Lambda^*/\Lambda$. The full set of Noether–Lefschetz numbers $NL^\pi_{h,d_1,\dots,d_r}$ defines a vector valued modular form

$$\Phi^\pi(q) = \sum_{\gamma\in G} \Phi^\pi_\gamma(q) v_\gamma \in \mathbb{C}[[q^{1/2l}]] \otimes \mathbb{C}[G],$$

of weight $(22 - r)/2$ and type ρ^*_Λ by results of Borcherds and Kudla–Millson [5, 31]. (While the results of the papers [5, 31] have considerable overlap, we will follow the point of view of Borcherds.) The Noether–Lefschetz numbers are the coefficients of the components of Φ^π,

$$NL^\pi_{h,(d_1,\dots,d_r)} = \Phi^\pi_\gamma\left[\frac{\Delta(h, d_1, \dots, d_r)}{2l}\right]$$

where $\delta(h, d_1, \dots, d_r) = \pm\gamma$. (If f is a series in q, $f[k]$ denotes the coefficient of q^k.) The modular form results significantly constrain the Noether–Lefschetz numbers.

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1.2.3. Refinements. If $d_1, \ldots, d_r$ do not simultaneously vanish, refined Noether–Lefschetz divisors are defined. If $\Delta(h, d_1, \ldots, d_r) > 0$,

$$D_{m,h,(d_1,\ldots,d_r)} \subset D_{h,(d_1,\ldots,d_r)}$$

is defined by requiring the class $\beta \in \mathrm{Pic}(S)$ to satisfy (1.2) and have divisibility $m > 0$. If $\Delta(h, d_1, \ldots, d_r) = 0$, then

$$D_{m,h,(d_1,\ldots,d_r)} = D_{h,(d_1,\ldots,d_r)}$$

if there exists $v \in \Lambda$ of divisibility $m > 0$ satisfying

$$\langle v_1, v\rangle = d_1, \quad \langle v_2, v\rangle = d_2, \ldots, \langle v_r, v\rangle = d_r.$$

If v satisfies the above degree conditions, v is unique. If no such $v \in \Lambda$ exists, then

$$D_{m,h,(d_1,\ldots,d_r)} = 0.$$

A necessary condition for the existence of v is the divisibility of each d_i by m. In case Λ is a unimodular lattice, v exists if and only if m is the greatest common divisor of $d_1, \ldots, d_r$.

Refined Noether–Lefschetz numbers are defined by

$$NL^{\pi}_{m,h,(d_1,\ldots,d_r)} = \int_C \iota_\pi^*[D_{m,h,(d_1,\ldots,d_r)}]. \tag{1.5}$$

The full set of Noether–Lefschetz numbers $NL^{\pi}_{h,(d_1,\ldots,d_r)}$ is easily shown to determine the refined numbers $NL^{\pi}_{m,h,(d_1,\ldots,d_r)}$; see [28].

1.3. GW/NL correspondence. The GW/NL correspondence intertwines three theories associated to a 1-parameter family

$$\pi : X \to C$$

of Λ-polarized $K3$ surfaces:

(i) the Noether–Lefschetz numbers of π;

(ii) the genus g Gromov–Witten invariants of X;

(iii) the genus g reduced Gromov–Witten invariants of the $K3$ fibers.

The Noether–Lefschetz numbers (i) are classical intersection products while the Gromov–Witten invariants (ii)–(iii) are quantum in origin. For (ii), we view

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the theory in terms the Gopakumar–Vafa invariants [16, 17]. (A review of the definitions will be given in Section 2.2.)

Let $n^X_{g,(d_1,\ldots,d_r)}$ denote the Gopakumar–Vafa invariant of X in genus g for π-vertical curve classes of degrees $d_1, \ldots, d_r$ with respect to the line bundles L_1, $\ldots$, L_r. (The invariant $n^X_{g,(d_1,\ldots,d_r)}$ may be a (finite) sum of $n^X_{g,\gamma}$ for π-vertical curve classes $\gamma \in H_2(X, \mathbb{Z})$.) Let $r_{g,\beta}$ denote the reduced $K3$ invariant defined in Section 0.3 for an effective curve class β. Since $r_{g,\beta}$ depends only upon the divisibility m and the norm square

$$\langle \beta, \beta \rangle = 2h - 2,$$

we will use the more efficient notation

$$r_{g,m,h} = r_{g,\beta}.$$

The following result is proven in [36] by a comparison of the reduced and usual deformation theories of maps of curves to the $K3$ fibers of π. (The result of the [36] is stated in the rank $r = 1$ case, but the argument is identical for arbitrary r.)

THEOREM 5. *For degrees $(d_1, \ldots, d_r)$ positive with respect to the quasipolarization λ^π,*

$$n^X_{g,(d_1,\ldots,d_r)} = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} r_{g,m,h} \cdot NL^\pi_{m,h,(d_1,\ldots,d_r)}.$$

For fixed g and $(d_1, \ldots, d_r)$, the sum over m is clearly finite since m must divide each d_i. The sum over h is also finite since, for fixed $(d_1, \ldots, d_r)$, $NL^\pi_{m,h,(d_1,\ldots,d_r)}$ vanishes for sufficiently high h by [36, Proposition 3]. By [36, Lemma 2], $r_{g,m,h}$ vanishes for $h < 0$ (and is therefore omitted from the sum in Theorem 5).

2. Anticanonical $K3$ surfaces in $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$

2.1. Polarization. Let $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$ be the blow-up of $\mathbb{P}^2 \times \mathbb{P}^1$ at a point,

$$\widetilde{\mathbb{P}^2 \times \mathbb{P}^1} \to \mathbb{P}^2 \times \mathbb{P}^1.$$

The Picard group is of rank 3:

$$\mathrm{Pic}(\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}) \cong \mathbb{Z}L_1 \oplus \mathbb{Z}L_2 \oplus \mathbb{Z}E,$$

where L_1 and L_2 are the pull-backs of $\mathcal{O}(1)$ from the factors $\mathbb{P}^2$ and $\mathbb{P}^1$, respectively and E is the exceptional divisor. The anticanonical class $3L_1+2L_2-2E$ is base point free.

A nonsingular anticanonical $K3$ hypersurface $S \subset \widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$ is naturally lattice polarized by L_1, L_2, and E. The lattice is

$$\Lambda = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

A general anticanonical Calabi–Yau 3-fold hypersurface,

$$\mathsf{X} \subset \widetilde{\mathbb{P}^2 \times \mathbb{P}^1} \times \mathbb{P}^1,$$

determines a 1-parameter family of anticanonical $K3$ surfaces in $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$,

$$\pi_3 : \mathsf{X} \to \mathbb{P}^1, \tag{2.1}$$

via projection π_3 onto the last $\mathbb{P}^1$. The fibers of π_3 have at worst nodal singularities. (There are 192 nodal fibers. We leave the elementary classical geometry here to the reader.) The Noether–Lefschetz theory of the Λ-polarized family

$$(\mathsf{X}, L_1, L_2, E, \pi_3)$$

plays a central role in our proof of Theorem 3. The quasipolarization λ^{π_3} (condition (ii) of Section 1.1) can be taken to be any very ample line bundle on $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$.

2.2. BPS states. Let $(\mathsf{X}, L_1, L_2, E, \pi_3)$ be the Λ-polarized family of anticanonical $K3$ surfaces of $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$ defined in Section 2.1. The vertical classes are the kernel of the push-forward map by π_3,

$$0 \to H_2(\mathsf{X}, \mathbb{Z})^{\pi_3} \to H_2(\mathsf{X}, \mathbb{Z}) \to H_2(\mathbb{P}^1, \mathbb{Z}) \to 0.$$

Let $\overline{M}_g(\mathsf{X}, \gamma)$ be the moduli space of stable maps from connected genus g curves to X of class γ. Gromov–Witten theory is defined by integration against the virtual class,

$$N^{\mathsf{X}}_{g,\gamma} = \int_{[\overline{M}_g(X,\gamma)]^{\mathrm{vir}}} 1. \tag{2.2}$$

The expected dimension of the moduli space is 0.

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The full genus Gromov–Witten potential F^{X} for nonzero vertical classes is the series

$$F^{\mathsf{X}} = \sum_{g\geqslant 0} \sum_{0\neq\gamma\in H_2(\mathsf{X},\mathbb{Z})^{\pi_3}} N^{\mathsf{X}}_{g,\gamma} \lambda^{2g-2} v^{\gamma},$$

where v is the curve class variable. The BPS counts $n^{\mathsf{X}}_{g,\gamma}$ of Gopakumar and Vafa are uniquely defined by the following equation:

$$F^{\mathsf{X}} = \sum_{g\geqslant 0} \sum_{0\neq\gamma\in H_2(\mathsf{X},\mathbb{Z})^{\pi_3}} n^{\mathsf{X}}_{g,\gamma} \lambda^{2g-2} \sum_{d>0} \frac{1}{d} \left(\frac{\sin(d\lambda/2)}{\lambda/2}\right)^{2g-2} v^{d\gamma}.$$

Conjecturally, the invariants $n^{\mathsf{X}}_{g,\gamma}$ are integral and obtained from the cohomology of an as yet unspecified moduli space of sheaves on X. We do *not* assume the conjectural properties hold.

Using the Λ-polarization, we label the classes $\gamma \in H_2(\mathsf{X},\mathbb{Z})^{\pi_3}$ by their pairings with L_i and E,

$$\gamma \mapsto \left(\int_\gamma [L_1], \int_\gamma [L_2], \int_\gamma [E]\right).$$

We write the BPS counts as $n^{\mathsf{X}}_{g,(d_1,d_2,d_3)}$. Since $\gamma \neq 0$, not all the d_i can vanish.

2.3. Invertibility of constraints. Let $\mathcal{P} \subset \mathbb{Z}^3$ be the set of triples $(d_1, d_2, d_3) \neq (0, 0, 0)$ which are positive with respect to the quasipolarization λ^{π_3} of the Λ-polarized family

$$\pi_3 : \mathsf{X} \to \mathbb{P}^1.$$

Theorem 5 applied to $(\mathsf{X}, L_1, L_2, E, \pi_3)$ yields the equation

$$n^{\mathsf{X}}_{g,(d_1,d_2,d_3)} = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} r_{g,m,h} \cdot NL^{\pi_3}_{m,h,(d_1,d_2,d_3)} \tag{2.3}$$

for $(d_1, d_2, d_3) \in \mathcal{P}$. We view (2.3) as linear constraints for the unknowns $r_{g,m,h}$ in terms of the BPS states on the left and the refined Noether–Lefschetz degrees.

The integrals $r_{g,m,h}$ are very simple in case $h \leqslant 0$. By [36, Lemma 2], $r_{g,m,h} = 0$ for $h < 0$,

$$r_{0,1,0} = 1,$$

and $r_{g,m,0} = 0$ otherwise.

Proposition 6. *The set of invariants $\{r_{g,m,h}\}_{g\geqslant 0, m\geqslant 1, h>0}$ is uniquely determined by the set of constraints (2.3) for $(d_1, d_2, d_3) \in \mathcal{P}$ and the integrals $r_{g,m,h\leqslant 0}$.*

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Proof. A certain subset of the linear equations will be shown to be upper-triangular in the variables $r_{g,m,h}$.

Let us fix in advance the values of $g \geqslant 0$, $m \geqslant 1$, and $h > 0$. We proceed by induction on h assuming the reduced invariants $r_{g,m',h'}$ have already been determined for all $h' < h$. If $2h-2$ is not divisible by $2m^2$, then we have $r_{g,m,h} = 0$ by definition, so we can further assume

$$2h - 2 = m^2(2s-2)$$

for an integer $s > 0$.

Consider the fiber class $\gamma \in H_2(\mathsf{X}, \mathbb{Z})^{\pi_3}$ given by the lattice element $msL_1 + mL_2 + m(s+1)E$,

$$\gamma = (ms[L_1] + m[L_2] + m(s+1)[E]) \cap [S],$$

where S is a $K3$-fiber of π_3. Since L_1, L_2 and E are effective on S, the class γ is effective and hence positive with respect to the quasipolarization of Λ. The degrees of γ are

$$(d_1, d_2, d_3) = (2ms + 3m, 3ms, -2m(s+1)), \tag{2.4}$$

and γ is of divisibility exactly m in the lattice Λ.

Consider Equation (2.3) for (d_1, d_2, d_3) given by (2.4). By the Hodge index theorem, we must have

$$\begin{aligned} 0 &\leqslant \Delta(h', 2ms + 3m, 3ms, -2m(s+1)) \\ &= 18(2 - 2h' + m^2(2s-2)) \\ &= 36(h - h') \end{aligned} \tag{2.5}$$

if $NL^{\pi_3}_{m',h',(2ms+3m,3ms,-2m(s+1))} \neq 0$. Inequality (2.5) implies $h' \leqslant h$. If $h' = h$, then

$$\Delta(h' = h, 2ms + 3m, 3ms, -2m(s+1)) = 0.$$

By the definition of Section 1.2.3,

$$NL^{\pi_3}_{m',h'=h,(2ms+3m,3ms,-2m(s+1))} = 0$$

unless there exists $v \in \Lambda$ of divisibility m' with degrees

$$(2ms + 3m, 3ms, -2m(s+1)).$$

But $\gamma \in \Lambda$ is the unique such lattice element, and γ has divisibility m. Therefore, the constraint (2.3) takes the form

$$n^X_{g,(2ms+3m,3ms,-2m(s+1))} = r_{g,m,h} NL^{\pi_3}_{m,h,(2ms+3m,3ms,-2m(s+1))} + \cdots,$$

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where the dots represent terms involving $r_{g,m',h'}$ with $h' < h$. The leading coefficient is given by

$$NL^{\pi_3}_{m,h,(2ms+3m,3ms,-2m(s+1))} = NL^{\pi_3}_{h,(2ms+3m,3ms,-2m(s+1))} = -2.$$

As the system is upper-triangular, we can invert to solve for $r_{g,m,h}$.

The calculation of $NL^{\pi_3}_{h,(2ms+3m,3ms,-2m(s+1))}$ is elementary. In the discriminant $\Delta = 0$ case, we must determine the degree of the dual of the Hodge line bundle $\mathcal{K}$ on the base $\mathbb{P}^1$. The relative dualizing sheaf ω_{π_3} is the pull-back of $\mathcal{O}_{\mathbb{P}^1}(2)$ from the base. Hence, the dual of the Hodge line has degree -2. See [**36**, Section 6.3] for many such calculations. □

The proof of Proposition 6 does *not* involve induction on the genus. The same argument will be used later in the theory of stable pairs.

3. Theorem 2

3.1. Strategy. We will prove Theorem 4 via the GW/P and Noether–Lefschetz correspondences for the family $(\mathsf{X}, L_1, L_2, E, \pi_3)$ of $K3$ surfaces defined in Section 2.1. While all of the necessary Gromov–Witten theory has been established in Sections 1 and 2, our proof here depends upon stable pairs results proven later in Sections 7 and 8.

3.2. Stable pairs. Let V be a nonsingular, projective 3-fold, and let $\beta \in H_2(V, \mathbb{Z})$ be a nonzero class. We consider the moduli space of stable pairs

$$[\mathcal{O}_V \xrightarrow{s} F] \in P_n(V, \beta)$$

where F is a pure sheaf supported on a Cohen–Macaulay subcurve of V, s is a morphism with 0-dimensional cokernel, and

$$\chi(F) = n, \quad [F] = \beta.$$

The space $P_n(V, \beta)$ carries a virtual fundamental class of dimension $\int_\beta c_1(T_V)$ obtained from the deformation theory of complexes with trivial determinant in the derived category [**40**].

We specialize now to the case where V is the total space of a $K3$-fibration (with at worst nodal fibers),

$$\pi : V \to C,$$

over a nonsingular projective curve and $\beta \in H_2(V, \mathbb{Z})^\pi$ is a vertical class. Then the expected dimension of $P_n(V, \beta)$ is always 0. For nonzero $\beta \in H_2(V, \mathbb{Z})^\pi$,

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define the stable pairs invariant

$$\widetilde{N}^{\bullet}_{n,\beta} = \int_{[P_n(V,\beta)]^{\mathrm{vir}}} 1.$$

The partition function is

$$\mathsf{Z}_{\mathsf{P}}(V;q)_\beta = \sum_n \widetilde{N}^{\bullet}_{n,\beta}\, q^n.$$

Since $P_n(V,\beta)$ is empty for sufficiently negative n, the partition function is a Laurent series in q. The following is a special case of [40, Conjecture 3.26].

CONJECTURE 7. *The partition function $\mathsf{Z}_{\mathsf{P}}(V;q)_\beta$ is the Laurent expansion of a rational function in q.*

If the total space V is a Calabi–Yau 3-fold, then Conjecture 7 has been proven in [6, 45]. In particular, Conjecture 7 holds for the anticanonical 3-fold

$$\mathsf{X} \subset \widetilde{\mathbb{P}^2 \times \mathbb{P}^1} \times \mathbb{P}^1$$

of Section 2.1.

In fact, if V is any complete intersection Calabi–Yau 3-fold in a toric variety which admits sufficient degenerations, Conjecture 7 has been proven in [39]. By factoring equations, there is no difficulty in constructing the degenerations of X into toric 3-folds required for [39]. Just as in the case of the quintic in $\mathbb{P}^4$, the geometries which occur are toric 3-folds, projective bundles over $K3$ and toric surfaces, and fibrations over curves. A complete discussion of the degeneration scheme for X is given in Appendix B.

3.3. GW/P correspondence for X.

Following the notation of Section 2.2, let $H_2(\mathsf{X},\mathbb{Z})^{\pi_3}$ denote the vertical classes of X and let

$$F^{\mathsf{X}} = \sum_{g\geqslant 0} \sum_{0\neq\gamma\in H_2(X,\mathbb{Z})^{\pi_3}} N^{\mathsf{X}}_{g,\gamma}\, \lambda^{2g-2} v^\gamma$$

be the potential of connected Gromov–Witten invariants. The partition function (of possibly disconnected) Gromov–Witten invariants is defined via the exponential,

$$\mathsf{Z}_{\mathsf{GW}}(\mathsf{X};\lambda) = \exp(F^{\mathsf{X}}).$$

Let $\mathsf{Z}_{\mathsf{GW}}(\mathsf{X};\lambda)_\gamma$ denote the coefficient of v^γ in $\mathsf{Z}_{\mathsf{GW}}(\mathsf{X};\lambda)$. The main result of [39] applied to X is the following GW/P correspondence for complete intersection Calabi–Yau 3-folds in products of projective spaces.

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GW/P correspondence. After the change of variable $-q = e^{i\lambda}$, we have

$$\mathsf{Z}_{\mathsf{GW}}(\mathsf{X};\lambda)_\gamma = \mathsf{Z}_{\mathsf{P}}(\mathsf{X};q)_\gamma.$$

The change of variables is well defined by the rationality of $\mathsf{Z}_{\mathsf{P}}(\mathsf{X};q)_\gamma$ of Conjecture 7. The GW/P correspondence is proven in [39] for *every* nonzero class in $H_2(\mathsf{X},\mathbb{Z})$, but we only will require here the statement for fiber classes γ.

3.4. $K3$ integrals. Let S be a nonsingular projective $K3$ surface with a nonzero class $\alpha \in \mathrm{Pic}(S)$ which is both effective and primitive. By the definitions of Section 0.3 in Gromov–Witten theory,

$$F_\alpha = \sum_{g\geqslant 0}\sum_{m>0} R_{g,m\alpha}\lambda^{2g-2}v^{m\alpha},$$

$$F_\alpha = \sum_{g\geqslant 0}\sum_{m>0} r_{g,m\alpha}\lambda^{2g-2}\sum_{d>0}\frac{1}{d}\left(\frac{\sin(d\lambda/2)}{\lambda/2}\right)^{2g-2} v^{dm\alpha}.$$

Via $K3$-fibrations over a pointed curve

$$\epsilon : T \to (\Delta, 0)$$

satisfying the conditions (i), (ii), and ($\star$) of Section 0.5, we have defined in (0.8) the series

$$\widetilde{F}_\alpha = \sum_{n\in\mathbb{Z}}\sum_{m>0} \widetilde{R}_{n,m\alpha}q^n v^{m\alpha}$$

in the theory of stable pairs. Using the identity

$$\begin{aligned}2^{2g-2}\sin(d\lambda/2)^{2g-2} &= \left(\frac{e^{id\lambda/2}-e^{-id\lambda/2}}{i}\right)^{2g-2}\\ &= (-1)^{g-1}((-q)^d-2+(-q)^{-d})^{g-1}\end{aligned}$$

under the change of variables $-q = e^{i\lambda}$, we define the stable pairs BPS invariants $\widetilde{r}_{g,m\alpha}$ by the relation

$$\widetilde{F}_\alpha = \sum_{g\in\mathbb{Z}}\sum_{m>0}\widetilde{r}_{g,m\alpha}\sum_{d>0}\frac{(-1)^{g-1}}{d}((-q)^d-2+(-q)^{-d})^{g-1}v^{dm\alpha}.$$

See [40, Section 3.4] for a discussion of such BPS expansions for stable pairs. The invariants $\widetilde{r}_{g,m\alpha}$ are integers.

Since $\widetilde{r}_{g,\beta}$ depends only upon the divisibility m and the norm square

$$\langle \beta, \beta \rangle = 2h - 2,$$

we will use, as before, the notation

$$\widetilde{r}_{g,m,h} = \widetilde{r}_{g,\beta}.$$

By definition in Gromov–Witten theory, $r_{g,m,h} = 0$ for $g < 0$. However, for fixed m and h, the definitions allow $r_{g,m,h}$ to be nonzero for all positive g. On the stable pairs side for fixed m and h, $\widetilde{r}_{g,m,h} = 0$ for sufficiently large g, but $\widetilde{r}_{g,m,h}$ may be nonzero for all negative g.

We will prove Theorem 4 by showing the BPS counts for $K3$ surfaces in Gromov–Witten theory and stable pairs theory exactly match:

$$r_{g,m,h} = \widetilde{r}_{g,m,h} \tag{3.1}$$

for all $g \in \mathbb{Z}$, $m \geqslant 1$, and $h \in \mathbb{Z}$.

3.5. Noether–Lefschetz theory for stable pairs. The stable pairs potential $\widetilde{F}^{\mathsf{X}}$ for nonzero vertical classes is the series

$$\widetilde{F}^{\mathsf{X}} = \log \left(1 + \sum_{0 \neq \gamma \in H_2(\mathsf{X},\mathbb{Z})^{\pi_3}} \mathsf{Z}_{\mathsf{P}}(\mathsf{X}; q)_\gamma v^\gamma \right),$$

where v is the curve class variable. The stable pairs BPS counts $\widetilde{n}^{\mathsf{X}}_{g,\gamma}$ are uniquely defined by

$$\widetilde{F}^{\mathsf{X}} = \sum_{g \in \mathbb{Z}} \sum_{0 \neq \gamma \in H_2(\mathsf{X},\mathbb{Z})^{\pi_3}} \widetilde{n}^{\mathsf{X}}_{g,\gamma} \sum_{d > 0} \frac{(-1)^{g-1}}{d} ((-q)^d - 2 + (-q)^{-d})^{g-1} v^{d\gamma},$$

following [**40**, Section 3.4].

The following stable pairs result is proven in Section 8. A central issue in the proof is the translation of the Noether–Lefschetz geometry of stable pairs to a precise relation constraining the *logarithm* $\widetilde{F}^{\mathsf{X}}$ of the stable pairs series.

THEOREM 8. *For degrees (d_1, d_2, d_3) positive with respect to the quasipolarization of the family $\pi_3 : \mathsf{X} \to \mathbb{P}^1$,*

$$\widetilde{n}^{\mathsf{X}}_{g,(d_1,d_2,d_3)} = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} \widetilde{r}_{g,m,h} \cdot NL^{\pi_3}_{m,h,(d_1,d_2,d_3)}.$$

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3.6. Proof of Theorem 4. We first match the BPS counts of X by using the GW/P correspondence. Then, the uniqueness statement of Proposition 6 implies (3.1).

PROPOSITION 9. *For all $g \in \mathbb{Z}$ and all $\gamma \in H_2(\mathsf{X}, \mathbb{Z})^{\pi_3}$, we have*

$$n^{\mathsf{X}}_{g,\gamma} = \widetilde{n}^{\mathsf{X}}_{g,\gamma}.$$

Proof. By Corollary 41 of Section 7, $\widetilde{r}_{g,m,h} = 0$ if $g < 0$. Theorem 8 then implies $\widetilde{n}^{\mathsf{X}}_{g,\gamma} = 0$ if $g < 0$. Hence, there are only finitely many nonzero BPS states for fixed γ since $\widetilde{n}^{\mathsf{X}}_{g,\gamma}$ vanishes for sufficiently large g by construction [**40**]. (The GW/P correspondence yields an equality of partition functions after the variable change $-q = e^{i\lambda}$ whether or not $\widetilde{n}_{g<0,\gamma}$ vanishes. Proposition 9 asserts a stronger result: the Gromov–Witten BPS expansion equals the stable pairs BPS expansion. Since these expansions are in opposite directions, finiteness is needed.) By the GW/P correspondence, the $\widetilde{n}^{\mathsf{X}}_{g,\gamma}$ then yield the Gromov–Witten BPS expansion. □

PROPOSITION 10. *For all $g \in \mathbb{Z}$, $m \geqslant 1$, and $h \in \mathbb{Z}$, we have*

$$r_{g,m,h} = \widetilde{r}_{g,m,h}.$$

Proof. The equality $r_{g,m,h} = \widetilde{r}_{g,m,h}$ holds in case $h \leqslant 0$ by following the argument of [**36**, Lemma 2] for stable pairs. (A different argument is given in Corollary 40 of Section 7.) For $h < 0$, the vanishing of $\widetilde{r}_{g,m,h}$ holds by the same geometric argument given in [**36**, Lemma 2]. The $h = 0$ case is the conifold for which the equality is well known (and a consequence of GW/P correspondence).

We view relation (2.3) and Theorem 8 as systems of linear equations for the unknowns $r_{g,m,h}$ and $\widetilde{r}_{g,m,h}$, respectively. By Proposition 9, we have

$$n^{X}_{g,\gamma} = \widetilde{n}^{X}_{g,\gamma}$$

for all g. Hence, the two systems of linear equations are the *same*.

We now apply the uniqueness established in Proposition 6. The initial conditions and the linear equations are identical. Therefore, the solutions must also agree. □

Theorem 4 follows immediately from Proposition 10 for the $K3$ invariants in Gromov–Witten theory and stable pairs. □

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4. $K3 \times \mathbb{C}$: Localization

4.1. Overview. We begin now our analysis of the moduli spaces of stable pairs related to $K3$ surfaces and $K3$-fibrations. Let S be a nonsingular projective $K3$ surface. We first study the trivial fibration

$$Y = S \times \mathbb{C} \longrightarrow \mathbb{C}$$

by $\mathbb{C}^*$-localization with respect to the scaling action on $\mathbb{C}$. Let $\mathfrak{t}$ denote the weight 1 representation of $\mathbb{C}^*$ on the tangent space to $\mathbb{C}$ at $0 \in \mathbb{C}$.

We compute here the $\mathbb{C}^*$-residue contribution to the reduced stable pairs theory of $S \times \mathbb{C}$ of the $\mathbb{C}^*$-fixed component parameterizing stable pairs supported on S and thickened uniformly k times about $0 \in \mathbb{C}$. (Throughout we use the term *component* to denote any open and closed subset. More formally, a component for us is a union of connected components in the standard sense.) In Section 5, all other $\mathbb{C}^*$-fixed components will be shown to have vanishing contributions to the virtual localization formula.

4.2. Uniformly thickened pairs. Define the following Artinian rings and schemes:

$$A_k = \mathbb{C}[x]/(x^k), \qquad B_k = \operatorname{Spec} A_k. \tag{4.1}$$

We have the obvious maps

$$\operatorname{Spec} \mathbb{C} \xleftarrow{\pi_k} B_k \xhookrightarrow{\iota_k} \mathbb{C} = \operatorname{Spec} \mathbb{C}[x].$$

For any variety Z, we define

$$Z_k = Z \times B_k,$$

and use the same symbols π_k, ι_k to denote the corresponding projections and inclusions,

$$Z \xleftarrow{\pi_k} Z_k \xhookrightarrow{\iota_k} Z \times \mathbb{C}. \tag{4.2}$$

We will often abbreviate ι_k to ι.

Let $\beta \in H_2(S, \mathbb{Z})$ be a curve class. Let $P_S = P_n(S, \beta)$ denote the moduli space of stable pairs on S with universal stable pair $(\mathbb{F}, s)$ and universal complex

$$\mathbb{I}^\bullet_S = \{\mathcal{O}_{S \times P_S} \xrightarrow{s} \mathbb{F}\}.$$

Using the maps (4.2) for $Z = S \times P_S$ (so $Z \times \mathbb{C} = Y \times P_S$) we define

$$\mathbb{F}_k = \pi_k^* \mathbb{F}, \quad \mathbb{I}^\bullet_{S_k} = \{\mathcal{O}_{S_k \times P_S} \xrightarrow{s_k} \mathbb{F}_k\} \tag{4.3}$$

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on $S_k \times P_S$, where $s_k = \pi_k^* s$. Pushing s_k forward to $Y \times P_S$ we obtain

$$\mathbb{I}_Y^\bullet = \{\mathcal{O}_{Y\times P_S} \xrightarrow{s_k} \iota_* \mathbb{F}_k\}. \tag{4.4}$$

Since we have constructed a flat family over P_S of stable pairs on Y of class $k\beta$ and holomorphic Euler characteristic kn, we obtain a classifying map from P_S to the moduli space of stable pairs on Y:

$$f: P_S = P_n(S,\beta) \longrightarrow P_{kn}(Y, k\beta) = P_Y. \tag{4.5}$$

(For any class $\gamma \in \mathrm{Pic}(S)$, we denote the push-forward to $H_2(Y,\mathbb{Z})$ also by γ.)

LEMMA 11. *The map* (4.5) *is an isomorphism onto an open and closed component of the* $\mathbb{C}^*$*-fixed locus of* P_Y.

Proof. Let P_f denote the open and closed component of $(P_Y)^{\mathbb{C}^*}$ containing the image of f. Certainly, f is a bijection on closed points onto P_f. There is a $\mathbb{C}^*$-fixed universal stable pair on $Y \times P_f$. We push down the universal stable pair to $S \times P_f$ and then take $\mathbb{C}^*$-invariant sections. The result is flat over P_f and hence classified by a map $P_f \to P_S$ which is easily seen to be the inverse map to f. □

4.3. Deformation theory of pairs. Let $P_Y = P_m(Y,\gamma)$ be the moduli space of stable pairs on Y of class $\gamma \in H_2(Y,\mathbb{Z})$ with holomorphic Euler characteristic m. There is a universal complex $\mathbb{I}_Y^\bullet$ over $Y \times P_Y$. We will soon take $m = kn$ and $\gamma = k\beta$, in which case $\mathbb{I}_Y^\bullet$ pulls back via the classifying map $f: P_S \to P_Y$ to (4.4).

We review here the basics of the deformation theory of stable pairs on the 3-fold Y [40]. Let

$$\pi_P: Y \times P_Y \to P_Y$$

be the projection, and define

$$E_Y^\bullet = (R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \mathbb{I}_Y^\bullet)_0)^\vee[-1] \cong R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \mathbb{I}_Y^\bullet \otimes \omega_{\pi_P})_0[2]. \tag{4.6}$$

Here $R\mathscr{H}om_{\pi_P} = R\pi_{P*}R\mathscr{H}om$, the subscript 0 denotes trace-free homomorphisms, and the isomorphism is Serre duality down π_P. (Although π_P is not proper, the compact support of $R\mathscr{H}om(\mathbb{I}_Y^\bullet, \mathbb{I}_Y^\bullet)_0$ ensures that Serre duality holds. This is proved in [37, Equations (15), (16)], for instance, by compactifying $Y = S \times \mathbb{C}$ to $Y = \mathbb{C} \times \mathbb{P}^1$.) Let $\mathbb{L}$ denote the truncated cotangent complex $\tau^{\geqslant -1}L^\bullet$. Using the Atiyah class of $\mathbb{I}_Y^\bullet$, we obtain a map [40, Section 2.3],

$$E_Y^\bullet \longrightarrow \mathbb{L}_{P_Y}, \tag{4.7}$$

exhibiting $E_Y^\bullet$ as a perfect obstruction theory for P_Y [20, Theorem 4.1].

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In fact, (4.7) is the natural obstruction theory of trivial-determinant objects $I^\bullet = \{\mathcal{O}_Y \xrightarrow{s} F\}$ of the derived category $D(Y)$. The more natural obstruction theory of pairs (F, s) is given by the complex

$$(R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \mathbb{F}_Y))^\vee \tag{4.8}$$

where $\mathbb{F}_Y$ is the universal sheaf. (This is essentially proved in [21] once combined with [3, Theorem 4.5]; see [22, Sections 12.3–12.5] for a full account.) However, (4.8) is *not* perfect in general. To define stable pair invariants, we must use (4.6). The two theories give the same tangent spaces, but different obstructions. On surfaces, however, the analogous obstruction theory

$$E_S^\bullet = (R\mathscr{H}om_{\pi_P}(\mathbb{I}_S^\bullet, \mathbb{F}))^\vee \tag{4.9}$$

is indeed perfect and is used to define invariants [29]. Here π_P denotes the projection $S \times P_S \to P_S$.

The following result describes the relationship between the above obstruction theories when pulled back via the map $f : P_S \to P_Y$ of (4.5).

PROPOSITION 12. *We have an isomorphism*

$$f^* E_Y^\bullet \cong E_S^\bullet \otimes A_k^* \ \oplus \ (E_S^\bullet)^\vee \otimes \mathfrak{t}^{-1} A_k[1],$$

where $A_k^* = 1 + \mathfrak{t} + \cdots + \mathfrak{t}^{k-1}$ and $\mathfrak{t}^{-1} A_k = \mathfrak{t}^{-1} + \mathfrak{t}^{-2} + \cdots + \mathfrak{t}^{-k}$. (We ignore here the ring structure (4.1) on A_k and considering A_k as just a vector space with $\mathbb{C}^*$-action. As such, $A_k^* \cong \mathfrak{t}^{k-1} A_k$, as we use below.)

Proof. We will need two preliminaries on pull-backs. First, over $S_k \times P_S$ there is a canonical exact triangle

$$\mathbb{F}_k \otimes N_k^*[1] \longrightarrow \iota^* \iota_* \mathbb{F}_k \longrightarrow \mathbb{F}_k, \tag{4.10}$$

where $\iota^* = L\iota_k^*$ is the *derived* pull-back functor, and

$$N_k \cong \mathcal{O}_{S_k \times P_S} \otimes \mathfrak{t}^k$$

denotes the normal bundle of $S_k \times P_S$ in $Y \times P_S$. Second, combine the first arrow of (4.10) with the obvious map $\mathcal{O}_{Y \times P_S} \to \iota_* \mathcal{O}_{S_k \times P_S}$:

$$\begin{aligned} R\mathscr{H}om(\iota_*\mathbb{F}_k, \mathcal{O}_{Y\times P_S}) &\longrightarrow R\mathscr{H}om(\iota_*\mathbb{F}_k, \iota_*\mathcal{O}_{S_k\times P_S}) \\ &\cong \iota_* R\mathscr{H}om(\iota^*\iota_*\mathbb{F}_k, \mathcal{O}_{S_k\times P_S}) \longrightarrow \iota_* R\mathscr{H}om(\mathbb{F}_k \otimes N_k^*[1], \mathcal{O}_{S_k\times P_S}). \end{aligned}$$

A local computation shows the above composition is an isomorphism:

$$R\mathscr{H}om(\iota_*\mathbb{F}_k, \mathcal{O}_{Y\times P_S}) \cong \iota_* R\mathscr{H}om(\mathbb{F}_k \otimes N_k^*[1], \mathcal{O}_{S_k\times P_S}). \tag{4.11}$$

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Now combine (4.10) with ι^* of the triangle

$$\mathbb{I}^\bullet_Y \to \mathcal{O}_{Y\times P_S} \to \iota_*\mathbb{F}_k \tag{4.12}$$

to give the following diagram of exact triangles on $S_k \times P_S$:

$$\begin{tikzcd} & \iota^*\mathbb{I}^\bullet_Y \arrow[d] & \\ & \mathcal{O}_{S_k\times P_S} \arrow[r, equal] \arrow[d] & \mathcal{O}_{S_k\times P_S} \arrow[d] \\ \mathbb{F}_k \otimes N_k^*[1] \arrow[r] & \iota^*\iota_*\mathbb{F}_k \arrow[r] & \mathbb{F}_k. \end{tikzcd} \tag{4.13}$$

The right hand column defines the complex $\mathbb{I}^\bullet_{S_k}$ (4.3), so by the octahedral axiom we can fill in the top row with the exact triangle

$$\mathbb{F}_k \otimes N_k^* \longrightarrow \iota^*\mathbb{I}^\bullet_Y \longrightarrow \mathbb{I}^\bullet_{S_k}. \tag{4.14}$$

Again letting π_P denote both projections

$$S_k \times P_S \to P_S \quad \text{and} \quad Y \times P_S \to P_S,$$

we apply $R\mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet_Y, \cdot)$ to $\mathbb{I}^\bullet_Y \to \mathcal{O}_{Y\times P_S} \to \iota_*\mathbb{F}_k$ to give the triangle

$$R\mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet_Y, \iota_*\mathbb{F}_k) \longrightarrow R\mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet_Y, \mathbb{I}^\bullet_Y)[1] \longrightarrow R\mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet_Y, \mathcal{O})[1] \tag{4.15}$$

relating the obstruction theory (4.8) to the obstruction theory (4.6) (without its trace part removed: we will deal with this presently).

Now use the following obvious diagram of exact triangles on $Y \times P_S$:

$$\begin{tikzcd} \mathbb{I}^\bullet_Y \arrow[r] \arrow[d] & \mathcal{O}_{Y\times P_S} \arrow[r] \arrow[d] & \iota_*\mathbb{F}_k \arrow[d, equal] \\ \iota_*\mathbb{I}^\bullet_{S_k} \arrow[r] & \iota_*\mathcal{O}_{S_k\times P_S} \arrow[r] & \iota_*\mathbb{F}_k. \end{tikzcd}$$

This maps the triangle (4.15) to the triangle

$$R\mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet_Y, \iota_*\mathbb{F}_k) \longrightarrow R\mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet_Y, \iota_*\mathbb{I}^\bullet_{S_k})[1] \longrightarrow R\mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet_Y, \iota_*\mathcal{O})[1].$$

By adjunction this is

$$R\mathscr{H}om_{\pi_P}(\iota^*\mathbb{I}^\bullet_Y, \mathbb{F}_k) \longrightarrow R\mathscr{H}om_{\pi_P}(\iota^*\mathbb{I}^\bullet_Y, \mathbb{I}^\bullet_{S_k})[1] \longrightarrow R\mathscr{H}om_{\pi_P}(\iota^*\mathbb{I}^\bullet_Y, \mathcal{O})[1],$$

which in turn maps to

$$R\mathscr{H}om_{\pi_P}(\mathbb{F}_k, \mathbb{F}_k)\mathfrak{t}^k \longrightarrow R\mathscr{H}om_{\pi_P}(\mathbb{F}_k, \mathbb{I}^\bullet_{S_k})\mathfrak{t}^k[1] \longrightarrow R\mathscr{H}om_{\pi_P}(\mathbb{F}_k, \mathcal{O})\mathfrak{t}^k[1] \tag{4.16}$$

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by the first arrow $\mathbb{F}_k \otimes N_k^* \to \iota^*\mathbb{I}_Y^\bullet$ of (4.14). Notice that since $\mathbb{I}_{S_k}^\bullet$ and $\mathbb{F}_k$ are the pull-backs of $\mathbb{I}_S^\bullet$ and $\mathbb{F}$ by $\pi_k\colon S_k \to S$, the central term simplifies to

$$R\mathscr{H}om_{\pi_P}(\mathbb{F}, \mathbb{I}_S^\bullet) \otimes \mathfrak{t}^k A_k[1] \cong E_S^\bullet \otimes \mathfrak{t}^k A_k[-1],$$

where $E_S^\bullet$ is the obstruction theory (4.9).

We next remove the trace component of (4.15) using the diagram

$$\begin{tikzcd} & R\pi_{P*}\mathcal{O}_{Y\times P_S}[1] \arrow[r, equal] \arrow[d, "\mathrm{id}"] & R\pi_{P*}\mathcal{O}_{Y\times P_S}[1] \arrow[d] \\ R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \iota_*\mathbb{F}_k) \arrow[r] & R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \mathbb{I}_Y^\bullet)[1] \arrow[r] \arrow[d] & R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \mathcal{O})[1] \arrow[d] \\ & R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \mathbb{I}_Y^\bullet)_0[1] \arrow[r] & R\mathscr{H}om_{\pi_P}(\iota_*\mathbb{F}_k, \mathcal{O})[2]. \end{tikzcd} \tag{4.17}$$

Here the right hand column is given by applying $R\mathscr{H}om_{\pi_P}(\cdot\ ,\mathcal{O})$ to (4.12). The top right hand corner commutes because the composition $\mathcal{O} \xrightarrow{\mathrm{id}} \mathscr{H}om(\mathbb{I}_Y^\bullet, \mathbb{I}_Y^\bullet) \to \mathscr{H}om(\mathbb{I}_Y^\bullet, \mathcal{O})$ takes 1 to the canonical map $\mathbb{I}_Y^\bullet \to \mathcal{O}$. Therefore, the whole diagram commutes.

The central row of (4.17) is (4.15), and our map from (4.15) to (4.16) kills the top row of (4.17) by $\mathbb{C}^*$-equivariance: $R\pi_{P*}\mathcal{O}_{Y\times P_S}[1]$ has $\mathbb{C}^*$-weights in $(-\infty, 0]$ while (4.16) has weights in $[1, k]$. Therefore, it descends to a map from the bottom row of (4.17) (completed using the octahedral axiom) to (4.16). The upshot is the following map of triangles

$$\begin{tikzcd} R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \iota_*\mathbb{F}_k) \arrow[r] \arrow[d] & f^*(E_Y^\bullet)^\vee \arrow[r] \arrow[d] & R\mathscr{H}om_{\pi_P}(\iota_*\mathbb{F}_k, \mathcal{O})[2] \arrow[d] \\ R\mathscr{H}om_{\pi_P}(\mathbb{F}_k, \mathbb{F}_k)\mathfrak{t}^k \arrow[r] & E_S^\bullet \otimes \mathfrak{t}^k A_k[-1] \arrow[r] & R\mathscr{H}om_{\pi_P}(\mathbb{F}_k, \mathcal{O})\mathfrak{t}^k[1]. \end{tikzcd} \tag{4.18}$$

Recall that the first column was induced from the triangle (4.14), so sits inside a triangle

$$R\mathscr{H}om_{\pi_P}(\mathbb{I}_{S_k}^\bullet, \mathbb{F}_k) \longrightarrow R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \iota_*\mathbb{F}_k) \longrightarrow R\mathscr{H}om_{\pi_P}(\mathbb{F}_k, \mathbb{F}_k)\mathfrak{t}^k.$$

Again we can simplify because $\mathbb{I}_{S_k}^\bullet$ and $\mathbb{F}_k$ are the pull-backs of $\mathbb{I}_S^\bullet$ and $\mathbb{F}$ by $\pi_k\colon S_k \to S$. That is,

$$R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \iota_*\mathbb{F}_k) \cong R\mathscr{H}om_{\pi_P}(\mathbb{I}_S^\bullet, \mathbb{F}) \otimes A_k \oplus R\mathscr{H}om_{\pi_P}(\mathbb{F}, \mathbb{F}) \otimes \mathfrak{t}^k A_k,$$

where the splitting follows from the $\mathbb{C}^*$-invariance of the connecting homomorphism: it must vanish because A_k has weights in $[-(k-1), 0]$ while $\mathfrak{t}^k A_k$ has weights in $[1, k]$.

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So this splits the first vertical arrow of (4.18); we claim the last vertical arrow is the isomorphism induced by (4.11). Altogether this gives the splitting

$$f^*(E_Y^\bullet)^\vee \cong R\mathscr{H}om_{\pi_P}(\mathbb{I}_S^\bullet, \mathbb{F}) \otimes A_k \oplus E_S^\bullet[-1] \otimes \mathfrak{t}^k A_k.$$

Dualizing gives

$$f^* E_Y^\bullet \cong E_S^\bullet \otimes A_k^* \oplus (E_S^\bullet)^\vee[1] \otimes \mathfrak{t}^{-1} A_k,$$

as required.

It remains to prove the claim that the third vertical arrow of (4.18) is induced by (4.11). By the construction of these maps, it is sufficient to prove the commutativity of the diagram

$$\begin{tikzcd}
R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \mathcal{O}_{Y\times P_S}) \arrow[r] \arrow[d, "\partial^*"'] & R\mathscr{H}om_{\pi_P}(\mathbb{F}_k \otimes N_k^*, \mathcal{O}_{S_k\times P_S}) \\
R\mathscr{H}om_{\pi_P}(\iota_*\mathbb{F}_k[-1], \mathcal{O}_{Y\times P_S}). \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Here the vertical arrow is induced by the connecting homomorphism ∂ of the standard triangle $\mathbb{I}_Y^\bullet \to \mathcal{O}_{Y\times P_S} \to \iota_*\mathbb{F}_k$, and the diagonal arrow is $R\pi_{P*}$ applied to (4.11).

The horizontal arrow is our map from (4.15) to (4.16) (restricted to the right hand term in each triangle). It is therefore the composition

$$\begin{aligned}
R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \mathcal{O}_{Y\times P_S}) &\longrightarrow R\mathscr{H}om_{\pi_P}(\mathbb{I}_Y^\bullet, \iota_*\mathcal{O}_{S_k\times P_S}) \\
&\cong R\mathscr{H}om_{\pi_P}(\iota^*\mathbb{I}_Y^\bullet, \mathcal{O}_{S_k\times P_S}) \xrightarrow{(4.14)} R\mathscr{H}om_{\pi_P}(\mathbb{F}_k \otimes N_k^*, \mathcal{O}_{S_k\times P_S}).
\end{aligned}$$

Via $\partial : \iota_*\mathbb{F}_k[-1] \to \mathbb{I}_Y^\bullet$, the above composition maps to the composition

$$\begin{aligned}
R\mathscr{H}om_{\pi_P}(\iota_*\mathbb{F}_k[-1], \mathcal{O}_{Y\times P_S}) &\longrightarrow R\mathscr{H}om_{\pi_P}(\iota_*\mathbb{F}_k[-1], \iota_*\mathcal{O}_{S_k\times P_S}) \\
&\cong R\mathscr{H}om_{\pi_P}(\iota^*\iota_*\mathbb{F}_k[-1], \mathcal{O}_{S_k\times P_S}) \xrightarrow{(4.10)} R\mathscr{H}om_{\pi_P}(\mathbb{F}_k \otimes N_k^*, \mathcal{O}_{S_k\times P_S}).
\end{aligned}$$

Therefore, the first two resulting squares commute. The last square is:

$$\begin{tikzcd}
R\mathscr{H}om_{\pi_P}(\iota^*\mathbb{I}_Y^\bullet, \mathcal{O}_{S_k\times P_S}) \arrow[r, "(4.14)"] \arrow[d, "\iota^*\partial^*"] & R\mathscr{H}om_{\pi_P}(\mathbb{F}_k \otimes N_k^*, \mathcal{O}_{S_k\times P_S}) \arrow[d, equal] \\
R\mathscr{H}om_{\pi_P}(\iota^*\iota_*\mathbb{F}_k[-1], \mathcal{O}_{S_k\times P_S}) \arrow[r, "(4.10)"] & R\mathscr{H}om_{\pi_P}(\mathbb{F}_k \otimes N_k^*, \mathcal{O}_{S_k\times P_S}).
\end{tikzcd}$$

By the construction of (4.14) from (4.10), the square commutes. $\square$

A virtual class on $P_{kn}(Y, k\beta)^{\mathbb{C}^*}$ induced by $(E_Y^\bullet)^{\mathrm{fix}}$ is defined in [18]. The moduli space $P_n(S, \beta)$ hence carries several virtual classes:

(i) via the intrinsic obstruction theory $E_S^\bullet$;

(ii) via $(E_Y^\bullet)^{\mathrm{fix}}$ and the local isomorphism

$$f : P_n(S, \beta) \hookrightarrow P_{kn}(Y, k\beta)^{\mathbb{C}^*}$$

for every $k \geqslant 1$.

PROPOSITION 13. *The virtual classes on $P_n(S, \beta)$ obtained from* (i) *and* (ii) *are all equal.*

Proof. By Proposition 12, there is an isomorphism,

$$f^*(E_Y^\bullet)^{\mathrm{fix}} \cong E_S^\bullet, \tag{4.19}$$

in the derived category. Since the virtual class is expressed in terms of the Fulton total Chern class of $P_n(S, \beta)$ and the Segre class of the dual of the obstruction theory, the isomorphism (4.19) implies the equality of the virtual classes. (The relationship of the virtual class with the Fulton total Chern class and the normal cone is reviewed in Appendix C.1.) □

In fact, Proposition 13 is trivial: the virtual classes of $P_n(S, \beta)$ obtained from the obstruction theories $E_S^\bullet$ and $(E_Y^\bullet)^{\mathrm{fix}}$ both vanish by the existence of the *reduced* theory.

The reduced obstruction theory for Y is constructed in, for instance, [**37**, Section 3]. We review the construction in a slightly more general setting in Section 6.6. For the reduced theory of S, we can either $\mathbb{C}^*$-localize the 3-fold reduced class, or equivalently, use the construction in [**29**]. In particular, [**29**, Proposition 3.4] shows the two constructions are compatible under the isomorphism (4.5). They both remove a trivial piece $\mathcal{O}_P[-1]$ (of $\mathbb{C}^*$-weight 0!) from the obstruction theory. The nontrivial version of Proposition 13 is the following.

PROPOSITION 14. *The reduced virtual classes on $P_n(S, \beta)$ obtained from* (i) *and* (ii) *are all equal.*

The proof of Proposition 14 is exactly the same as the proof of Proposition 13 given above.

4.4. Localization calculation I. We can now evaluate the residue contribution of the locus of k-times uniformly thickened stable pairs

$$P_n(S, \beta) \subset P_{kn}(Y, k\beta)^{\mathbb{C}^*}$$

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of (4.5) to the $\mathbb{C}^*$-equivariant integral

$$\int_{[P_{kn}(Y,k\beta)]^{\mathrm{red}}_{\mathbb{C}^*}} 1.$$

We will see in Section 5 that the contributions of all other $\mathbb{C}^*$-fixed loci to the virtual localization formula vanish.

By Proposition 14, the reduced virtual class on $P_{kn}(Y, k\beta)^{\mathbb{C}^*}$ obtained from $(E_Y^\bullet)^{\mathrm{fix}}$ matches the reduced virtual class of $P_n(S, \beta)$ obtained from the obstruction theory $E_S^\bullet$. The virtual normal bundles are the same for the reduced and standard obstruction theories (since the semiregularity map is $\mathbb{C}^*$-invariant here).

Writing A_k as $\mathbb{C} \oplus \mathfrak{t}^{-1} A_{k-1}$, we can read off the virtual normal bundle to $P_n(S, \beta) \subset P_{kn}(Y, k\beta)$ from Proposition 12:

$$N^{\mathrm{vir}} = (E_S^\bullet)^\vee \otimes \mathfrak{t}^{-1} A_{k-1} \oplus E_S^\bullet \otimes \mathfrak{t} A_k^*[-1].$$

After writing $\mathfrak{t}A_k^*$ as $\mathfrak{t}A_{k-1}^* \oplus \mathfrak{t}^k$, the residue contribution of $P_n(S, \beta)$ to the $\mathbb{C}^*$-equivariant integral $\int_{[P_{kn}(Y,k\beta)]^{\mathrm{red}}_{\mathbb{C}^*}} 1$ is

$$\int_{[P_n(S,\beta)]^{\mathrm{red}}} \frac{1}{e(N^{\mathrm{vir}})} = \int_{[P_n(S,\beta)]^{\mathrm{red}}} \frac{e(E_S^\bullet \otimes \mathfrak{t}A_{k-1}^*)}{e((E_S^\bullet \otimes \mathfrak{t}A_{k-1}^*)^\vee)}\, e(E_S^\bullet \otimes \mathfrak{t}^k).$$

The rank of $E_S^\bullet$ is the virtual dimension

$$\langle \beta, \beta \rangle + n = 2h_\beta - 2 + n$$

of $P_n(S, \beta)$ before reduction. Therefore, the rank of tensor product $E_S^\bullet \otimes \mathfrak{t}A_{k-1}^*$ is $(k-1)(2h_\beta - 2 + n)$, and the quotient in the integrand is

$$(-1)^{(k-1)(2h_\beta-2+n)} = (-1)^{(k-1)n}.$$

Let t denote the $\mathbb{C}^*$-equivariant first Chern class of the representation $\mathfrak{t}$. We have proven the following result.

PROPOSITION 15. *The residue contribution of*

$$P_n(S, \beta) \subset P_{kn}(Y, k\beta)^{\mathbb{C}^*}$$

to the integral $\int_{[P_{kn}(Y,k\beta)]^{\mathrm{red}}_{\mathbb{C}^*}} 1$ *is:*

$$\begin{aligned}\int_{[P_n(S,\beta)]^{\mathrm{red}}} \frac{1}{e(N^{\mathrm{vir}})} &= (-1)^{(k-1)n} \int_{[P_n(S,\beta)]^{\mathrm{red}}} e(E_S^\bullet \otimes \mathfrak{t}^k) \\ &= \frac{(-1)^{(k-1)n}}{kt} \int_{[P_n(S,\beta)]^{\mathrm{red}}} c_{\langle \beta,\beta\rangle+n+1}(E_S^\bullet).\end{aligned}$$

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4.5. Dependence. Let S be a $K3$ surface equipped with an ample primitive polarization L. Let $\beta \in \mathrm{Pic}(S)$ be a positive class with respect to L,

$$\langle L, \beta \rangle > 0.$$

If β is nonzero and effective, β must be positive. The integral

$$\int_{[P_n(S,\beta)]^{\mathrm{red}}} c_{\langle \beta,\beta\rangle+n+1}(E_S^\bullet) \tag{4.20}$$

is deformation invariant as (S, β) varies so long as β remains an algebraic class. Hence, the integral depends *only* upon n, the divisibility of β, and $\langle \beta, \beta \rangle$.

If β is effective, then $H^2(S, \beta) = 0$; otherwise $-\beta$ would also be effective by Serre duality. Hence, by the results of [**30**], the integral (4.20) depends *only* upon n and

$$\langle \beta, \beta \rangle = 2h_\beta - 2$$

in the effective case.

If β is effective and $h_\beta < 0$, then the integral vanishes since the virtual number of sections of β is negative. A proof is given below in Section 4.6 following [**29**, **30**]. Finally, if $h_\beta \geqslant 0$, then β must be effective (since β is positive) by Riemann–Roch.

If β is not effective, then the integral (4.20) vanishes. In the ineffective case, $h_\beta < 0$ must hold. The discussion of cases is summarized by the following result, whose final statement will be proved in Proposition 17 below.

PROPOSITION 16. *For a positive class $\beta \in \mathrm{Pic}(S)$, the integral*

$$\int_{[P_n(S,\beta)]^{\mathrm{red}}} c_{\langle \beta,\beta\rangle+n+1}(E_S^\bullet) \tag{4.21}$$

depends only upon n and h_β. Moreover, if $h_\beta < 0$, the integral vanishes.

4.6. Vanishing. Let S be a $K3$ surface with an effective curve class $\beta \in \mathrm{Pic}(S)$ satisfying

$$\beta^2 = 2h - 2 \quad \text{with } h < 0.$$

Let $\mathbb{P}_\beta$ be the linear system of all curves of class β. Since h is the reduced virtual dimension of the moduli space

$$P_{1-h}(S, \beta) = \mathbb{P}_\beta,$$

the corresponding virtual cycle vanishes,

$$[P_{1-h}(S, \beta)]^{\mathrm{red}} = 0. \tag{4.22}$$

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We would like to conclude

$$[P_{1-h+k}(S,\beta)]^{\mathrm{red}}=0 \tag{4.23}$$

for all k.

If $k<0$, then $P_{1-h+k}(S,\beta)$ is empty, so (4.23) certainly holds. If $k>0$, the moduli space $P_{1-h+k}(S,\beta)$ fibers over $P_{1-h}(S,\beta)$:

$$P_{1-h+k}(S,\beta)\cong \mathrm{Hilb}^k(\mathcal{C}/\mathbb{P}_\beta)\xrightarrow{\pi}\mathbb{P}_\beta\cong P_{1-h}(S,\beta),$$

where $\mathcal{C}\to\mathbb{P}_\beta$ is the universal curve. By [1] and [29, Footnote 22], the projection π is flat of relative dimension n. Therefore, the vanishing (4.23) follows from (4.22) and Proposition 17 below. (The result was implicit in [29] but never actually stated there, so we provide a proof. The result holds more generally for any surface S and class $\beta\in H^2(S,\mathbb{Z})$ for which $H^2(L)=0$ whenever $c_1(L)=\beta$.) Since π is flat, pull-back is well defined on algebraic cycles. Also, as we have noted, β effective implies $H^2(S,\beta)=0$.

PROPOSITION 17. *For $H^2(S,\beta)=0$ and $k\geqslant 0$, we have*

$$[P_{1-h+k}(S,\beta)]^{\mathrm{red}}=\pi^*[P_{1-h}(S,\beta)]^{\mathrm{red}}.$$

Proof. In [29, Appendix A], the moduli space $P_{1-h+k}(S,\beta)$ is described by equations as follows. (Since $H^1(S,\mathcal{O}_S)=0$ the description here is simpler.) Let A be a sufficiently ample divisor on S. The inclusion,

$$\mathbb{P}_\beta\subset\mathbb{P}_{\beta+A},$$

is described as the zero locus of a section of a vector bundle E. Next,

$$\mathrm{Hilb}^k(\mathcal{C}/\mathbb{P}_\beta)\subset\mathbb{P}_\beta\times\mathrm{Hilb}^k S \tag{4.24}$$

is described as the zero locus of a section of a bundle F which extends to $\mathbb{P}_{\beta+A}\times\mathrm{Hilb}^k S$.

Let $\mathcal{A}$ denote the nonsingular ambient space $\mathbb{P}_{\beta+A}\times\mathrm{Hilb}^k S$ which contains our moduli space

$$P=P_{1-h+k}(S,\beta)\cong\mathrm{Hilb}^k(\mathcal{C}/\mathbb{P}_\beta).$$

The above description then defines a natural virtual cycle on P via

$$[P]^{\mathrm{red}}=[s(P\subset\mathcal{A})c(\pi^*E)c(F)\cap[P]]_{h+k}, \tag{4.25}$$

the refined top Chern class of $\pi^*E\oplus F$ on P. Here,

$$h+k=\dim\mathcal{A}-\mathrm{rank}\,E-\mathrm{rank}\,F$$

is the virtual dimension of the construction. By the main result of [29, Appendix A], the class (4.25) is, as the notation suggests, equal to the reduced virtual cycle of P.

By [1] and [29, Footnote 22], the section of F cutting out (4.24) is in fact regular. Hence, the resulting normal cone

$$C_{P\subset\mathbb{P}_\beta\times\mathrm{Hilb}^k S}\cong F$$

is locally free and isomorphic to F. We have the following exact sequence of cones:

$$0\longrightarrow C_{P\subset\mathbb{P}_\beta\times\mathrm{Hilb}^k S}\longrightarrow C_{P\subset\mathcal{A}}\longrightarrow \pi^*C_{\mathbb{P}_\beta\subset\mathbb{P}_{\beta+A}}\longrightarrow 0.$$

(Since $H^1(S,\mathcal{O}_S)=0$ and the Hilbert scheme of curves is just the nonsingular linear system $\mathbb{P}_\beta$, all three terms are locally free. In general only the first is.) After substitution in (4.25), we obtain

$$\begin{aligned}[P]^{\mathrm{red}} &= \left[s(F)\pi^*s\left(\mathbb{P}_\beta\subset\mathbb{P}_{\beta+A}\right)\pi^*c(E)c(F)\cap\pi^*\left[\mathbb{P}_\beta\right]\right]_{h+n}\\ &=\pi^*\left(\left[s\left(\mathbb{P}_\beta\subset\mathbb{P}_{\beta+A}\right)c(E)\cap\left[\mathbb{P}_\beta\right]\right]_h\right)\\ &=\pi^*\left[\mathbb{P}_\beta\right]^{\mathrm{red}}. \quad \square\end{aligned}$$

5. $K3\times\mathbb{C}$: vanishing

5.1. Overview. We will show the components of $(P_Y)^{\mathbb{C}^*}$ which do not correspond to the thickenings studied in Section 4 do *not* contribute to the localization formula.

Recall first the proof of the vanishing of the ordinary (nonreduced) $\mathbb{C}^*$-localized invariants of

$$Y=S\times\mathbb{C}.$$

Translation along the $\mathbb{C}$-direction in Y induces a vector field on P_Y which has $\mathbb{C}^*$-weight 1. By the symmetry of the obstruction theory, such translation induces a $\mathbb{C}^*$-weight 0 cosection: a surjection from the obstruction sheaf Ω_{P_Y} to $\mathcal{O}_{P_Y}$. Since the cosection is $\mathbb{C}^*$-fixed, it descends to a cosection for the $\mathbb{C}^*$-fixed obstruction theory on $(P_Y)^{\mathbb{C}^*}$, forcing the virtual cycle to vanish [26].

To apply the above strategy to the *reduced* obstruction theory, we need to find *another* weight 1 vector field on the moduli space. We will describe such a vector field which is proportional to the original translational vector field along $(P_Y)^{\mathbb{C}^*}\subset P_Y$ precisely on the components of uniformly thickened stable pairs of Section 4. On the other components of $(P_Y)^{\mathbb{C}^*}$, the linear independence of the two weight 1 vector fields forces the reduced localized invariants to vanish.

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5.2. Basic model. Our new vector field will again arise from an action in the $\mathbb{C}$-direction on Y, pulling apart a stable pair supported on a thickening of the central fiber $S \times \{0\}$. The S-direction plays little role, so we start by explaining the basic model on $\mathbb{C}$ itself. For clarity, we will here use the notation

$$\mathbb{C}_x = \operatorname{Spec} \mathbb{C}[x],$$

where the subscript denotes the corresponding parameter. The space $\mathbb{C}_x$ carries its usual $\mathbb{C}^*$-action, with the coordinate function x having weight -1.

Consider the k-times thickened origin $B_k \subset \mathbb{C}_x$. We wish to fix $B_{k-1} \subset B_k$ and move the remaining point away through $\mathbb{C}_x$ at unit speed. In other words, we consider the flat family of subschemes of $\mathbb{C}_x$ parameterized by $t \in \mathbb{C}_t$ given by

$$Z = \{x^{k-1}(t-x) = 0\} \subset \mathbb{C}_x \times \mathbb{C}_t. \tag{5.1}$$

Specializing to $t = 0$ indeed gives the subscheme $\{x^k = 0\} = B_k$, while for $t \neq 0$ we have $\{x^{k-1} = 0\} \sqcup \{x = t\}$.

5.3. Extension class. Consider $\mathcal{O}_Z$ as a flat family of sheaves over $\mathbb{C}_t$ defining a deformation of fiber $\mathcal{O}_{B_k}$ over $t = 0$. After restriction to $\operatorname{Spec} \mathbb{C}[t]/(t^2)$, we obtain a first order deformation $\mathcal{O}_Z/(t^2)$ of the sheaf

$$\mathcal{O}_Z/(t) = \mathcal{O}_{B_k}.$$

Such deformations are classified by an element e of the group

$$\operatorname{Ext}^1_{\mathbb{C}_x}(\mathcal{O}_{B_k}, \mathcal{O}_{B_k}) \tag{5.2}$$

described as follows. The exact sequence

$$0 \longrightarrow \mathcal{O}_Z/(t) \xrightarrow{t} \mathcal{O}_Z/(t^2) \longrightarrow \mathcal{O}_Z/(t) \longrightarrow 0,$$

is isomorphic to

$$0 \longrightarrow \mathcal{O}_{B_k} \xrightarrow{t} \frac{\mathbb{C}[x,t]}{(t^2, x^{k-1}(x-t))} \longrightarrow \mathcal{O}_{B_k} \longrightarrow 0. \tag{5.3}$$

Considering (5.3) as a sequence of $\mathbb{C}[x]$-modules (pushing it down by $\pi_2 : \mathbb{C}_x \times \operatorname{Spec} \mathbb{C}[t]/(t^2) \to \mathbb{C}_x$) an extension class e in (5.2) is determined.

Using the resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}_x}(-B_k) \xrightarrow{x^k} \mathcal{O}_{\mathbb{C}_x} \longrightarrow \mathcal{O}_{B_k} \longrightarrow 0$$

of $\mathcal{O}_{B_k}$ we compute (5.2) as

$$\begin{aligned} \operatorname{Ext}^1_{\mathbb{C}_x}(\mathcal{O}_{B_k}, \mathcal{O}_{B_k}) &\cong \operatorname{Hom}(\mathcal{O}_{\mathbb{C}_x}(-B_k), \mathcal{O}_{B_k}) \\ &\cong H^0(\mathcal{O}_{B_k}(B_k)) \cong H^0(\mathcal{O}_{B_k}) \otimes \mathfrak{t}^k. \end{aligned} \tag{5.4}$$

PROPOSITION 18. *The class $e \in \mathrm{Ext}^1_{\mathbb{C}_x}(\mathcal{O}_{B_k}, \mathcal{O}_{B_k})$ of the first order deformation $\mathcal{O}_Z/(t^2)$ of $\mathcal{O}_{B_k}$ is*

$$x^{k-1} \otimes \mathfrak{t}^k \in H^0(\mathcal{O}_{B_k}) \otimes \mathfrak{t}^k.$$

It coincides (to first order) with the deformation given by moving $\mathcal{O}_{B_k}$ by the translational vector field $(1/k)\partial_x$. In particular, e has $\mathbb{C}^*$-weight 1.

Proof. Consider the generator of $\mathrm{Hom}(\mathcal{O}_{\mathbb{C}_x}(-B_k), \mathcal{O}_{B_k})$ multiplied by x^{k-1}. From the description (5.4), it corresponds to the extension E coming from the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{C}_x}(-B_k) \arrow[r, "x^k"] \arrow[d, "x^{k-1}"'] & \mathcal{O}_{\mathbb{C}_x} \arrow[r] \arrow[d] & \mathcal{O}_{B_k} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{O}_{B_k} \arrow[r] & E \arrow[r] & \mathcal{O}_{B_k} \arrow[r] & 0. \end{tikzcd} \tag{5.5}$$

On the other hand, e is defined by the push-down to $\mathbb{C}_x$ of the extension (5.3). The latter sits inside the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}[x] \arrow[r, "x^k"] \arrow[d, "x^{k-1}"'] & \mathbb{C}[x] \arrow[r] \arrow[d, "\pi^*"] & \mathbb{C}[x]/(x^k) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{C}[x]/(x^k) \arrow[r, "t"] & \mathbb{C}[x,t]/(t^2, x^{k-1}(x-t)) \arrow[r] & \mathbb{C}[x]/(x^k) \arrow[r] & 0. \end{tikzcd}$$

Here the central vertical arrow π^* takes a polynomial in x to the same polynomial in x (with no t-dependence). This is indeed a map of $\mathbb{C}[x]$-modules (though not $\mathbb{C}[x,t]$-modules) and makes the left hand square commute since $x^k = x^{k-1}t$ in the ring

$$\mathcal{O}_Z/(t^2) = \mathbb{C}[x,t]/(t^2, x^{k-1}(x-t)).$$

Since the second diagram is isomorphic to the first diagram (5.5), we find e is indeed $x^{k-1} \otimes \mathfrak{t}^k$.

Next we observe that moving $\mathcal{O}_{B_k}$ by the translation vector field $(1/k)\partial_x$ yields the structure sheaf of the different family

$$\{(x - t/k)^k = 0\} \subset \mathbb{C}_x \times \mathbb{C}_t.$$

Restricting to Spec $\mathbb{C}[t]/(t^2)$ gives the first order deformation

$$\frac{\mathbb{C}[x,t]}{(t^2, (x-t/k)^k)} = \frac{\mathbb{C}[x,t]}{(t^2, x^k - tx^{k-1})},$$

the same as $\mathcal{O}_Z/(t^2)$.

Finally, the $\mathbb{C}^*$-weight of $(1/k)\partial_x$ is clearly 1. More directly, $x^{k-1} \otimes \mathfrak{t}^k$ has weight $-(k-1) + k = 1$. □

The conceptual reason for the surprising result of Proposition 18 is that, *to first order*, weight 1 deformations only see the corresponding deformation of the center of mass of the subscheme. The two deformations in Proposition 18 clearly deform the center of mass in the same way.

We will next apply a version of the above deformation to $\mathbb{C}^*$-fixed stable pairs. The first order part of the deformation will describe a weight 1 vector field on P_Y along $(P_Y)^{\mathbb{C}^*} \subset P_Y$ and thus a $\mathbb{C}^*$-invariant cosection of the obstruction theory.

On the stable pairs which are *uniformly* thickened as in Section 4.2, Proposition 18 will show the new vector field to be proportional to the standard vector field given by the translation ∂_x. Thus, our cosection is proportional to the cosection we have already reduced by, and provides us nothing new. Hence, the nonzero contributions of Section 4.4 are permitted.

For nonuniformly thickened stable pairs, however, the new vector field will be seen to be linearly independent of the translational vector field.

5.4. Full model. The basic model of Section 5.2 gives a deformation of the $\mathbb{C}[x]$-modules $A_k = \mathbb{C}[x]/(x^k)$. We now extend this to describe a deformation of any $\mathbb{C}^*$-equivariant torsion $\mathbb{C}[x]$-module M which is a (possibly infinite) direct sum of finite-dimensional $\mathbb{C}^*$-equivariant $\mathbb{C}[x]$-modules. By the classification of modules over PIDs, M is a direct sum of $\mathfrak{t}^j$-twists of the standard modules $A_k = \mathbb{C}[x]/(x^k)$. Since these were treated in Section 5.2, the extension is a simple matter. However, by describing our deformations intrinsically, we will be able to apply the construction to $\mathbb{C}^*$-fixed stable pairs (F, s) on $S \times \mathbb{C}_x$. Let

$$U \subset S$$

be an affine open set. Then, $F|_{U\times\mathbb{C}_x}$ is equivalent to a $\mathbb{C}^*$-equivariant torsion $\mathbb{C}[x]$-module carrying an action of the ring $\mathcal{O}(U)$. The model developed here will sheafify over S and determines a deformation of (F, s).

Since the sheaf $F|_{U\times\mathbb{C}_x}$ has only finitely many weights (all nonpositive), we restrict attention to torsion $\mathbb{C}^*$-equivariant $\mathbb{C}[x]$-modules M with weights lying in the interval $[-(k-1), 0]$ for some $k \geqslant 1$. Examples include A_k and $A_j\mathfrak{t}^{-(k-j)}$ for $j \leqslant k$. Write

$$M = \bigoplus_{i=0}^{k-1} M_i$$

as a sum of weight spaces, where M_i has weight $-i$. Multiplication by x is encoded in the weight (-1) operators

$$X\colon M_i \longrightarrow M_{i+1}.$$

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Since X annihilates the most negative weight space, $M_{k-1} \subset M$ is an equivariant $\mathbb{C}[x]$-submodule. We will define a deformation which moves M_{k-1} away at unit speed while leaving the remaining M/M_{k-1} fixed.

To do so, notice that the basic model (5.1) of Section 5.2 can be described as follows. Take the direct sum of the structure sheaves of the two irreducible components of Z (or, equivalently, the structure sheaf $\mathcal{O}_{\overline{Z}}$ of the normalization of Z), then $\mathcal{O}_Z$ is the subsheaf of sections which agree over the intersection $\Delta(B_{k-1})$ of the two components:

$$\mathcal{O}_Z = \ker\left(\pi^*\mathcal{O}_{B_{k-1}} \oplus \Delta_*\mathcal{O} \xrightarrow{(r,-r)} \Delta_*\mathcal{O}_{B_{k-1}}\right). \tag{5.6}$$

Here, $\Delta_*\mathcal{O}_{B_{k-1}} = \Delta_* A_{k-1}$ and r denotes restriction to $\Delta(B_{k-1})$. Finally

$$\pi : \mathbb{C}_x \times \mathbb{C}_t \longrightarrow \mathbb{C}_x,$$
$$\Delta : \mathbb{C} \hookrightarrow \mathbb{C}_x \times \mathbb{C}_t$$

are the projection and the inclusion of the diagonal, respectively.

We define the $\mathbb{C}^*$-equivariant $\mathbb{C}[x,t]$-module $\widetilde{M}$ to be the kernel of the map

$$\begin{aligned}\pi^*(M/M_{k-1}) \oplus \Delta_*(M_{k-1}\mathfrak{t}^{k-1} \otimes_{\mathbb{C}} \mathbb{C}[x])&\\ \xrightarrow{(\psi\circ r,-r)} \Delta_*(M_{k-1}\mathfrak{t}^{k-1} \otimes_{\mathbb{C}} A_{k-1}),&\end{aligned} \tag{5.7}$$

where ψ is the map

$$M/M_{k-1} = \bigoplus_{i=0}^{k-2} M_i \xrightarrow{\bigoplus_i X^{k-1-i}\mathfrak{t}^{k-1}\otimes x^i} M_{k-1}\mathfrak{t}^{k-1} \otimes_{\mathbb{C}} A_{k-1}.$$

By construction this is a weight 0 map of equivariant $\mathbb{C}[x]$-modules.

By splitting M into direct sums of irreducible modules $A_n\mathfrak{t}^m$, comparing with (5.6), and using Proposition 18, we obtain the following result.

PROPOSITION 19. *The sheaf $\widetilde{M}$ defined by (5.7) is flat over $\mathbb{C}_t$ and specializes to $\widetilde{M}/t\widetilde{M} = M$ over $t = 0$. The first order deformation*

$$e \in \mathrm{Ext}^1_{\mathbb{C}_x}(M, M) \quad \text{classifying } \widetilde{M}/t^2\widetilde{M}$$

is proportional to the first order translation deformation ∂_x on any irreducible module M with weights in $[-(k-1), 0]$ as follows:

- *For $M = A_k = \mathcal{O}_{B_k}$ we have $e = \partial_x/k$.*
- *For $M = A_j\mathfrak{t}^{-(k-j)} = \mathcal{O}_{B_j}\mathfrak{t}^{-(k-j)}$ with $j \leqslant k$ we have $e = \partial_x/j$.*
- *For M with $M_{k-1} = 0$ we have $e = 0$.* □

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Proposition 19 is the foundation of our localization to *uniformly thickened* stable pairs. The above deformation applied to $\mathbb{C}^*$-fixed stable pairs will describe a weight 1 vector field on P_Y along $(P_Y)^{\mathbb{C}^*} \subset P_Y$, and thus a $\mathbb{C}^*$-invariant cosection of the obstruction theory.

For any stable pair which is not uniformly thickened, the new vector field acts as a translation which operates at different speeds along different parts of the stable pair. The corresponding cosection is therefore linearly independent of the pure translation ∂_x and descends to give a nowhere vanishing cosection of the reduced obstruction theory.

5.5. Second cosection.

Fix a component of $(P_Y)^{\mathbb{C}^*}$ over which the $\mathbb{C}^*$-fixed stable pairs are k-times thickened, supported on

$$(P_Y)^{\mathbb{C}^*} \times S \times B_n \subset (P_Y)^{\mathbb{C}^*} \times S \times \mathbb{C}_x$$

for $n = k$ but not for any $n < k$.

We now apply the results of Section 5.4 to the universal sheaf

$$\mathbb{F} \quad \text{on } (P_Y)^{\mathbb{C}^*} \times S \times \mathbb{C}_x$$

to produce a flat deformation over $(P_Y)^{\mathbb{C}^*} \times S \times \mathbb{C}_x \times \mathbb{C}_t$ by the formula (5.7). The universal section s of $\mathbb{F}$ also deforms to the $\mathbb{C}^*$-invariant section

$$([s], X^{k-1}s.\mathfrak{t}^{k-1} \otimes 1)$$

of (5.7). Restricting to $\operatorname{Spec}\mathbb{C}[t]/(t^2)$ defines a P_Y tangent vector field along $(P_Y)^{\mathbb{C}^*}$ of weight 1:

$$v \in \mathfrak{t}^* \otimes H^0\big((P_Y)^{\mathbb{C}^*}, \mathscr{E}xt^1_{\pi_P}(\mathbb{I}^\bullet_Y, \mathbb{I}^\bullet_Y)_0\big).$$

From v and the isomorphism $\omega_Y \cong \mathfrak{t}^{-1}$, we construct a weight 0 cosection over $(P_Y)^{\mathbb{C}^*}$:

$$\mathscr{E}xt^2_{\pi_P}(\mathbb{I}^\bullet_Y, \mathbb{I}^\bullet_Y)_0 \xrightarrow{\cup v} \mathscr{E}xt^3_{\pi_P}(\mathbb{I}^\bullet_Y, \mathbb{I}^\bullet_Y) \otimes \mathfrak{t}^{-1} \xrightarrow{\mathrm{tr}} \mathcal{O}_{(P_Y)^{\mathbb{C}^*}},$$

where the final arrow is dual to the identity

$$\mathcal{O}_{(P_Y)^{\mathbb{C}^*}} \to \mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet_Y, \mathbb{I}^\bullet_Y)$$

under the Serre duality of (4.6). Combined with our standard cosection, we obtain a map

$$\mathscr{E}xt^2_{\pi_P}(\mathbb{I}^\bullet_Y, \mathbb{I}^\bullet_Y)_0 \xrightarrow{\mathrm{tr}(\,\cdot\,\cup v) \oplus \mathrm{tr}(\,\cdot\,\cup(\partial_x \lrcorner \operatorname{At}(\mathbb{I}^\bullet_Y)))} \mathcal{O}^{\oplus 2}_{(P_Y)^{\mathbb{C}^*}}. \tag{5.8}$$

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PROPOSITION 20. *The map (5.8) is surjective over $(P_Y)^{\mathbb{C}^*}$ away from stable pairs which are uniformly thickened as in Section 4.2.*

Proof. By the Nakayama Lemma, we can check surjectivity at closed points (F, s). By Serre duality, we need only show that if our two elements of the Zariski tangent space to P_Y at (F, s),

$$v,\ \partial_x \lrcorner\ \mathrm{At}(I_Y^\bullet) \in \mathrm{Ext}^1(I_Y^\bullet, I_Y^\bullet)_0,$$

are linearly dependent then (F, s) is uniformly thickened.

Pick an affine open set $U \subset S$ and consider the restriction of (F, s) to $U \times \mathbb{C}_x$ as a $\mathbb{C}^*$-equivariant $\mathbb{C}[x]$-module. Since F is k-times thickened and pure, the support C of F is also k-times thickened over the open set where $\mathcal{O}_C \xrightarrow{s} F$ is an isomorphism. In particular, the equivariant $\mathbb{C}[x]$-module $F|_{U\times\mathbb{C}_x}$ contains copies of A_k as summands. By Proposition 18, the deformation v is the same as ∂_x/k on A_k.

Therefore, if v and ∂_x are linearly dependent at (F, s) then in fact v must equal ∂_x/k at (F, s). In particular, by Proposition 19 all of the irreducible equivariant $\mathbb{C}[x]$-submodule summands of $F|_{U\times\mathbb{C}_x}$ are isomorphic to A_k (for any U). Therefore, we get an isomorphism

$$F \xrightarrow{\sim} F_{k-1}\mathfrak{t}^{k-1} \otimes_{\mathbb{C}} A_k \tag{5.9}$$

by the map defined in terms of the weight space decomposition of F as

$$\bigoplus_{i=0}^{k-1} F_i \xrightarrow{\ x^{k-1-i}\mathfrak{t}^{k-1}\otimes x^i\ } F_{k-1}\mathfrak{t}^{k-1} \otimes_{\mathbb{C}} A_k.$$

The isomorphism (5.9) implies F is uniformly thickened.

Finally, the $\mathbb{C}^*$-invariant section maps 1 to a degree 0 element of the module $F|_{U\times\mathbb{C}_x} \cong F|_{U\times\{0\}} \otimes_{\mathbb{C}} A_k$. Such elements are of the form $f \otimes 1$. Hence, the elements are pulled back from S to $S \times B_k$, and the pair (F, s) is uniformly thickened. □

COROLLARY 21. *The invariants calculated in Section 4.4 are the only nonzero contributions to the reduced localized invariants of $Y = S \times \mathbb{C}$.*

Proof. We work on a component of $(P_Y)^{\mathbb{C}^*}$ parameterizing pairs which are *not* uniformly thickened. (By $\mathbb{C}^*$-invariance, one pair in a component is uniformly thickened if and only if they all are.)

Since (5.8) is $\mathbb{C}^*$-invariant it factors through the $\mathbb{C}^*$-fixed part of the obstruction sheaf, which by [18] is the obstruction sheaf of the induced perfect

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obstruction theory on $(P_Y)^{\mathbb{C}^*}$. The *reduced* obstruction sheaf is given by taking the kernel of the first factor of this map:

$$\mathrm{Ob}^{\mathrm{red}}_{(P_Y)^{\mathbb{C}^*}} = \ker\left(\mathscr{E}xt^2_{\pi_P}(\mathbb{I}^\bullet_Y, \mathbb{I}^\bullet_Y)_0^{\mathbb{C}^*} \xrightarrow{\mathrm{tr}(\,\cdot\,\cup(\partial_x \lrcorner\, \mathrm{At}(\mathbb{I}^\bullet_Y)))} \mathcal{O}_{(P_Y)^{\mathbb{C}^*}}\right).$$

Proposition 20 then states that (5.8) gives a surjection

$$\mathrm{Ob}^{\mathrm{red}}_{(P_Y)^{\mathbb{C}^*}} \xrightarrow{\mathrm{tr}(\,\cdot\,\cup v)} \mathcal{O}_{(P_Y)^{\mathbb{C}^*}}.$$

Therefore, the reduced class vanishes. □

5.6. Localization calculation II. The results of Sections 4 and 5 together yield a complete localization calculation.

Let S be a $K3$ surface equipped with an ample primitive polarization L. Let $\alpha \in \mathrm{Pic}(S)$ be a primitive and positive class. (No further conditions are placed on α: the Picard rank of S may be high and α may be the sum of effective classes.) Define

$$\langle 1\rangle^{\mathrm{red}}_{Y,m\alpha} = \sum_n y^n \int_{[P_n(Y,m\alpha)]^{\mathrm{red}}_{\mathbb{C}^*}} 1. \tag{5.10}$$

The integral on the right side of (5.10), denoting the $\mathbb{C}^*$-residue, is well defined since the $\mathbb{C}^*$-fixed loci of $P_n(Y, m\alpha)$ are compact. Since the *reduced* virtual dimension of $P_n(Y, m\alpha)$ is 1, the residues are of degree -1 in t (see Proposition 15),

$$\langle 1\rangle^{\mathrm{red}}_{Y,m\alpha} \in \frac{1}{t}\mathbb{Q}((y)).$$

PROPOSITION 22. *Let $\alpha \in \mathrm{Pic}(S)$ be a primitive and positive class. Then $\langle 1\rangle^{\mathrm{red}}_{Y,\alpha}$ depends only upon*

$$\langle \alpha, \alpha\rangle = 2h - 2.$$

Moreover, if $h < 0$, then $\langle 1\rangle^{\mathrm{red}}_{Y,\alpha} = 0$.

Proof. By the vanishing of Corollary 21 and the residue formula of Section 4.4, we have

$$\langle 1\rangle^{\mathrm{red}}_{Y,\alpha} = \frac{1}{t}\sum_n y^n \int_{[P_n(S,\alpha)]^{\mathrm{red}}} c_{\langle\alpha,\alpha\rangle+n+1}(E^\bullet_S). \tag{5.11}$$

By Proposition 16, the integral over $[P_n(S,\alpha)]^{\mathrm{red}}$ occurring in (5.11) depends only upon n and $\langle\alpha,\alpha\rangle$ and vanishes if $h < 0$. □

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To isolate the dependence of Proposition 22, we define, for primitive and positive α,

$$I_h = \langle 1\rangle^{\mathrm{red}}_{Y,\alpha}, \quad \langle \alpha, \alpha\rangle = 2h-2. \tag{5.12}$$

In case α is *also* irreducible (which we may assume), the moduli space $P_n(S, \alpha)$ is nonsingular [**25**, **41**]. The evaluation of (5.11) reduces to the Euler characteristic calculation of Kawai and Yoshioka [**25**] as explained in the [**41**, Appendix C] and reviewed in Section 5.7 below.

PROPOSITION 23. *The following multiple cover formula holds:*

$$\langle 1\rangle^{\mathrm{red}}_{Y,m\alpha} = \sum_{k|m} \frac{1}{k} I_{(m^2/k^2)(h-1)+1}(-(-y)^k).$$

Proof. The result follows from the vanishing of Corollary 21, the residue formula of Section 4.4, the dependence result of Proposition 16, and the definition (5.12). $\square$

5.7. Kawai–Yoshioka evaluation. Let $P_n(S, h)$ denote the nonsingular moduli space of stable pairs for an irreducible class α satisfying

$$2h-2 = \langle \alpha, \alpha\rangle.$$

The cotangent bundle Ω_P of the moduli space $P_n(S, h)$ is the obstruction bundle of the reduced theory. Since the dimension of $P_n(S, h)$ is $2h-2+n+1$,

$$I_h(y) = \frac{1}{t}\sum_n (-1)^{2h-1+n} e(P_n(S,h))y^n.$$

The topological Euler characteristics of $P_n(S, h)$ have been calculated by Kawai–Yoshioka. By [**25**, Theorem 5.80],

$$\sum_{h=0}^{\infty}\sum_{n=1-h}^{\infty} e(P_n(S,h))y^n q^h$$
$$= \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^{-2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}.$$

For our pairs invariants, we require the signed Euler characteristics,

$$\sum_{h=0}^{\infty} I_h(y)q^h = \frac{1}{t}\sum_{h=0}^{\infty}\sum_{n=1-h}^{\infty}(-1)^{2h-1+n} e(P_n(S,h))y^n q^h.$$

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Therefore, $\sum_{h=0}^{\infty} tI_h(y)q^h$ equals

$$-\left(\sqrt{-y}-\frac{1}{\sqrt{-y}}\right)^{-2}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{20}(1+yq^n)^2(1+y^{-1}q^n)^2}.$$

The above formula implies $tI_h(y)$ is the Laurent expansion of a rational function of y.

6. Relative theory and the logarithm

6.1. Overview. Our goal here is to define and study the analogue $\widetilde{R}_{n,\beta}$ for stable pairs of the Gromov–Witten integrals $R_{g,\beta}$ associated to $K3$ surfaces. Though the definition is via the stable pairs theory of $K3$-fibrations, the main idea is to move the integration to the rubber of an associated relative geometry. The interplay with various rubber theories allows for a geometric interpretation of the logarithm occurring in the definition of $\widetilde{R}_{n,\beta}$.

6.2. Definition. Let S be a $K3$ surface equipped with an ample primitive polarization L. Let $\alpha \in \mathrm{Pic}(S)$ be a primitive, positive class not proportional to L with norm square

$$\langle \alpha, \alpha \rangle = 2h - 2.$$

(Positivity, $\langle L, \alpha \rangle > 0$, is with respect to the polarization L.) Let $m > 0$ be an integer. By replacing L with $\widehat{L} = xL + \alpha$ for large x, we can assume $\widehat{L}$ is ample and primitive, α is positive with respect to $\widehat{L}$, and the inequality

$$\langle \widehat{L}, \widehat{L} \rangle > m \langle \widehat{L}, \alpha \rangle \tag{6.1}$$

holds. Condition (6.1) forbids effective summands of $m\alpha$ to be multiples of $\widehat{L}$.

Let $(S, \widehat{L}) \in \mathcal{M}$ denote the corresponding moduli point in the associated moduli space of polarized $K3$ surfaces. (We require $\widehat{L}$ to be ample, so $\mathcal{M}$ is an open set of the moduli of quasipolarized $K3$ surfaces considered in Section 1.) Since no effective summand γ of $m\alpha$ is a multiple of $\widehat{L}$, every such summand corresponds to a nondegenerate local Noether–Lefschetz locus $\mathrm{NL}(\gamma)$ near $(S, \widehat{L})$ of codimension 1. Let

$$\Delta \subset \mathcal{M}$$

be a quasiprojective curve passing through $(S, \widehat{L})$ and transverse to the local Noether–Lefschetz loci corresponding to all the (finitely many) effective summands of $m\alpha$.

Associated to Δ is a 3-fold X fibered in polarized $K3$ surfaces,

$$\epsilon : X \to (\Delta, 0). \tag{6.2}$$

We summarize the conditions we have as follows:

(i) Δ is a nonsingular quasiprojective curve;

(ii) ϵ is smooth, projective, and $\epsilon^{-1}(0) \stackrel{\sim}{=} S$;

($\star$) for every *effective* decomposition

$$m\alpha = \sum_{i=1}^{l} \gamma_i \in \mathrm{Pic}(S),$$

the local Noether–Lefschetz locus $\mathrm{NL}(\gamma_i) \subset \Delta$ corresponding to each class $\gamma_i \in \mathrm{Pic}(S)$ is the *reduced* point $0 \in \Delta$.

The class $\alpha \in \mathrm{Pic}(S)$ is m-rigid with respect to the $K3$-fibration ϵ.

In Section 0.6, m-rigidity was defined for effective α. The above definition is for positive α. Since effective implies positive, the definition here extends the definition of Section 0.6.

At the special fiber $\epsilon^{-1}(0) \cong S$, the Kodaira–Spencer class

$$\kappa \in H^1(T_S) \tag{6.3}$$

associated to ϵ is the extension class of the exact sequence

$$0 \longrightarrow T_S \longrightarrow T_X|_S \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

After fixing a holomorphic symplectic form $\sigma \in H^0(\Omega_S^2)$, we obtain the $(1, 1)$ class

$$\kappa \lrcorner \sigma \in H^1(\Omega_S).$$

The transversality of Δ to the local Noether–Lefschetz locus corresponding to the class $\gamma \in \mathrm{Pic}(S)$ is equivalent to the condition

$$\int_\gamma \kappa \lrcorner \sigma \neq 0. \tag{6.4}$$

Let $\mathrm{Eff}(m\alpha) \subset \mathrm{Pic}(S)$ denote the subset of effective summands of $m\alpha$. By property ($\star$), there is a compact, open, and closed component

$$P_n^\star(X, \gamma) \subset P_n(X, \gamma)$$

parameterizing stable pairs supported set-theoretically over the point $0 \in \Delta$ for every effective summand $\gamma \in \mathrm{Eff}(m\alpha)$.

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DEFINITION. Let $\alpha \in \mathrm{Pic}(S)$ be a primitive, positive class. Given a family $\epsilon : X \to (\Delta, 0)$ satisfying conditions (i), (ii), and ($\star$) for $m\alpha$, let

$$\sum_{n\in\mathbb{Z}} \widetilde{R}_{n,m\alpha}(S)q^n = \mathrm{Coeff}_{v^{m\alpha}}\left[\log\left(1+\sum_{n\in\mathbb{Z}}\sum_{\gamma\in\mathrm{Eff}(m\alpha)} q^n v^\gamma \int_{[P_n^\star(X,\gamma)]^{\mathrm{vir}}} 1\right)\right]. \tag{6.5}$$

An immediate geometric consequence of the above definition is the following vanishing statement: *if* $m\alpha \in \mathrm{Pic}(S)$ *is not effective, then* $\widetilde{R}_{n,m\alpha}(S) = 0$ *for all* n.

The main result of our study here will be a geometric interpretation of the logarithm on the right. As a consequence, we will see that $\widetilde{R}_{n,m\alpha}(S)$ depends *only* upon n, m, and $\langle\alpha,\alpha\rangle$ and *not* upon S nor the family ϵ. We therefore drop S from the notation. Also, $\widetilde{R}_{n,m\alpha}$ is well defined for all m by the existence of m-rigid families ϵ for suitable $\widehat{L}$ (as we have constructed).

The integrals over $P_n^\star(X, m\alpha)$ appearing on the right side of (6.5) play a central role,

$$P_{n,\gamma}^\star(X) = \int_{[P_n^\star(X,\gamma)]^{\mathrm{vir}}} 1, \qquad \gamma \in \mathrm{Eff}(m\alpha).$$

6.3. Relative moduli spaces. Let $\alpha \in \mathrm{Pic}(S)$ be a primitive class. Let ϵ be a family of polarized $K3$ surfaces

$$\epsilon : X \to (\Delta, 0)$$

for which α is positive and m-rigid. We will consider the relative geometry associated to

$$X/S = X/\epsilon^{-1}(0).$$

Let $\beta \in \mathrm{Eff}(m\alpha) \subset \mathrm{Pic}(S)$. We recall the definition [35, 40] of the moduli space $P_n(X/S, \beta)$ parameterizing stable relative pairs

$$\mathcal{O}_{X[k]} \xrightarrow{s} F \tag{6.6}$$

on k-step degenerations $X[k]$ of X along S [33]. ($X[k]$ is the union of X with a chain of $k \geqslant 0$ copies of $S \times \mathbb{P}^1$, where the ith copy of $S \times \mathbb{P}^1$ is attached along $S \times \{\infty\}$ to the $(i+1)$st along $S \times \{0\}$. Contracting the chain to $S \subset X$ defines a projection $X[k] \to X$ with automorphism group $(\mathbb{C}^*)^k$. There is a distinguished divisor $S_\infty \subset X[k]$ at $S \times \{\infty\}$ in the extremal component. If $k = 0$, then S_∞ is just $S \subset X$.) Here, F is a sheaf on $X[k]$ with

$$\chi(F) = n$$

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and support $[F]$ which pushes down to the class $\beta \in H_2(X, \mathbb{Z})$. The pair (F, s) satisfies the following stability conditions:

(i) F is pure with finite locally free resolution;

(ii) the higher derived functors of the restriction of F to the singular loci of $X[k]$, and the divisor at infinity, vanish;

(iii) the section s has 0-dimensional cokernel supported away from the singular loci of $X[k]$ and away from S_∞;

(iv) the pair (6.6) has only finitely many automorphisms covering the automorphisms of $X[k]/X$.

The moduli space $P_n(X/S, \beta)$ is a Deligne–Mumford stack with a perfect obstruction theory which we describe in Section 6.5.

In our situation, β is a fiber class and the nearby fibers $X_{t\neq 0}$ contain no curves of class β (by the transversality condition of ϵ). Hence, there is a compact, open and closed substack

$$P_n^\star(X/S, \beta) \subset P_n(X/S, \beta)$$

parameterizing stable pairs (F, s) lying over the central fiber. By condition (ii) of stability, the target must be bubbled, with (F, s) living on some $X[k]$ with $k \geqslant 1$. Restricted to the ith bubble $S \times \mathbb{P}^1$, (F, s) determines a stable pair

$$(F_i, s_i) \quad \text{disjoint from } S \times \{0, \infty\}, \tag{6.7}$$

with invariants

$$\chi(F_i) = n_i, \quad [F_i] = \beta_i \in \mathrm{Eff}(m\alpha) \subset H_2(S, \mathbb{Z}).$$

(For the class of the supports $[F_i]$, we always push down to S.) Since F is a disjoint union of the F_i,

$$\sum_{i=1}^{k} n_i = n, \quad \sum_{i=1}^{k} \beta_i = \beta. \tag{6.8}$$

Let $\mathscr{B}$ denote the stack of (n, β)-*marked* expanded degenerations [33, 35] of X/S, with universal family

$$\mathcal{X} \to \mathscr{B}.$$

(To emphasize the marking, we will sometimes denote $\mathscr{B}$ by $\mathscr{B}_{n,\beta}$.) Over a closed point of $\mathscr{B}$ with stabilizer $(\mathbb{C}^*)^k$, the fiber of $\mathcal{X}$ is the scheme $X[k]$ acted on by $(\mathbb{C}^*)^k$, covering the identity on X, with markings

$$(n_0, \beta_0) \in \mathbb{Z} \oplus H_2(X, \mathbb{Z}), \quad (n_i, \beta_i) \in \mathbb{Z} \oplus H_2(S, \mathbb{Z}),$$

satisfying

$$n_0 + \sum_{i=1}^{k} n_i = n, \qquad \beta_0 + \sum_{i=1}^{k} \beta_i = \beta. \tag{6.9}$$

All the stable pairs parameterized by $P_n^\star(X/S, \beta)$ lie over the substack where $(n_0, \beta_0) = (0, 0)$. By (6.9), we see the (n_i, β_i) are required to satisfy (6.8).

We view $P_n^\star(X/S, \beta)$ as a moduli space of stable pairs on the fibers of $\mathcal{X} \to \mathscr{B}$ with a map

$$P_n^\star(X/S, \beta) \to \mathscr{B}$$

taking a pair to the marked support.

6.4. Rubber. The universal family over the divisor of $\mathscr{B}$ corresponding to a nontrivial degeneration of X over S with

$$(n_0, \beta_0) = (0, 0)$$

is called *rubber*.

Alternatively, rubber geometry arises from the following construction. Consider the stack $\mathscr{B}_{0,\infty}$ of (n, β)-marked expanded degenerations of

$$S \times \mathbb{P}^1 / S \times \{0, \infty\}. \tag{6.10}$$

The markings (n_i, β_i) on the components are required to satisfy (6.8). Let

$$\mathscr{B}_\infty \subset \mathscr{B}_{0,\infty}$$

be the open substack where $S \times \{0\}$ has not been bubbled. The standard $\mathbb{C}^*$-action on $\mathbb{P}^1$ induces a $\mathbb{C}^*$-action on $\mathscr{B}_\infty$ and the associated universal family. Quotienting by the $\mathbb{C}^*$-action yields the *rubber target*: the universal family

$$\mathcal{S} \longrightarrow \mathscr{B}_r \tag{6.11}$$

over the *rubber stack*

$$\mathscr{B}_r = \mathscr{B}_\infty / \mathbb{C}^*.$$

The universal family $\mathcal{S}$ carries the canonical divisors

$$S_0 \times \mathscr{B}_r, \qquad S_\infty \times \mathscr{B}_r \subset \mathcal{S}.$$

Gluing S_0 of the rubber target to the central fiber S of X embeds the rubber stack into the stack of (n, β)-marked expanded degenerations of X. We obtain

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the commutative diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathscr{B}_r \arrow[r, hook] & \mathscr{B}. \end{tikzcd} \tag{6.12}$$

Let $R(n, \beta)$ denote the moduli space of stable pairs on the fibers of (6.11). Concretely, $R(n, \beta)$ is the moduli space of relative stable pairs on

$$S \times \mathbb{P}^1 / S \times \{0, \infty\}$$

with Euler characteristic n, class β, and no bubbling over $S \times \{0\}$ – all modulo the action of $\mathbb{C}^*$. The compactness of $R(n, \beta)$ is a consequence of the $\mathbb{C}^*$-quotient geometry.

We have seen that relative stable pairs on X/S near $0 \in \Delta$ are in fact supported on the rubber target (6.12). Pushing forward from rubber to the expanded degenerations of X/S yields a morphism

$$\iota : R(n, \beta) \to P_n^\star(X/S, \beta) \tag{6.13}$$

which is a closed embedding of Deligne–Mumford stacks *and* a bijection on closed points. The equation which cuts out $R(n, \beta) \subset P_n^\star(X/S, \beta)$ is the smoothing parameter of the first bubble.

We will prove ι is *almost* an isomorphism: ι satisfies the curvilinear lifting property. To prove ι is an isomorphism, the smoothing parameter of the first bubble must be shown to vanish in all flat families associated to the moduli space $P_n^\star(X/S, \beta)$. We leave the isomorphism question open.

6.5. Deformation theory. Following Section 6.1, let

$$\epsilon : X \to (\Delta, 0)$$

be a $K3$-fibration for which α is m-rigid. Let $\beta \in \mathrm{Eff}(m\alpha)$. We study here the deformation theory of

$$P = P_n^\star(X/S, \beta) \longrightarrow \mathscr{B},$$

the moduli space of stable pairs on the fibers of the right hand side of the diagram (6.12). Identical arguments apply to the left hand side of (6.12), replacing $\mathscr{B}$ by the substack $\mathscr{B}_r$ and P by $R(n, \beta)$ to give the deformation theory of stable pairs on the rubber target.

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Over $\mathcal{X} \times_{\mathscr{B}} P$ there is a universal stable pair

$$\mathbb{I}^\bullet = \{\mathcal{O}_{\mathcal{X}\times_{\mathscr{B}} P} \xrightarrow{s} \mathbb{F}\},$$

where the complex $\mathbb{I}^\bullet$ has $\mathcal{O}$ in degree 0. Let π_P denote the projection $\mathcal{X} \times_{\mathscr{B}} P \to P$ and

$$E^\bullet = (R\mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1])^\vee. \tag{6.14}$$

(Here $R\mathscr{H}om_{\pi_P} = R\pi_{P*}R\mathscr{H}om$ is the right derived functor of $\mathscr{H}om_{\pi_P} = \pi_{P*}\mathscr{H}om$.) By [20, 35], $P \to \mathscr{B}$ admits a relative perfect obstruction theory

$$E^\bullet \longrightarrow \mathbb{L}_{P/\mathscr{B}} \tag{6.15}$$

described as follows. Under the map $\mathbb{L}_{\mathcal{X}\times_{\mathscr{B}} P} \to \mathbb{L}_{(\mathcal{X}\times_{\mathscr{B}} P)/\mathscr{B}}$, the Atiyah class $\mathrm{At}(\mathbb{I}^\bullet)$ of $\mathbb{I}^\bullet$ projects to the *relative* Atiyah class:

$$\begin{array}{ccc} \mathrm{Ext}^1\left(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \mathbb{L}_{\mathcal{X}\times_{\mathscr{B}} P}\right) & \longrightarrow & \mathrm{Ext}^1\left(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \left(\mathbb{L}_{\mathcal{X}/\mathscr{B}} \oplus \mathbb{L}_{P/\mathscr{B}}\right)\right), \\ \mathrm{At}(\mathbb{I}^\bullet) & \longmapsto & (\mathrm{At}_{\mathcal{X}/\mathscr{B}}(\mathbb{I}^\bullet), \mathrm{At}_{P/\mathscr{B}}(\mathbb{I}^\bullet)). \end{array} \tag{6.16}$$

The map (6.15) is given by the partial Atiyah class $\mathrm{At}_{P/\mathscr{B}}(\mathbb{I}^\bullet)$ via the following identifications:

$$\begin{aligned} \mathrm{Ext}^1(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \pi_P^*\mathbb{L}_{P/\mathscr{B}}) &= H^1\left(R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet) \otimes \pi_P^*\mathbb{L}_{P/\mathscr{B}}\right) \\ &= H^0\left(R\pi_{P*}R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)[1] \otimes \mathbb{L}_{P/\mathscr{B}}\right) \\ &\longrightarrow H^0\left(R\pi_{P*}R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1] \otimes \mathbb{L}_{P/\mathscr{B}}\right) \\ &= \mathrm{Hom}\left(E^\bullet, \mathbb{L}_{P/\mathscr{B}}\right). \end{aligned} \tag{6.17}$$

Defining $\mathcal{E}^\bullet$ to be the cone on the induced map $E^\bullet[-1] \to \mathbb{L}_{\mathscr{B}}$, we obtain a commutative diagram of exact triangles:

$$\begin{tikzcd} \mathcal{E}^\bullet \arrow[r] \arrow[d] & E^\bullet \arrow[r] \arrow[d] & \mathbb{L}_{\mathscr{B}}[1] \arrow[d, equal] \\ \mathbb{L}_P \arrow[r] & \mathbb{L}_{P/\mathscr{B}} \arrow[r] & \mathbb{L}_{\mathscr{B}}[1]. \end{tikzcd} \tag{6.18}$$

The Artin stack $\mathscr{B}$ is smooth with $\mathbb{L}_{\mathscr{B}}$ a 2-term complex of bundles supported in degrees 0 and 1. The induced map

$$\Omega_P = h^0(E^\bullet) \longrightarrow h^1(\mathbb{L}_{\mathscr{B}})$$

is onto because the points of P all have finite stabilizers by condition (iv) above. Therefore, the long exact sequence of sheaf cohomologies of the top row of

(6.18) shows $\mathcal{E}^\bullet$ has cohomology in degrees -1 and 0 only. The complex $\mathcal{E}^\bullet$ is perfect because $E^\bullet$ and $\mathbb{L}_{\mathscr{B}}$ are. Since the Deligne–Mumford stack P is projective, $\mathcal{E}^\bullet$ is quasiisomorphic to a 2-term complex of locally free sheaves on P. Finally, the 5-Lemma applied to the long exact sequences in cohomology of (6.18) implies

$$\mathcal{E}^\bullet \longrightarrow \mathbb{L}_P \tag{6.19}$$

is an isomorphism on h^0 and onto on h^{-1}. Therefore, (6.19) is a perfect obstruction theory for P.

The virtual dimension of $P_n(X/S, \beta)$ is 0. The open and closed component $P_n^\star(X/S, \beta) \subset P_n(X/S, \beta)$ hence carries a virtual class of dimension 0. We define

$$P^\star_{n,\beta}(X/S) = \int_{[P_n^\star(X/S,\beta)]^{\mathrm{vir}}} 1.$$

LEMMA 24. *We have* $P^\star_{n,\beta}(X) = P^\star_{n,\beta}(X/S)$.

Proof. Consider the degeneration of the total space X to the normal cone of the special fiber $S = \epsilon^{-1}(0)$. By the degeneration formula for stable pairs invariants, $P^\star_{n,\beta}(X)$ is expressed as a product of integrals over $P^\star_{n_1}(X/S, \beta_1)$ and $P_{n_2}(S \times \mathbb{P}^1/S \times \{0\}, \beta_2)$ where

$$n = n_1 + n_2, \qquad \beta = \beta_1 + \beta_2.$$

Since the virtual class of $P_{n_2}(S \times \mathbb{P}^1/S \times \{0\}, \beta_2)$ vanishes by the existence of the reduced theory (see Section 6.6 below), the only surviving term of the degeneration formula is $n_1 = n$ and $\beta_1 = \beta$. □

6.6. Reduced obstruction theory. Let $R = R(n, \beta)$ be the moduli space of stable pairs on the rubber target.

The construction of the obstruction theory in Section 6.5 applies to $\mathcal{S} \times_{\mathscr{B}_r} R$ with the associated universal complex $\mathbb{I}^\bullet$ and projection π_R to R. The result is a relative obstruction theory for R given by a similar formula:

$$F^\bullet = (R\mathscr{H}om_{\pi_R}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1])^\vee \longrightarrow \mathbb{L}_{R/\mathscr{B}_r}, \tag{6.20}$$

and an absolute obstruction theory

$$\mathcal{F}^\bullet = \mathrm{Cone}\big(F^\bullet[-1] \longrightarrow \mathbb{L}_{\mathscr{B}_r}\big). \tag{6.21}$$

The relative obstruction sheaf of (6.20) contains $H^{0,2}(S)$ which can be thought of as the topological or Hodge theoretic part of the obstruction to deforming a

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stable pair. So long as S remains fixed, $H^{0,2}(S)$ is trivial and can be removed. After removal, we obtain the *reduced obstruction theory*. By now, there are many approaches to the reduced theory; see [29] for an extensive account and references. We include here a brief treatment.

We fix a holomorphic symplectic form σ on S. Let the 2-form $\bar\sigma$ denote the pull-back of σ to $\mathcal{S}$. The semiregularity map from the relative obstruction sheaf $\mathrm{Ob}_F = h^1((F^\bullet)^\vee)$ to $\mathcal{O}_R$ plays a central role:

$$\begin{aligned}\mathscr{E}xt^2_{\pi_R}(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0 &\xrightarrow{\cup \mathrm{At}(\mathbb{I}^\bullet)} \mathscr{E}xt^3_{\pi_R}(\mathbb{I}^\bullet,\mathbb{I}^\bullet\otimes\mathbb{L}_{(\mathcal{S}\times_{\mathscr{B}_r}R)/\mathscr{B}_r})\\ &\longrightarrow \mathscr{E}xt^3_{\pi_R}(\mathbb{I}^\bullet,\mathbb{I}^\bullet\otimes\Omega_{\mathcal{S}/\mathscr{B}_r}) \xrightarrow{\wedge\bar\sigma} \mathscr{E}xt^3_{\pi_R}(\mathbb{I}^\bullet,\mathbb{I}^\bullet\otimes\Omega^3_{\mathcal{S}/\mathscr{B}_r})\\ &\xrightarrow{\mathrm{tr}} R^3\pi_{R*}(\Omega^3_{\mathcal{S}/\mathscr{B}_r}) \longrightarrow R^3\pi_{R*}(\omega_{\mathcal{S}/\mathscr{B}_r}) \cong \mathcal{O}_R. \end{aligned} \tag{6.22}$$

In the last line, $\omega_{\mathcal{S}/\mathscr{B}_r}$ is the fiberwise canonical sheaf. Using the simple structure of the singularities, we see $\omega_{\mathcal{S}/\mathscr{B}_r}$ is the sheaf of fiberwise 3-forms with logarithmic poles along the singular divisors in each fiber with opposite residues along each branch. The canonical sheaf $\omega_{\mathcal{S}/\mathscr{B}_r}$ inherits a natural map from $\Omega^3_{\mathcal{S}/\mathscr{B}_r}$.

PROPOSITION 25. *The semiregularity map* (6.22) *is onto.*

Proof. We work at a closed point (F,s) of R where F is a sheaf on $S\times\mathbb{P}^1[k]$. In (6.22), we replace $\mathbb{I}^\bullet$ by

$$I^\bullet = \{\mathcal{O}\xrightarrow{s}F\}$$

and each $\mathscr{E}xt_{\pi_R}$ sheaf by the corresponding $\mathrm{Ext}_{S\times\mathbb{P}^1[k]}$ group. We will show the result is a surjection

$$\mathrm{Ext}^2(I^\bullet,I^\bullet)_0 \longrightarrow \mathbb{C}. \tag{6.23}$$

By the vanishing of the higher trace-free Ext sheaves, base change and the Nakayama Lemma, the surjection (6.23) implies the claimed result.

We use the first order deformation $\kappa\in H^1(T_S)$ of S of (6.3) and the holomorphic symplectic form σ. By (6.4), we have

$$\int_\beta \kappa \lrcorner \sigma \neq 0. \tag{6.24}$$

The pull-back of the Kodaira–Spencer class κ to $S\times\mathbb{P}^1[k]$ is

$$\bar\kappa \in \mathrm{Ext}^1(\mathbb{L}_{S\times\mathbb{P}^1[k]},\mathcal{O}_{S\times\mathbb{P}^1[k]}),$$

the class of the corresponding deformation of $S\times\mathbb{P}^1[k]$. We consider

$$\bar\kappa\circ\mathrm{At}(I^\bullet)\in\mathrm{Ext}^2(I^\bullet,I^\bullet), \tag{6.25}$$

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which by [9, 21] is the obstruction to deforming $I^\bullet$ to first order with the deformation κ of S. Since $\det(I^\bullet)$ is trivial, (6.25) lies in the subgroup of trace-free Exts. We will show the map (6.23) is nonzero on the element (6.25) of $\mathrm{Ext}^2(I^\bullet, I^\bullet)_0$.

The semiregularity map is entirely local to the support

$$\mathrm{supp}(F) \subset \coprod_i S \times \{p_i\},$$

where the p_i lie in the interiors of the $\mathbb{P}^1$ bubbles. Using the $(\mathbb{C}^*)^k$ action, we may assume the p_i are all different points of $\mathbb{C}^* = \mathbb{P}^1 \backslash \{0, \infty\}$. By moving all of the p_i to a single bubble, we may compute the same map on $S \times \mathbb{P}^1$.

By [9, Proposition 4.2],

$$\mathrm{tr}\left(\bar{\kappa} \circ \mathrm{At}(I^\bullet) \circ \mathrm{At}(I^\bullet)\right) \in H^3(\Omega_{S\times\mathbb{P}^1})$$

equals $2\bar{\kappa} \lrcorner \mathrm{ch}_2(I^\bullet)$. Therefore, the image of $\bar{\kappa} \circ \mathrm{At}(I^\bullet)$ under the map (6.23) is

$$2\int_{S\times\mathbb{P}^1} (\bar{\kappa} \lrcorner \mathrm{ch}_2(I^\bullet)) \wedge \bar{\sigma} = -2\int_{S\times\mathbb{P}^1} (\bar{\kappa} \lrcorner \bar{\sigma}) \wedge \mathrm{ch}_2(I^\bullet), \tag{6.26}$$

by the homotopy formula

$$0 = \bar{\kappa} \lrcorner (\mathrm{ch}_2(I^\bullet) \wedge \bar{\sigma}) = (\bar{\kappa} \lrcorner \mathrm{ch}_2(I^\bullet)) \wedge \bar{\sigma} + (\bar{\kappa} \lrcorner \bar{\sigma}) \wedge \mathrm{ch}_2(I^\bullet). \tag{6.27}$$

Since $\mathrm{ch}_2(I^\bullet)$ is Poincaré dual to $-\beta$, we conclude (6.26) equals

$$2\int_\beta \kappa \lrcorner \sigma,$$

which is nonzero (6.24) by the choice of κ. □

Composing (6.22) with the truncation map $(F^\bullet)^\vee \to h^1((F^\bullet)^\vee)[-1]$ and dualizing gives a map

$$\mathcal{O}_R[1] \longrightarrow F^\bullet. \tag{6.28}$$

PROPOSITION 26. *The map (6.28) lifts uniquely to the absolute obstruction theory $\mathcal{F}^\bullet$ of (6.21).*

Proof. To obtain a lifting, we must show the composition

$$\mathcal{O}_R[1] \longrightarrow F^\bullet \longrightarrow \mathbb{L}_{R/\mathscr{B}_r} \longrightarrow \mathbb{L}_{\mathscr{B}_r}[1]$$

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is zero. In fact, the composition of the second and third arrows is already zero on R. We will show the vanishing of the dual composition

$$(\mathbb{L}_{\mathscr{B}_r})^\vee[-1] \longrightarrow (\mathbb{L}_{R/\mathscr{B}_r})^\vee \longrightarrow (F^\bullet)^\vee. \tag{6.29}$$

We work with the universal complex $\mathbb{I}^\bullet$ on $\mathcal{S} \times_{\mathscr{B}_r} R$. By (6.16), we have the diagram (in which we have suppressed some pull-back maps):

$$\begin{tikzcd}
\mathbb{L}^\vee_{\mathscr{B}_r}[-1] \arrow[r] & \mathbb{L}^\vee_{\mathcal{S}/\mathscr{B}_r} \oplus \mathbb{L}^\vee_{R/\mathscr{B}_r} \arrow[r] \arrow[d, "\mathrm{At}_{\mathcal{S}/\mathscr{B}_r}(\mathbb{I}^\bullet)\oplus\mathrm{At}_{R/\mathscr{B}_r}(\mathbb{I}^\bullet)"] & \mathbb{L}^\vee_{\mathcal{S}\times_{\mathscr{B}_r} R} \arrow[d, "\mathrm{At}(\mathbb{I}^\bullet)"] \\
& R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1] \arrow[r, equal] & R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1].
\end{tikzcd}$$

The top row is an exact triangle, so the induced map

$$\mathbb{L}^\vee_{\mathscr{B}_r}[-1] \longrightarrow R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1]$$

vanishes. (There is no obstruction to deforming as we move through R over the base $\mathscr{B}_r$: there indeed exists a complex $\mathbb{I}^\bullet$ over all of $\mathcal{S} \times_{\mathscr{B}_r} R$.) Therefore, the composition

$$\pi_R^*\mathbb{L}^\vee_{\mathscr{B}_r}[-1] \longrightarrow \pi_{\mathcal{S}}^*\mathbb{L}^\vee_{\mathcal{S}/\mathscr{B}_r} \xrightarrow{\mathrm{At}_{\mathcal{S}/\mathscr{B}_r}(\mathbb{I}^\bullet)} R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1] \tag{6.30}$$

equals minus the composition

$$\pi_R^*\mathbb{L}^\vee_{\mathscr{B}_r}[-1] \longrightarrow \pi_R^*\mathbb{L}^\vee_{R/\mathscr{B}_r} \xrightarrow{\mathrm{At}_{R/\mathscr{B}_r}(\mathbb{I}^\bullet)} R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1]. \tag{6.31}$$

By adjunction the composition (6.31) gives the composition

$$\mathbb{L}^\vee_{\mathscr{B}_r}[-1] \longrightarrow \mathbb{L}^\vee_{R/\mathscr{B}_r} \xrightarrow{\mathrm{At}_{R/\mathscr{B}_r}(\mathbb{I}^\bullet)} R\pi_{R*}R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1], \tag{6.32}$$

which by (6.15), (6.17), (6.20) is precisely the composition (6.29) we want to show is zero. So it is sufficient to show (6.30) vanishes.

The first arrow of (6.30) is (the pull-back from $\mathscr{B}_r$ to R of) the Kodaira–Spencer class of $\mathcal{S}/\mathscr{B}_r$: the final arrow in the exact triangle

$$\mathbb{L}_{\mathscr{B}_r} \longrightarrow \mathbb{L}_{\mathcal{S}} \longrightarrow \mathbb{L}_{\mathcal{S}/\mathscr{B}_r} \longrightarrow \mathbb{L}_{\mathscr{B}_r}[1].$$

Away from the singularities $S \times \{0, \infty\}$ in each $S \times \mathbb{P}^1$-bubble, $\mathcal{S}$ is locally a trivial family over $\mathscr{B}_r$: it is isomorphic to

$$S \times \mathbb{C}^* \times \mathscr{B}_r$$

locally over $\mathscr{B}_r$. (But not globally due to the nontrivial $\mathbb{C}^*$-action on the $\mathbb{C}$ factor giving a nontrivial line bundle over $\mathscr{B}_r$.) Therefore, this Kodaira–Spencer map vanishes in a neighborhood of the support of the universal sheaf $\mathbb{F}$. But the second arrow of (6.30) – the Atiyah class $\mathrm{At}_{S/\mathscr{B}_r}(\mathbb{I}^\bullet)$ of $\mathbb{I}^\bullet$ – is nonzero only on the support of $\mathbb{F}$, so the composition is zero.

Finally, choices of lift are parameterized by $\mathrm{Hom}(\mathcal{O}_R[1], \mathbb{L}_{\mathscr{B}_r})$. This vanishes because $\mathbb{L}_{\mathscr{B}_r}$ is concentrated in degrees 0 and 1. Therefore, the lift is unique. □

The relative and absolute *reduced* obstruction theories are defined, respectively by:

$$F^\bullet_{\mathrm{red}} = \mathrm{Cone}\Big(\mathcal{O}_R[1] \longrightarrow F^\bullet\Big), \quad \mathcal{F}^\bullet_{\mathrm{red}} = \mathrm{Cone}\Big(\mathcal{O}_R[1] \longrightarrow \mathcal{F}^\bullet\Big) \tag{6.33}$$

The associated obstruction sheaves

$$\mathrm{Ob}^{\mathrm{red}}_F = h^1((F^\bullet_{\mathrm{red}})^\vee), \quad \mathrm{Ob}^{\mathrm{red}}_{\mathcal{F}} = h^1((\mathcal{F}^\bullet_{\mathrm{red}})^\vee)$$

are the kernels of the induced semiregularity maps

$$\mathrm{Ob}_F \longrightarrow \mathcal{O}_R, \quad \mathrm{Ob}_{\mathcal{F}} \longrightarrow \mathcal{O}_R, \tag{6.34}$$

with the first given by (6.22).

Though not required here, one can show [29, 42] the complexes (6.33) do indeed define perfect obstruction theories for R. For our purposes of extracting invariants, the simpler cosection method of Kiem–Li [26] is sufficient to produce the reduced virtual cycle $[R]^{\mathrm{red}}$ as in [37].

We summarize here the cosection method for the reader. Writing

$$(\mathcal{F}^\bullet)^\vee = \{F_0 \longrightarrow F_1\}$$

as a global two-term complex of locally free sheaves on R, Behrend and Fantechi [3] produce a cone

$$C \subset F_1 \quad \text{such that } [R]^{\mathrm{vir}} = 0^!_{F_1} C = [s(C)c(F_1)]_{\mathrm{virdim}}.$$

Here, s is the Segre class, c is the total Chern class, and we take the piece in degree equal to the virtual dimension

$$\mathrm{virdim} = \mathrm{rank}\, F_0 - \mathrm{rank}\, F_1.$$

Kiem and Li show the cone C lies cycle theoretically (rather than scheme-theoretically) in the kernel K of the composition

$$F_1 \longrightarrow \mathrm{Ob} \longrightarrow \mathcal{O}_R.$$

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We define the reduced virtual cycle in the reduced virtual dimension (virdim+1) by

$$[R]^{\mathrm{red}} = 0^!_K C = [s(C)c(K)]_{\mathrm{virdim}+1}. \tag{6.35}$$

The reduced class is much more interesting than the standard virtual class from the point of view of invariants: the exact sequence

$$0 \longrightarrow K \longrightarrow F_1 \longrightarrow \mathcal{O}_R \longrightarrow 0$$

implies the vanishing of the standard virtual class,

$$[R]^{\mathrm{vir}} = c_1(\mathcal{O}_R).[R]^{\mathrm{red}} + [s(C)c(K)]_{\mathrm{virdim}} = 0.$$

(The second term vanishes because C is a cycle inside K, so $s(C)c(K)$ is a sum of cycles in dimension $\geqslant$ rank F_0 − rank K = virdim + 1.) In our particular situation, the vanishing is even more obvious since the standard virtual dimension is −1. Since the reduced virtual dimension is 0, the reduced virtual class is nonetheless nontrivial in general.

The vanishing reflects the fact that we can deform S along κ_S (6.24): β does not remain of type (1, 1) so there can be no holomorphic curves in class β.

We define the reduced rubber invariants of S via integration over the dimension 0 class (6.35):

$$R^{\mathrm{red}}_{n,\beta}(S \times \mathcal{R}) = \int_{[R(n,\beta)]^{\mathrm{red}}} 1. \tag{6.36}$$

Here $\mathcal{R}$ denotes the rubber, the quotient by $\mathbb{C}^*$ of the relative geometry $\mathbb{P}^1/\{0, \infty\}$.

6.7. Comparison of obstruction theories. We have constructed three obstruction theories:

(i) $\mathcal{F}^\bullet$ on the rubber moduli space R;

(ii) $\mathcal{F}^\bullet_{\mathrm{red}}$ on the rubber moduli space R;

(iii) $\mathcal{E}^\bullet$ on the moduli space $P^\star$ of stable pairs on X/S over $0 \in \Delta$.

Our goal in Sections 6.7–6.9 is to relate (i), (ii), and (iii).

By pushing stable pairs forward from the rubber to the expanded degenerations of X/S, we get a map (6.13):

$$\iota : R \longrightarrow P^\star.$$

Since $E^\bullet$ and $F^\bullet$ were defined by essentially the same formulas (6.14) and (6.20), respectively, we see

$$\iota^* E^\bullet \cong F^\bullet. \tag{6.37}$$

The definitions of $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ (6.18) and (6.21) then yield the following diagram of exact triangles on R:

$$\begin{tikzcd} N^* \arrow[r, equal] \arrow[d] & N^* \arrow[d] & \\ \iota^*\mathbb{L}_{\mathscr{B}} \arrow[r] \arrow[d] & \iota^*\mathcal{E}^\bullet \arrow[r] \arrow[d] & \iota^* E^\bullet \arrow[d, equal] \\ \mathbb{L}_{\mathscr{B}_r} \arrow[r] & \mathcal{F}^\bullet \arrow[r] & F^\bullet, \end{tikzcd} \tag{6.38}$$

where $N^* = \mathbb{L}_{\mathscr{B}_r/\mathscr{B}}[-1]$ is the conormal bundle of the divisor $\mathscr{B}_r \subset \mathscr{B}$. Dualizing the central column and passing to cohomology gives a map

$$N \longrightarrow \mathrm{Ob}_{\mathcal{F}} = h^1((\mathcal{F}^\bullet)^\vee) \tag{6.39}$$

which describes the obstruction to deforming a pair in the image of ι off the rubber and into the bulk of X/S. Composing with the semiregularity map (6.34) gives

$$N \longrightarrow \mathcal{O}_R. \tag{6.40}$$

The rest of Section 6.7 will be devoted to proving the following result.

PROPOSITION 27. *The maps* (6.39), (6.40) *are injections of sheaves on* R. *Moreover,* (6.39) *has no zeros.*

Connected case. We first work at a stable pair (F, s) with connected support. The sheaf F is therefore supported on $S \times (\mathbb{P}^1 \backslash \{0, \infty\})$ with no further bubbles in the rubber. (If we view the stable pair as lying in X/S, there is a single bubble and (F, s) is supported in its interior.) We will show that the composition (6.40) is an isomorphism at the point (F, s). By the vanishing of the higher trace-free Exts, base change and the Nakayama lemma, the Proposition will follow in the connected support case.

A chart for the stack $\mathscr{B}$ of (n, β)-marked expanded degenerations of X/S in a neighborhood of the 1-bubble locus is $\mathbb{C}/\mathbb{C}^*$ with universal family $\mathcal{X} \to \mathscr{B}$ given by [33]

$$\begin{tikzcd} \mathrm{Bl}_{S\times\{0\}}(X \times \mathbb{C}) \arrow[d] & \circlearrowleft \mathbb{C}^* \\ \mathbb{C} & \circlearrowleft \mathbb{C}^*. \end{tikzcd} \tag{6.41}$$

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Here, the trivial $\mathbb{C}^*$-action on X and the usual weight 1 action on $\mathbb{C}$ yield a $\mathbb{C}^*$-action on $X \times \mathbb{C}$. The blow-up along the $\mathbb{C}^*$-fixed subvariety $S \times \{0\}$ has a canonically induced $\mathbb{C}^*$-action. The exceptional divisor $S \times \mathbb{P}^1$ inherits a $\mathbb{C}^*$-action. The central fiber is

$$X[1] = X \cup_S (S \times \mathbb{P}^1).$$

More explicitly, let x denote the coordinate on X pulled back locally at $0 \in \varDelta$ from the base of the $K3$-fibration

$$X \to \varDelta, \tag{6.42}$$

and let t denote the coordinate pulled back from the $\mathbb{C}$-base of (6.41). By definition,

$$\mathrm{Bl}_{S\times\{0\}}(X \times \mathbb{C}_t) \quad \text{is } \{t = \lambda x\} \subset X \times \mathbb{C}_t \times \mathbb{P}^1_\lambda, \tag{6.43}$$

where x has $\mathbb{C}^*$-weight 0 while t and λ have weight -1. Here, λ is the usual coordinate on $\mathbb{P}^1$ which takes the value ∞ on the relative divisor $S \times \{\infty\}$ in the central fiber, and the value 0 on the proper transform $\overline{X \times \{0\}}$ of the central fiber. Removing these loci, which are disjoint from the support of (F, s), our universal family over $\mathscr{B}$ becomes the quotient by $\mathbb{C}^*$ of

$$X \times \mathbb{C}^*_\lambda \xrightarrow{t=\lambda x} \mathbb{C}_t. \tag{6.44}$$

The key to Proposition 27 is the following observation: as we move in the direction ∂_t in the base of (6.44), we move away from the central fiber $S \subset X$ in the direction $\lambda^{-1}\partial_x$ over the base of the $K3$-fibration (6.42). In other words, on the central fiber $S \times \mathbb{C}^*_\lambda$, the Kodaira–Spencer class of the family (6.44) applied to ∂_t is

$$\lambda^{-1}\kappa \in \varGamma(\mathcal{O}_{\mathbb{C}^*}) \otimes H^1(T_S) \cong H^1(T_X|_{S\times\mathbb{C}^*}). \tag{6.45}$$

Here, as usual, $\kappa \in H^1(T_S)$ is the Kodaira–Spencer class (6.3) of (6.42) on the central fiber S. Since $\lambda \neq \infty$ on the support of (F, s), the $(0, 2)$-part of the class β of F immediately becomes nonzero along $\lambda^{-1}\partial_x$, just as in the proof of Proposition 25. Thus the semiregularity map is nonzero. We now make the argument more precise.

The semiregularity map (6.34) of Proposition 26 was first defined on $F^\bullet$. After rearranging (6.38), we see the composition (6.40) we require is induced from the composition

$$\iota^*\mathbb{L}^\vee_{\mathscr{B}} \longrightarrow (\iota^* E^\bullet)^\vee[1] \cong (F^\bullet)^\vee[1] \longrightarrow \mathcal{O}_R. \tag{6.46}$$

The last arrow is (6.28). By (6.38), the first arrow is zero on composition with $\mathbb{L}^\vee_{\mathscr{B}_r} \to \iota^*\mathbb{L}^\vee_{\mathscr{B}}$, so the first arrow factors through the cone N as required. In fact, we even have a splitting

$$\iota^*\mathbb{L}^\vee_{\mathscr{B}} \cong N \oplus \mathbb{L}^\vee_{\mathscr{B}_r}, \tag{6.47}$$

obtained from expressing $\mathscr{B}$ locally as $\mathbb{C}/\mathbb{C}^*$, with the substack $\mathscr{B}_r$ given by $\{0\}/\mathbb{C}^*$. Therefore,

$$\mathbb{L}_{\mathscr{B}} \cong \{\Omega_{\mathbb{C}} \to \mathfrak{g}^* \otimes \mathcal{O}_{\mathscr{B}}\},$$

where $\mathfrak{g}$ is the Lie algebra of $\mathbb{C}^*$, and, in the standard trivialization, the map takes dt to t. On applying ι^* the map therefore vanishes, leaving

$$\iota^*\Omega_{\mathbb{C}} \oplus (\mathfrak{g}^* \otimes \mathcal{O}_{\mathscr{B}_r})[-1] = N^* \oplus \mathbb{L}_{\mathscr{B}_r}$$

as claimed.

By the same argument as in (6.30)–(6.32), the first arrow of (6.46) is (up to a sign) the composition of the following Kodaira–Spencer and Atiyah classes:

$$\iota^*\mathbb{L}^\vee_{\mathscr{B}} \longrightarrow R\pi_{R*}(\iota^*\mathbb{L}^\vee_{\mathcal{X}/\mathscr{B}})[1] \xrightarrow{\mathrm{At}_{\mathcal{X}/\mathscr{B}}(\mathbb{I}^\bullet)} R\pi_{R*}R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[2].$$

Together with the splitting (6.47) and the description (6.46) of our map, we find that at a point $I^\bullet$ the map (6.40) is the composition

$$N|_{I^\bullet} \longrightarrow \mathrm{Ext}^1(\mathbb{L}_{X[1]}, \mathcal{O}_{X[1]}) \longrightarrow \mathrm{Ext}^2(I^\bullet, I^\bullet)_0 \longrightarrow \mathbb{C}. \qquad (6.48)$$

The first arrow is the Kodaira–Spencer class of the family (6.41) on the central fiber $X[1]$. The connected support of our stable pair (F, s) is contained in $S\times\{1\}$, without loss of generality. On restriction to this support, the Kodaira–Spencer class is κ (6.44), (6.45). (If we act by $\lambda \in \mathbb{C}^*$ the relevant statement becomes that for a stable pair supported in $S \times \{\lambda\}$, the value of the Kodaira–Spencer class on $\lambda\partial_t$ is $\lambda.\lambda^{-1}\kappa = \kappa$ (6.45). The point is that there is no natural trivialization of N, and $\lambda \in \mathbb{C}^*$ takes the trivialization ∂_t to the trivialization $\lambda\partial_t$.)

The second arrow is composition with the Atiyah class of the complex $I^\bullet$ on $X[1]$. The Atiyah class vanishes on the complement of the support $S \times \{1\}$. We may restrict $I^\bullet$ to the bubble $S \times \mathbb{P}^1_\lambda$ and calculate there. The final arrow is the semiregularity map (6.22). So (6.48) simplifies to the composition

$$\mathbb{C} \xrightarrow{\kappa} H^1(T_{S\times\mathbb{P}^1}) \xrightarrow{\mathrm{At}(I^\bullet)} \mathrm{Ext}^2_{S\times\mathbb{P}^1}(I^\bullet, I^\bullet)_0 \xrightarrow{\mathrm{tr}(\,\cdot\,\circ \mathrm{At}(I^\bullet)\wedge\bar\sigma)} \mathbb{C}, \qquad (6.49)$$

where again we have trivialized N by the section ∂_t. The composition is therefore

$$\int_{S\times\mathbb{P}^1} \mathrm{tr}(\kappa \circ \mathrm{At}(I^\bullet) \circ \mathrm{At}(I^\bullet)) \wedge \bar\sigma.$$

By [9, Proposition 4.2], this is

$$2\int_{S\times\mathbb{P}^1} \kappa \lrcorner \mathrm{ch}_2(I^\bullet) \wedge \bar\sigma = -2\int_{S\times\mathbb{P}^1} (\kappa\lrcorner\bar\sigma) \wedge \mathrm{ch}_2(I^\bullet) = 2\int_\beta \kappa\lrcorner\sigma,$$

just as in (6.26). Since $\kappa\lrcorner\sigma$ is nonzero on β by design (6.4), the composition (6.49) is nonzero. Proposition 27 is established in the connected case.

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Disconnected case. To deal with the case of arbitrary support, we write a stable pair (F, s) on the rubber target as a direct sum of stable pairs with connected supports:

$$(F, s) = \bigoplus_i (F_i, s_i). \tag{6.50}$$

By stability we may assume, without loss of generality, that (F_1, s_1) is supported on the interior of the first bubble.

The decomposition (6.50) holds in a neighborhood of (F, s) in the moduli space R (though the ith summand need not have connected support for pairs not equal to (F, s)). The obstruction sheaf $\mathrm{Ob}_{\mathcal{F}}$ is additive with respect to the decomposition: $\mathscr{E}xt^2_{\pi_R}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$ splits into a corresponding direct sum. We will prove (6.40) is an isomorphism on the summand (F_1, s_1). The isomorphism will follow from the connected case after we have set up appropriate notation. Since the map (6.39) is linear, the result will prove (6.39) has no zeros. We will address the injectivity claim for the map (6.40) in the statement of Proposition 27 at the end of the proof.

Suppose (F, s) is supported on $X[k]$. In a neighborhood of $X[k]$, a chart for the stack of (n, β)-marked expanded degenerations of X is given by

$$\mathbb{C}^k / (\mathbb{C}^*)^k \tag{6.51}$$

where the group acts diagonally [33]. We let $t_1, \ldots, t_n$ denote the coordinates on the base $\mathbb{C}^k$. Let x be the coordinate pulled back from the base Δ of the $K3$-fibration X.

The universal family

$$\mathcal{X} \to \mathscr{B}, \tag{6.52}$$

restricted to the chart (6.51), is constructed by the following sequence of $(\mathbb{C}^*)^k$-equivariant blow-ups of $X \times \mathbb{C}^k$:

- Blow up $X \times \mathbb{C}^k$ along $x = 0 = t_1$ (the product of the surface $S \subset X$ and the first coordinate hyperplane).
- Blow up the result along the *proper transform* of $x = 0 = t_2$, (the proper transform of S times the second coordinate hyperplane).
- At the ith stage, blow up the result of the previous step in the proper transform of $x = 0 = t_i$.

After k steps, we obtain the universal family (6.52) over the chart (6.51).

The fiber of the universal family over the origin of (6.51) is $X[k]$ with marking

$$(n_0, \beta_0) \in \mathbb{Z} \oplus H_2(X, \mathbb{Z})$$

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on the first component and marking $(n_i, \beta_i) \in \mathbb{Z} \oplus H_2(S, \mathbb{Z})$ on the ith bubble for $1 \leqslant i \leqslant k$. The data satisfy (6.9):

$$n_0 + \sum_{i=1}^{k} n_i = n, \quad \beta_0 + \sum_{i=1}^{k} \beta_i = \beta.$$

Over a point of $\mathbb{C}^k$ with precisely j of the coordinates t_i vanishing, the fiber is $X[j]$. The j vanishing coordinates are in bijective ordered correspondence with the j bubbles and the j creases of $X[j]$. (The ith crease of $X[j]$ is the copy of S at the bottom of the ith bubble: the intersection of the $(i-1)$th and ith bubbles of $X[j]$.) Moving away from this point of the base, a crease smooths if and only if the corresponding coordinate becomes nonzero. If the ith and $(i+1)$th vanishing coordinates are t_a and t_b, then the marking on the ith bubble of $X[j]$ is

$$\left(\sum_{i=a}^{b-1} n_i, \sum_{i=a}^{b-1} \beta_i \right) \in \mathbb{Z} \oplus H^2(S).$$

Similarly, if the first vanishing coordinate is t_c, then the marking on $X \subset X[j]$ is

$$\left(n_0 + \sum_{i=1}^{t_c-1} n_i, \beta_0 + \sum_{i=a}^{t_c-1} \beta_i \right) \in \mathbb{Z} \oplus H^2(X). \tag{6.53}$$

Relative stable pairs – which cannot lie in X – all lie over the locus

$$t_1 = 0, \quad n_0 = 0, \quad \beta_0 = 0,$$

where $c = 1$ in (6.53) and the marking on X vanishes. The inclusion of the hyperplane $t_1 = 0$,

$$\iota \colon \mathbb{C}^{k-1}/(\mathbb{C}^*)^k = \{t_1 = 0\}/(\mathbb{C}^*)^k \hookrightarrow \mathbb{C}^k/(\mathbb{C}^*)^k \tag{6.54}$$

describes the inclusion (6.12) of the corresponding chart of the stack $\mathscr{B}_r \subset \mathscr{B}$ over which the rubber target $\mathcal{S}$ lies.

On the chart (6.51),

$$\mathbb{L}_{\mathscr{B}} = \{\Omega_{\mathbb{C}^k} \longrightarrow (\mathfrak{g}^*)^k \otimes \mathcal{O}_{\mathscr{B}}\},$$

where the map is diag $(t_1, \ldots, t_k)$ in the natural trivializations. Pulling back by (6.54) gives

$$\begin{aligned} \iota^* \mathbb{L}_{\mathscr{B}} &= (N^* \oplus \mathfrak{g}^* \otimes \mathcal{O}_{\mathscr{B}_r}[1]) \oplus \left\{ \Omega_{\mathbb{C}^{k-1}} \xrightarrow{\operatorname{diag}(t_2,\ldots,t_k)} (\mathfrak{g}^*)^{k-1} \otimes \mathcal{O}_{\mathscr{B}_r} \right\} \\ &= N^* \oplus \mathbb{L}_{\mathscr{B}_r}. \end{aligned}$$

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Thus $\iota^*\mathbb{L}^\vee_{\mathscr{B}} \cong N \oplus \mathbb{L}^\vee_{\mathscr{B}_r}$ just as in (6.47). The element ∂_{t_1} lies in – and generates – the first summand (but, just as before, is not a global trivialization as it is not $\mathbb{C}^*$-invariant).

We have to work out the composition (6.48) as before, replacing ∂_t by ∂_{t_1}. The stable pair (F_1, s_1) is supported on some $S \times \{1\}$ of the first bubble of $X[k]$ just as before. Restricting to the first bubble, the Kodaira–Spencer class evaluated on the section ∂_{t_1} of N is κ just as in the single bubble case: all further blow-ups in the construction of $\mathcal{X} \to \mathscr{B}$ occur at $S \times \{\infty\}$ in the bubble and hence do not affect the interior of the first bubble or its Kodaira–Spencer class. The same calculation then shows the map (6.40) at the point (F_1, s_1) takes ∂_{t_1} to

$$2 \int_{\beta_1} \kappa \lrcorner \sigma, \tag{6.55}$$

where $\beta_1 = [F_1]$. Since (6.55) is nonzero by (6.4), the map (6.40) is an isomorphism on the first summand (F_1, s_1) as claimed. As explained above, this implies that (6.39) has no zeros.

Consider now the summand (F_2, s_2). If the support of (F_2, s_2) is in the first bubble, the support lies in $S \times \{\lambda\}$ for some $\lambda \neq 1$, and the work we have already done shows that applied to (F_2, s_2) the map (6.40) takes ∂_{t_1} to

$$2\lambda^{-1} \int_{\beta_2} \kappa \lrcorner \sigma \neq 0. \tag{6.56}$$

Here, the nonvanishing is by (6.4) applied to $\beta_2 = [F_2]$.

For summands (F_i, s_i) not in the first bubble we can do a similar calculation, blowing up (6.43) once more and using local coordinates again. The result is that the Kodaira–Spencer class in the higher bubbles is 0 (this is effectively the $\lambda \to \infty$ limit of the above calculation).

To prove the injectivity claim for the map (6.40) in the statement of Proposition 27, we consider two possibilities.

- If (F_1, s_1) is the only summand in the first bubble, all others contribute zero to (6.40), so in total (6.40) is nonzero by (6.55).
- If there is another summand (F_2, s_2) in the first bubble, then, by linearity, the nonzero contribution (6.56) of (F_2, s_2) is added to that of the other summands, and can be varied by perturbing its support

$$\lambda \in \mathbb{C}^* \subset \mathbb{P}^1.$$

Therefore, even though the map (6.40) might be zero at (F, s), the map (6.40) is nonzero at a nearby perturbation. Since the perturbation by moving the support λ is along an étale trivial factor in the moduli space R, the map (6.40) must be injective as a morphism of sheaves.

The proof of Proposition 27 is complete. $\square$

6.8. Curvilinear lifting. Proposition 27 does not imply the moduli spaces R and $P^\star$ are isomorphic. Our analysis of $\widetilde{R}_{n,\beta}$ is crucially dependent upon a weaker curvilinear lifting relationship between R and $P^\star$ which does follow from Proposition 27.

LEMMA 28. *The map $\iota\colon R \to P^\star$ of (6.13) induces an isomorphism $\iota^*\Omega_{P^\star} \cong \Omega_R$ of cotangent sheaves.*

Proof. The obstruction theories $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ are related by the exact triangle

$$N^* \longrightarrow \iota^*\mathcal{E}^\bullet \longrightarrow \mathcal{F}^\bullet$$

of (6.38), giving the exact sequence

$$h^{-1}(\mathcal{F}^\bullet) \longrightarrow N^* \longrightarrow h^0(\iota^*\mathcal{E}^\bullet) \longrightarrow h^0(\mathcal{F}^\bullet) \longrightarrow 0.$$

Since $\mathcal{E}^\bullet$ vanishes in strictly positive degrees and ι^* is right exact, $h^0(\iota^*\mathcal{E}^\bullet) = \iota^* h^0(\mathcal{E}^\bullet)$. Therefore, we obtain

$$h^{-1}(\mathcal{F}^\bullet) \longrightarrow N^* \longrightarrow \iota^*\Omega_{P^\star} \longrightarrow \Omega_R \longrightarrow 0. \tag{6.57}$$

By Proposition 27, the first map is surjective. $\square$

COROLLARY 29. *Suppose A is a subscheme of B with ideal J satisfying*

$$d\colon J \to \Omega_B|_A \quad \text{injective.} \tag{6.58}$$

(In particular $J^2 = 0$.) Then any extension $\tilde{f}\colon B \to P^\star$ of a map $f\colon A \to R$ factors through R.

Proof. Let I denote the ideal of R inside P. To show the factorization of $\tilde{f}$ through R, we must show the image of

$$\tilde{f}^* I \to J$$

vanishes. Since $J^2 = 0$, the above image can be evaluated after restriction to A,

$$\tilde{f}^* I|_A = f^* I \to J. \tag{6.59}$$

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By Lemma 28, the first map d vanishes in the exact sequence of Kähler differentials

$$I \xrightarrow{d} \iota^* \Omega_{P^\star} \longrightarrow \Omega_R \longrightarrow 0.$$

Pulling back via f gives the top row of the following commutative diagram with exact rows.

$$\begin{tikzcd}
 & f^*I \arrow[r, "0"] \arrow[d, "\tilde{f}^*"] & f^*\iota^*\Omega_{P^\star} \arrow[r, "\sim"] \arrow[d, "\tilde{f}^*"] & f^*\Omega_R \arrow[r] \arrow[d, "f^*"] & 0 \\
0 \arrow[r] & J \arrow[r, "d"] & \Omega_B|_A \arrow[r] & \Omega_A \arrow[r] & 0.
\end{tikzcd}$$

Here, the central map uses the isomorphism $f^*\iota^*\Omega_{P^\star} \cong (\tilde{f}^*\Omega_{P^\star})|_A$. As a result, the first vertical arrow (given by (6.59)) is zero. Hence, $\tilde{f}$ has image in R. $\square$

The basic relationship between R and $P^\star$ which we need is the *curvilinear lifting property* proven in the following corollary.

COROLLARY 30. *Every map* $\operatorname{Spec}\mathbb{C}[x]/(x^k) \to P^\star$ *factors through* R.

Proof. Since $R \subset P^\star$ is a bijection of sets, we have the result for $k = 1$. Since

$$A = \operatorname{Spec}\frac{\mathbb{C}[x]}{(x^k)} \subset \operatorname{Spec}\frac{\mathbb{C}[x]}{(x^{k+1})} = B$$

satisfies (6.58), the result for higher k follows by Corollary 29 and induction. $\square$

We summarize the above results in the following proposition.

PROPOSITION 31. *The map* $\iota\colon R \to P^\star$ *of* (6.13) *is a closed embedding of Deligne–Mumford stacks which satisfies the curvilinear lifting property.* $\square$

The complexes $\iota^*\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$, and $\mathcal{F}^\bullet_{\mathrm{red}}$ on R are related by the exact triangles

$$\begin{tikzcd}
 & (\mathcal{F}^\bullet_{\mathrm{red}})^\vee \arrow[r, equal] \arrow[d] & (\mathcal{F}^\bullet_{\mathrm{red}})^\vee \arrow[d] \\
N[-1] \arrow[r] \arrow[d, equal] & (\mathcal{F}^\bullet)^\vee \arrow[r] \arrow[d] & (\iota^*\mathcal{E}^\bullet)^\vee \arrow[d] \\
N[-1] \arrow[r] & \mathcal{O}_R[-1] \arrow[r] & \mathcal{O}_D[-1].
\end{tikzcd} \tag{6.60}$$

Here, D is the Cartier divisor on which the injection $N \to \mathcal{O}_R$ (6.40) vanishes.

In particular, in K-theory the classes of $(\iota^*\mathcal{E}^\bullet)^\vee$ and $(\mathcal{F}^\bullet_{\mathrm{red}})^\vee$ differ by $\mathcal{O}_D[-1]$. The K-theory classes determine the corresponding virtual cycles via the formula of [15, 44]. For the virtual class $[P^\star]^{\mathrm{vir}}$ associated to $\mathcal{E}^\bullet$, the formula is

$$[P^\star]^{\mathrm{vir}} = [s((\mathcal{E}^\bullet)^\vee) \cap c_F(P^\star)]_{\mathrm{virdim}}, \tag{6.61}$$

where $c_F(P^\star)$ is the Fulton total Chern class [15, 4.2.6.(a)] of the scheme $P^\star$, s denotes the total Segre class, and the subscript denotes the term in degree specified by the virtual dimension (equal to 0 here). The homology class $c_F(P^\star)$ is *not* of pure degree. The expression (6.61) is a sum of different degree parts of the cohomology class $s((\mathcal{E}^\bullet)^\vee)$ capped with the different degree parts of $c_F(P^\star)$ to give the virtual class.

Since the reduced scheme structure of R and $P^\star$ is the same, we may view $c_F(P^\star)$ and $[P^\star]^{\mathrm{vir}}$ as cycles on R. The curvilinear lifting property of Proposition 31 implies a basic relation between the Fulton Chern classes of R and $P^\star$ explained in Appendix C.2,

$$c_F(R) = c_F(P^\star) \in A_*(R). \tag{6.62}$$

Formulas (6.61) and (6.62) allow us to study $[P^\star]^{\mathrm{vir}}$ via the geometry of R.

Since $s(\mathcal{O}_D) = 1 - D$, the rightmost column of (6.60) yields the identity

$$s\big((\iota^*\mathcal{E}^\bullet)^\vee\big) = s\big((\mathcal{F}^\bullet_{\mathrm{red}})^\vee\big) + D.\, s\big((\iota^*\mathcal{E}^\bullet)^\vee\big).$$

After substituting in (6.61), we obtain

$$\begin{aligned}[P^\star]^{\mathrm{vir}} &= \big[s\big((\mathcal{F}^\bullet_{\mathrm{red}})^\vee\big) \cap c_F(P^\star)\big]_0 + \big[s\big((\iota^*\mathcal{E}^\bullet|_D)^\vee\big) \cap c_F(P^\star)|_D\big]_0 \\ &= \big[s\big((\mathcal{F}^\bullet_{\mathrm{red}})^\vee\big) \cap c_F(R)\big]_0 + \big[s\big((\iota^*\mathcal{E}^\bullet|_D)^\vee\big) \cap c_F(R)|_D\big]_0 \\ &= [R]^{\mathrm{red}} + \big[s\big((\iota^*\mathcal{E}^\bullet|_D)^\vee\big) \cap c_F(R)|_D\big]_0. \end{aligned} \tag{6.63}$$

We may replace D in the rightmost term by any other Cartier divisor in the same linear equivalence class, since the replacement leaves the K-theory class $[\mathcal{O}_D]$ unchanged. We will work on a cover of the rubber moduli space R on which D becomes linearly equivalent to a rather more tractable divisor.

6.9. Rigidification. Let $R = R(n, \beta)$ be the moduli space of stable pairs on the rubber target. Let

$$\pi : U(n, \beta) = \mathcal{S} \times_{\mathscr{B}_r} R \to R \tag{6.64}$$

be the universal target over the rubber moduli space R. Let

$$W(n, \beta) \subset U(n, \beta) \tag{6.65}$$

denote the open set on which the morphism π is smooth.

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We view U as a moduli of pairs (r, p) where $r \in R$ and p is a point in the rubber target associated to r. For pairs $(r, p) \in W$, the point p is not permitted to lie on *any* creases. Hence, the restriction

$$\pi : W \to R$$

is a smooth morphism. The rubber target admits a natural map,

$$\rho : W \to S,$$

to the underlying $K3$ surface.

Viewing $[R]^{\mathrm{red}}$ and $[P^\star]^{\mathrm{vir}}$ as cycle classes on R, we define classes $[W]^{\mathrm{red}}$ and $[W]^{\mathrm{vir}}$ on W by flat pull-back:

$$[W]^{\mathrm{red}} = \pi^*[R]^{\mathrm{red}}, \quad [W]^{\mathrm{vir}} = \pi^*[P^\star]^{\mathrm{vir}}.$$

By the definitions of the cones and the Fulton class,

$$[W]^{\mathrm{red}} = \Big[s\big((\pi^*\mathcal{F}^\bullet_{\mathrm{red}})^\vee\big)s\big(\mathcal{T}_\pi\big) \cap c_F(W)\Big]_3, \tag{6.66}$$

where $\mathcal{T}_\pi$ on W is the relative tangent bundle of π. Similarly,

$$[W]^{\mathrm{vir}} = \Big[s\big((\pi^*\iota^*\mathcal{E}^\bullet)^\vee\big)s\big(\mathcal{T}_\pi\big) \cap c_F(W)\Big]_3. \tag{6.67}$$

Integrals over R may be moved to W by the following procedure. To a class $\delta \in H^2(S, \mathbb{Q})$, we associate a primary insertion

$$T_0(\delta) = \mathrm{ch}_2(\mathbb{F}) \cup \rho^*(\delta) \in H^6\big(U, \mathbb{Q}\big),$$

where $\mathbb{F}$ is the universal sheaf. The key identity is the divisor formula obtained by integrating down the fibers of (6.64):

$$\pi_*\big(T_0(\delta)\big) = \langle \delta, \beta \rangle = \int_\beta \delta \in H^0(R, \mathbb{Q}). \tag{6.68}$$

The push-forward π_* is well defined since $\mathrm{ch}_2(\mathbb{F})$ and $T_0(\delta)$ are supported on $\mathrm{Supp}(\mathbb{F})$ which is projective over R via π.

The derivation in Section 6.8 of (6.63) can be pulled back via π to W to yield:

$$[W]^{\mathrm{vir}} = [W]^{\mathrm{red}} + \Big[s\big((\pi^*\iota^*\mathcal{E}^\bullet|_D)^\vee\big)s\big(\mathcal{T}_\pi\big) \cap c_F(W)|_D\Big]_3. \tag{6.69}$$

As before, D is any divisor representing the first Chern class of the pull-back of N^*, the conormal bundle of the divisor $\mathscr{B}_r \subset \mathscr{B}$. By integration against (6.69), formula (6.68) yields

$$P^\star_{n,\beta}(X) - R^{\mathrm{red}}_{n,\beta}(S \times \mathcal{R}) = \frac{1}{\langle \delta, \beta \rangle} \int_D \Big(s\big((\pi^*\iota^*\mathcal{E}^\bullet|_D)^\vee\big)s\big(\mathcal{T}_\pi\big) \cap c_F(W)|_D\Big) \cdot T_0(\delta). \tag{6.70}$$

We describe next a geometric representative for D.

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Attaching the infinity section S_∞ of the rubber over $R(n_1, \beta_1)$ to the zero section S_0 of rubber over W defines a divisor

$$D_{n_1,\beta_1} = R(n_1, \beta_1) \times W(n_2, \beta_2) \subset W(n, \beta), \tag{6.71}$$

whenever $(n_1, \beta_1) + (n_2, \beta_2) = (n, \beta)$. The following result is a form of *topological recursion*.

LEMMA 32. *The line bundle $\pi^* N^*$ has a section with zeros given by the divisor*

$$D = \sum_{n_1,\beta_1} D_{n_1,\beta_1}.$$

The sum is over the finitely many $(n_1, \beta_1) \in \mathbb{Z} \oplus H_2(S, \mathbb{Z})$ for which D_{n_1,β_1} is nonempty.

Proof. The universal family $\mathcal{X} \to \mathcal{B}$ has smooth total space. Moving normal to $\mathcal{B}_r$ smooths the first crease in the expanded degeneration $\mathcal{X}$: the crease where X joins the rubber $\mathcal{S}$ across $S \subset X$ and the 0-section S_0 of the rubber. Therefore, the normal bundle N, pulled back to $\mathcal{X}$ and restricted to the first crease, is isomorphic to

$$N_{S \subset X} \otimes N_{S_0 \subset \mathcal{S}}.$$

Fixing once and for all a trivialization of $N_{S \subset X}$, we find that

$$N \cong \psi_0^*$$

is isomorphic to the tangent line to $\mathbb{P}^1$ on the zero section. (More precisely, fix any point $s \in S$. Then the corresponding point of the zero section S_0 of $\mathcal{S}$ defines a section s_0 of $\mathcal{S} \to \mathcal{B}_r$. The first bubble is $S_0 \times \mathbb{P}^1$, and the conormal bundle to S_0 is the restriction of $T^*_{\mathbb{P}^1} \subset T^*_{S_0 \times \mathbb{P}^1}$. Pulling back by s_0 gives the cotangent line ψ_0.)

Now pull back to $W(n, \beta)$ via (6.64). By forgetting S (but remembering the $\mathbb{Z} \oplus H_2(S, \mathbb{Z})$ marking), $W(n, \beta)$ maps to the stack $\mathcal{B}^p_{n,\beta}$ of (n, β)-marked expanded degenerations of $\mathbb{P}^1/\{0, \infty\}$,

$$W(n, \beta) \to \mathcal{B}^p_{n,\beta}. \tag{6.72}$$

The moduli space $W(n, \beta)$ parameterizes pairs (r, p). The map (6.72) is defined by using p to select the rigid component and to determine $1 \in \mathbb{P}^1$. The cotangent line at the relative divisor 0 defines a line bundle on $\mathcal{B}^p_{n,\beta}$ which we also call ψ_0. On $W(n, \beta)$, ψ_0 pulls back to

$$\pi^* \psi_0 \cong \pi^* N^*$$

above.

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Pick a nonzero element of $T^*_{\mathbb{P}^1}|_0$, pull back to any expanded degeneration, and restrict to the new 0-section. We have constructed a section of ψ_0 over $\mathscr{B}^p_{n,\beta}$ which vanishes precisely on the divisor where 0 has been bubbled. The latter divisor is the sum of the divisors D_{n_1,β_1} as required. $\square$

After combining Lemma 32 with (6.70), we obtain the formula:

$$\begin{aligned} &P^\star_{n,\beta}(X) - R^{\mathrm{red}}_{n,\beta}(S\times\mathcal{R}) \\ &\quad = \frac{1}{\langle\delta,\beta\rangle}\sum_{n_1,\beta_1}\int_{D_{n_1,\beta_1}}\Big(s\big((\pi^*\iota^*\mathcal{E}^\bullet)^\vee\big)s\big(\mathcal{T}_\pi\big)|_{D_{n_1,\beta_1}}\cap c_F(W)|_{D_{n_1,\beta_1}}\Big)\cdot T_0(\delta). \end{aligned} \tag{6.73}$$

To proceed, we must compute the restriction of $\pi^*\iota^*\mathcal{E}^\bullet$ to the divisor D_{n_1,β_1}.

The *relative* obstruction theory is given by the same formula (6.15) on the moduli spaces R and $P^\star$. On R, the relative obstruction theory was denoted $F^\bullet$ (6.20). Since the relative obstruction theory is additive over the connected components of a stable pair,

$$\pi^*\iota^*E^\bullet|_{D_{n_1,\beta_1}} \cong \iota^*E^\bullet_{R(n_1,\beta_1)} \oplus \pi^*F^\bullet_{W(n_2,\beta_2)}, \tag{6.74}$$

where we have split D_{n_1,β_1} as $R(n_1,\beta_1)\times W(n_2,\beta_2)$ using (6.71).

As in (6.18), (6.21) the relationship of the relative to the absolute obstruction theories $\mathcal{E}^\bullet_W$ and $\mathcal{F}^\bullet$ is through the usual exact triangles:

$$\iota^*\mathcal{E}^\bullet \longrightarrow \iota^*E^\bullet \longrightarrow \iota^*\mathbb{L}_{\mathscr{B}_{n,\beta}}[1] \tag{6.75}$$

on the moduli space R and

$$\begin{aligned} \iota^*\mathcal{E}^\bullet_{R(n_1,\beta_1)} \oplus \pi^*\mathcal{F}^\bullet_{W(n_2,\beta_2)} &\longrightarrow \iota^*E^\bullet_{R(n_1,\beta_1)} \oplus \pi^*F^\bullet_{W(n_2,\beta_2)} \\ &\longrightarrow \iota^*\mathbb{L}_{\mathscr{B}_{n_1,\beta_1}}[1] \oplus \pi^*\mathbb{L}_{\mathscr{B}_r}[1] \end{aligned} \tag{6.76}$$

on $D_{n_1,\beta_1} = R(n_1,\beta_1)\times W(n_2,\beta_2)$. In K-theory, the isomorphism (6.74) and the exact sequences (6.75)–(6.76) yield

$$\begin{aligned} \big[\pi^*\iota^*\mathcal{E}^\bullet|_{D_{n_1,\beta_1}}\big] &= [\iota^*\mathcal{E}^\bullet_{R(n_1,\beta_1)}] + [\pi^*\mathcal{F}^\bullet_{W(n_2,\beta_2)}] \\ &\quad - [\iota^*\mathbb{L}_{\mathscr{B}_{n_1,\beta_1}}] - [\pi^*\mathbb{L}_{\mathscr{B}_r}] + [\iota^*\mathbb{L}_{\mathscr{B}_{n,\beta}}]. \end{aligned} \tag{6.77}$$

Since D_{n_1,β_1} is pulled back from $\mathscr{B}_{n,\beta}$, the standard divisor conormal bundle sequence yields

$$[\mathbb{L}_{\mathscr{B}_{n_1,\beta_1}}] + [\mathbb{L}_{\mathscr{B}_r}] - [\mathbb{L}_{\mathscr{B}_{n,\beta}}] = -[\mathcal{O}_{D_{n_1,\beta_1}}(-D_{n_1,\beta_1})].$$

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(More precisely, $D_{n_1,\beta_1} \subset W$ is an open and closed component of the pull-back.) Hence, we obtain

$$s\Big(\big(\pi^*\iota^*\mathcal{E}^\bullet|_{D_{\beta_1,n_1}}\big)^\vee\Big)c\big(\mathcal{O}_{D_{n_1,\beta_1}}(D_{n_1,\beta_1})\big) = s\big((\iota^*\mathcal{E}^\bullet_{R(n_1,\beta_1)})^\vee\big)s\big((\pi^*\mathcal{F}^\bullet_{W(n_2,\beta_2)})^\vee\big). \tag{6.78}$$

By the basic properties of cones and the embedding $D_{n_1,\beta_1} \subset W(n,\beta)$ discussed in Appendix C.3, we have

$$c_F(W(n,\beta))|_{D_{n_1,\beta_1}} = c_F(D_{n_1,\beta_1})c(\mathcal{O}_{D_{n_1,\beta_1}}(D_{n_1,\beta_1})). \tag{6.79}$$

We replace $c_F(W(n,\beta))|_{D_{n_1,\beta_1}}$ by (6.79) in (6.73). After using (6.78) to cancel the $c(\mathcal{O}_{D_{n_1,\beta_1}}(D_{n_1,\beta_1}))$ term, we obtain

$$\frac{1}{\langle\delta,\beta\rangle}\sum_{n_1,\beta_1}\int_{R(n_1,\beta_1)} s((\iota^*\mathcal{E}^\bullet)^\vee)\cap c_F(R(n_1,\beta_1))$$
$$\times\int_{W(n_2,\beta_2)}\Big(s((\pi^*\mathcal{F}^\bullet)^\vee)s\big(\mathcal{T}_\pi\big)\cap c_F(W(n_2,\beta_2))\Big)\cdot T_0(\delta).$$

(The integration over $W(n_2,\beta_2)$ is well defined since the insertions $\tau_0(\delta)$ yields a complete cycle as before.) Since the two obstruction theories $\mathcal{F}^\bullet$ and $\mathcal{F}^\bullet_{\mathrm{red}}$ on $W(n_2,\beta_2)$ differ only by the trivial line bundle,

$$s((\mathcal{F}^\bullet)^\vee) = s((\mathcal{F}^\bullet_{\mathrm{red}})^\vee).$$

By another application of the rigidification formula (6.68), we obtain

$$P^\star_{n,\beta}(X) - R^{\mathrm{red}}_{n,\beta}(S\times\mathcal{R}) = \frac{1}{\langle\delta,\beta\rangle}\sum_{n_1,\beta_1} P^\star_{n_1,\beta_1}(X)\cdot\langle\delta,\beta_2\rangle\, R^{\mathrm{red}}_{n_2,\beta_2}(S\times\mathcal{R}).$$

6.10. Logarithm. Let $\alpha \in \mathrm{Pic}(S)$ is a primitive class which is positive with respect to a polarization, and let $\beta \in \mathrm{Eff}(m\alpha)$. The only effective decompositions of β are

$$\beta = \beta_1 + \beta_2, \quad \beta_i \in \mathrm{Eff}(m\alpha).$$

We formulate the last equation of Section 6.9 as the following result.

THEOREM 33. *For the family $\epsilon : X \to (\Delta, 0)$ satisfying conditions* (i), (ii), ($\star$) *of Section 6.1 for $m\alpha$, we have*

$$P^\star_{n,\beta}(X) - R^{\mathrm{red}}_{n,\beta}(S\times\mathcal{R}) = \frac{1}{\langle\delta,\beta\rangle}\sum_{n_1+n_2=n}\sum_{\beta_1+\beta_2=\beta} P^\star_{n_1,\beta_1}(X)\cdot\langle\delta,\beta_2\rangle\, R^{\mathrm{red}}_{n_2,\beta_2}(S\times\mathcal{R})$$

for every $\beta \in \mathrm{Eff}(m\alpha)$ and $\delta \in H^2(S,\mathbb{Z})$ satisfying $\langle\delta,\beta\rangle \neq 0$.

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The basic relationship between $P^\star_{n,\beta}(X)$ and $R^{\mathrm{red}}_{n,\beta}(S\times\mathcal{R})$ is an immediate Corollary of Theorem 33. Let

$$P^\star_\beta(X)=\sum_{n\in\mathbb{Z}}P^\star_{n,\beta}(X)q^n,\quad R^{\mathrm{red}}_\beta(S\times\mathcal{R})=\sum_{n\in\mathbb{Z}}R^{\mathrm{red}}_{n,\beta}(S\times\mathcal{R})q^n.$$

COROLLARY 34. *For $\beta\in\mathrm{Eff}(m\alpha)$,*

$$P^\star_\beta(X)=\mathrm{Coeff}_{v^\beta}\left[\exp\left(\sum_{\widehat{\beta}\in\mathrm{Eff}(m\alpha)}v^{\widehat{\beta}}R^{\mathrm{red}}_{\widehat{\beta}}(S\times\mathcal{R})\right)\right].$$

Proof. Since $\delta\in H^2(S,\mathbb{Z})$ is arbitrary, Theorem 33 uniquely determines $P^\star_\beta(X)$ in terms of $R^{\mathrm{red}}_{\beta_i}(S\times\mathcal{R})$ for $\beta_i\in\mathrm{Eff}(m\alpha)$. We see Corollary 34 implies exactly the recursion of Theorem 33 by differentiating the exponential. To write the differentiation explicitly, let

$$v_1,\ldots,v_{22}\in H^2(S,\mathbb{Z})$$

be a basis. For $\beta=\sum_{i=1}^{22}b_iv_i$, we write

$$v^\beta=\prod_{i=1}^{22}v_i^{b_i}.$$

Let $\langle\delta,v_i\rangle=c_i$. Then, differentiation of the equation of Corollary 34 by

$$\sum_{i=1}^{22}c_iv_i\frac{\partial}{\partial v_i}$$

yields the recursion of Theorem 33. □

6.11. Definition of $\widetilde{R}_{n,\beta}$. We now return to the definition of stable pairs invariants for $K3$ surface given in Section 6.2.

Let $\alpha\in\mathrm{Pic}(S)$ be a primitive class which is positive with respect to a polarization. Let

$$\epsilon:X\to(\Delta,0)$$

satisfying the conditions of Section 6.1: (i), (ii), and ($\star$) for $m\alpha$. We have

$$\widetilde{R}_{m\alpha}(S)=\mathrm{Coeff}_{v^{m\alpha}}\left[\log\left(1+\sum_{\beta\in\mathrm{Eff}(m\alpha)}v^\beta P^\star_\beta(X)\right)\right].$$

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Let $\widetilde{R}_{n,m\alpha}(S)$ be the associated q coefficients:

$$\widetilde{R}_{m\alpha}(S) = \sum_{n\in\mathbb{Z}} \widetilde{R}_{n,m\alpha}(S)q^n.$$

By Corollary 34 of Theorem 33, we can take the logarithm.

PROPOSITION 35. *We have* $\widetilde{R}_{m\alpha}(S) = R^{\mathrm{red}}_{m\alpha}(S \times \mathcal{R})$.

Proof. By Corollary 34 and the above definition,

$$\begin{aligned}\widetilde{R}_{m\alpha}(S) &= \mathrm{Coeff}_{v^{m\alpha}}\left[\log\exp\left(\sum_{\widehat{\beta}\in\mathrm{Eff}(m\alpha)} v^{\widehat{\beta}} R^{\mathrm{red}}_{\widehat{\beta}}(S\times\mathcal{R})\right)\right]\\ &= \mathrm{Coeff}_{v^{m\alpha}}\left[\sum_{\widehat{\beta}\in\mathrm{Eff}(m\alpha)} v^{\widehat{\beta}} R^{\mathrm{red}}_{\widehat{\beta}}(S\times\mathcal{R})\right].\end{aligned}$$

Hence, if $m\alpha \in \mathrm{Pic}(S)$ is effective, $\widetilde{R}_{m\alpha}(S) = R^{\mathrm{red}}_{m\alpha}(S\times\mathcal{R})$. If $m\alpha \in \mathrm{Pic}(S)$ is not effective, both $\widetilde{R}_{m\alpha}(S)$ and $R^{\mathrm{red}}_{m\alpha}(S\times\mathcal{R})$ vanish. □

Proposition 35 is the main point of Section 6. The complete geometric interpretation of the logarithm as integration over a moduli of stable pairs is crucial for our proof of the KKV conjecture.

6.12. Dependence. Let $\alpha \in \mathrm{Pic}(S)$ be a primitive class which is positive with respect to a polarization. As a consequence of Proposition 35, we obtain a dependence result.

PROPOSITION 36. $\widetilde{R}_{m\alpha}(S)$ *depends* only *upon m and*

$$\langle\alpha,\alpha\rangle = 2h-2$$

and not *upon S or the family ϵ.*

From the definition of $\widetilde{R}_{m\alpha}(S)$, the dependence statement is not immediate (since the argument of the logarithm depends upon invariants of effective summands of $m\alpha$ which may or may not persist in deformations of S for which α stays algebraic). However, $R^{\mathrm{red}}_{m\alpha}(S\times\mathcal{R})$ depends only upon m and the deformation class of the pair (S,α) – and the latter depends only upon $\langle\alpha,\alpha\rangle$.

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We finally have a well-defined analogue in stable pairs of the Gromov–Witten invariants $R_{n,m\alpha}$. We drop S from the notation, and to make the dependence clear, we define

$$\widetilde{R}_{m,m^2(h-1)+1} = \widetilde{R}_{m\alpha} \tag{6.80}$$

where right side is obtained from *any* $K3$ surface and family ϵ satisfying conditions (i), (ii), and ($\star$) for $m\alpha$. Here, $m\alpha$ is of divisibility m and has norm square

$$\langle m\alpha, m\alpha\rangle = m^2(2h-2) = 2(m^2(h-1)+1)-2.$$

In case $m=1$, we will use the abbreviation

$$\widetilde{R}_h = \widetilde{R}_{1,h}. \tag{6.81}$$

7. Multiple covers

7.1. Overview. By Proposition 35, the stable pairs invariants $\widetilde{R}_{m\alpha}(S)$ equal the reduced rubber invariants $R^{\mathrm{red}}_{m\alpha}(S\times\mathcal{R})$. Our goal here is to express the latter in terms of the reduced residue invariants $\langle 1\rangle^{\mathrm{red}}_{Y,m\alpha}$ of

$$Y = S\times\mathbb{C}$$

studied in Sections 4 and 5. The explicit calculations of Section 5.6 then determine $\widetilde{R}_{m\alpha}(S)$ and can be interpreted in terms of multiple cover formulas.

7.2. Rigidification. Let S be a $K3$ surface equipped with an ample polarization L. Let $\alpha\in\mathrm{Pic}(S)$ be a primitive, positive class with norm square

$$\langle\alpha,\alpha\rangle = 2h-2.$$

(Positivity, $\langle L,\alpha\rangle>0$, is with respect to the polarization L.) Let $m>0$ be a integer. The invariants

$$R^{\mathrm{red}}_{m\alpha}(S\times\mathcal{R}) = \sum_{n\in\mathbb{Z}} R^{\mathrm{red}}_{n,m\alpha}(S\times\mathcal{R})q^n$$

have been defined in Section 6.5 by integration over the rubber moduli spaces $R(n,m\alpha)$.

Following the notation of Section 6.9, let

$$\pi : U(n,m\alpha) = \mathcal{S}\times_{\mathscr{B}_r} R(n,m\alpha)\to R(n,m\alpha)$$

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be the universal target with virtual class pulled back from $R(n, m\alpha)$. We also consider here

$$V(n, m\alpha/0, \infty) = P_n(S \times \mathbb{P}^1/S_0 \cup S_\infty, m\alpha),$$

the moduli space of stable pairs on $S \times \mathbb{P}^1$ relative to the fibers over $0, \infty \in \mathbb{P}^1$. There is a standard rigidification map

$$\rho : U(n, m\alpha) \to V(n, m\alpha/0, \infty)$$

with the point in universal target determining $1 \in \mathbb{P}^1$. (We identify the point in the universal target $\mathcal{S}$ with the corresponding distinguished point of $S \times \mathbb{P}^1/S_0 \cup S_\infty$ lying over $1 \in \mathbb{P}^1$. The resulting identification of the two universal targets (for the rubber and for $S \times \mathbb{P}^1/S_0 \cup S_\infty$) is used to transport stable pairs from the former to the latter.)

As in Section 6.9, let $T_0(L)$ be the primary insertion in the rubber theory obtained from $L \in H^2(S, \mathbb{Z})$. By the divisor property (6.68),

$$m\langle L, \alpha\rangle \, R^{\mathrm{red}}_{m\alpha}(S \times \mathcal{R}) = \sum_n q^n \int_{[U(n,m\alpha)]^{\mathrm{red}}} T_0(L).$$

By rigidification, we find

$$\sum_n q^n \int_{[U(n,m\alpha)]^{\mathrm{red}}} T_0(L) = \sum_n q^n \int_{[V(n,m\alpha/0,\infty)]^{\mathrm{red}}} T_0(\mathcal{L})$$

where $\mathcal{L} \in H^4(S \times \mathbb{P}^1, \mathbb{Z})$ is dual to the cycle

$$\ell \times \{1\} \subset S \times \mathbb{P}^1$$

and $\ell \subset S$ represents $L \cap [S] \in H_2(S, \mathbb{Z})$. We define

$$V^{\mathrm{red}}_{m\alpha}(S \times \mathbb{P}^1/S_0 \cup S_\infty) = \sum_n q^n \int_{[V(n,m\alpha/0,\infty)]^{\mathrm{red}}} T_0(\mathcal{L})$$

and conclude

$$m\langle L, \alpha\rangle \, R^{\mathrm{red}}_{m\alpha}(S \times \mathcal{R}) = V^{\mathrm{red}}_{m\alpha}(S \times \mathbb{P}^1/S_0 \cup S_\infty). \tag{7.1}$$

7.3. Degeneration. We now study $V^{\mathrm{red}}_{m\alpha}(S \times \mathbb{P}^1/S_0 \cup S_\infty)$ on the right side of (7.1) via the degeneration formula.

We consider the degeneration of the relative geometry $S \times \mathbb{P}^1/S_\infty$ to the normal cone of

$$S_0 \subset S \times \mathbb{P}^1.$$

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The degeneration formula for the virtual class of the moduli of stable pairs under

$$S \times \mathbb{P}^1/S_\infty \rightsquigarrow S_0 \times \mathbb{P}^1/S_0 \bigcup S \times \mathbb{P}^1/S_0 \cup S_\infty \tag{7.2}$$

is easily seen to be compatible with the reduced class (since all the geometries project to S). Let

$$V(n, m\alpha/\infty) = P_n(S \times \mathbb{P}^1/S_\infty, m\alpha),$$
$$V^{\mathrm{red}}_{m\alpha}(S \times \mathbb{P}^1/S_\infty) = \sum_n q^n \int_{[V(n,m\alpha/\infty)]^{\mathrm{red}}} T_0(\mathcal{L})$$

for the insertion $T_0(\mathcal{L})$ defined in Section 7.2.

PROPOSITION 37. *We have*

$$V^{\mathrm{red}}_{m\alpha}(S \times \mathbb{P}^1/S_\infty) = V^{\mathrm{red}}_{m\alpha}(S \times \mathbb{P}^1/S_0 \cup S_\infty).$$

Proof. By the degeneration formula, the reduced virtual class of the moduli space $V(n, m\alpha/\infty)$ distributes to the products

$$V(n_1, \beta_1/\infty) \times V(n_2, \beta/0, \infty), \quad n_1 + n_2 = n,\ \beta_1 + \beta_2 = m\alpha, \tag{7.3}$$

associated to the reducible target (7.2). If β_1 and β_2 are *both* nonzero, then the product (7.3) admits a double reduction of the standard virtual class. Hence, the (singly) reduced virtual class of (7.3) vanishes unless β_1 or β_2 is 0.

When the degeneration formula is applied to $V^{\mathrm{red}}_{m\alpha}(S \times \mathbb{P}^1/S_\infty)$, the insertion $T_0(\mathcal{L})$ requires β_2 to be nonzero. Hence, only the $n_2 = n$ and $\beta_2 = m\alpha$ term contributes to the reduced degeneration formula. □

7.4. Localization. The next step is to apply $\mathbb{C}^*$-equivariant localization to $V^{\mathrm{red}}_{m\alpha}(S \times \mathbb{P}^1/S_\infty)$.

Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ with tangent weight $\mathfrak{t}$ at 0. We lift the class $\mathcal{L}$ to $\mathbb{C}^*$-equivariant cohomology by selecting the $\mathbb{C}^*$-fixed representative

$$L \times \{0\} \subset S \times \mathbb{P}^1.$$

The virtual localization formula expresses $V^{\mathrm{red}}_{m\alpha}(S \times \mathbb{P}^1/S_\infty)$ as a sum over products of residue contributions over S_0 and S_∞ in $S \times \mathbb{P}^1/S_\infty$. The $\mathbb{C}^*$-fixed loci admit a double reduction of their virtual class unless $m\alpha$ is distributed entirely to 0 or ∞. Since the insertion $T_0(\mathcal{L})$ requires the distribution to be over 0, we conclude

$$\begin{aligned} V^{\mathrm{red}}_{m\alpha}(S \times \mathbb{P}^1/S_\infty) &= \langle T_0(\mathcal{L})\rangle^{\mathrm{red}}_{Y,m\alpha}(q) \\ &= m\mathfrak{t}\langle L, \alpha\rangle\, \langle 1\rangle^{\mathrm{red}}_{Y,m\alpha}(q), \end{aligned} \tag{7.4}$$

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where we have followed the notation of Section 5.6 except for writing $\langle 1\rangle^{\mathrm{red}}_{Y,m\alpha}$ as a series in q instead of y.

7.5. Multiple cover formula. Sections 7.2–7.4 together with Proposition 23 imply

$$\begin{aligned} R^{\mathrm{red}}_{m\alpha}(S\times\mathcal{R}) &= t\langle 1\rangle^{\mathrm{red}}_{Y,m\alpha}(q)\\ &= \sum_{k|m}\frac{t}{k}I_{(m^2/k^2)(h-1)+1}(-(-q)^k)\end{aligned}\tag{7.5}$$

where $\langle\alpha,\alpha\rangle = 2h-2$. By the formulas of Section 5.7, $tI(q)$ is a rational function of q. (The variable q here is the variable y in Section 5.7.) Hence, $R^{\mathrm{red}}_{m\alpha}(S\times\mathcal{R})$ is a rational function of q.

We define R^{red}_h to equal the generating series $R^{\mathrm{red}}_\alpha(S\times\mathcal{R})$ associated to a primitive and positive class α with norm square $2h-2$. By (7.5) in the $m=1$ case,

$$R^{\mathrm{red}}_h = tI_h(q),$$

so $R^{\mathrm{red}}_h = 0$ for $h<0$ by Proposition 22. Rewriting (7.5), we obtain the fundamental multiple cover formula governing $R^{\mathrm{red}}_{m\alpha}$:

$$R^{\mathrm{red}}_{m\alpha}(q) = \sum_{k|m}\frac{1}{k}R^{\mathrm{red}}_{(m^2/k^2)(h-1)+1}(-(-q)^k).\tag{7.6}$$

The terms on the right correspond to lower-degree primitive contributions to $m\alpha$.

Finally, we write the multiple cover formula in terms of the invariants $\widetilde{R}_{m\alpha}$. Following the notation (6.81), let $\widetilde{R}_h$ equal the series $\widetilde{R}_\alpha(S)$ associated to a primitive and positive class α with norm square $2h-2$. By Proposition 35, we see

$$\widetilde{R}_h = 0 \quad \text{for } h<0.\tag{7.7}$$

The multiple cover formula (7.6) implies the following result.

THEOREM 38. *The series $\widetilde{R}_{m\alpha}(q)$ is the Laurent expansion of a rational function of q, and*

$$\widetilde{R}_{m\alpha}(q) = \sum_{k|m}\frac{1}{k}\widetilde{R}_{(m^2/k^2)(h-1)+1}(-(-q)^k).$$

7.6. Stable pairs BPS counts. The stable pairs potential $\widetilde{F}_\alpha(q,v)$ for classes proportional to α is

$$\widetilde{F}_\alpha = \sum_{n\in\mathbb{Z}}\sum_{m>0}\widetilde{R}_{n,m\alpha}q^n v^{m\alpha}.$$

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The stable pairs BPS counts $\widetilde{r}_{g,m\alpha}$ are uniquely defined [40] by:

$$\widetilde{F}_\alpha = \sum_{g\in\mathbb{Z}} \sum_{m>0} \widetilde{r}_{g,m\alpha} \sum_{d>0} \frac{(-1)^{g-1}}{d}((-q)^d - 2 + (-q)^{-d})^{g-1} v^{dm\alpha}.$$

Because $\widetilde{R}_{m\alpha}$ is a Laurent series in q, we see

$$\widetilde{r}_{g,m\alpha} = 0$$

for sufficiently high g and fixed m. In the primitive case,

$$\widetilde{r}_{g,\alpha} = 0 \quad \text{for } g < 0 \tag{7.8}$$

by [41, Equation (2.10)].

Since we know $\widetilde{R}_{m\alpha}$ only depends upon m and the norm square $\langle m\alpha, m\alpha\rangle$, the same is true for the associated BPS counts. Following the notation (6.80), we define

$$\widetilde{r}_{g,m,m^2(h-1)+1} = \widetilde{r}_{g,m\alpha}.$$

PROPOSITION 39. *The stable pairs BPS counts do not depend upon the divisibility:*

$$\widetilde{r}_{g,m,m^2(h-1)+1} = \widetilde{r}_{g,1,m^2(h-1)+1}.$$

Proof. By Theorem 38, we can write $\widetilde{F}_\alpha$ as

$$\begin{aligned}\widetilde{F}_\alpha &= \sum_{m>0} \widetilde{R}_{m\alpha} v^{m\alpha} \\ &= \sum_{m>0} \sum_{k|m} \frac{1}{k} \widetilde{R}_{(m^2/k^2)(h-1)+1}(-(-q)^k) v^{m\alpha}.\end{aligned}$$

Next, using the definition of BPS counts for primitive classes, we find

$$\widetilde{F}_\alpha = \sum_{m>0} \sum_{k|m} \sum_{g\geqslant 0} \frac{(-1)^{g-1}}{k} \widetilde{r}_{g,1,(m^2/k^2)(h-1)+1}((-q)^k - 2 + (-q)^{-k})^{g-1} v^{m\alpha}.$$

After a reindexing of the summation on the right, we obtain

$$\sum_{g\geqslant 0} \sum_{m>0} \widetilde{r}_{g,1,m^2(h-1)+1} \sum_{d>0} \frac{(-1)^{g-1}}{d}((-q)^d - 2 + (-q)^{-d})^{g-1} v^{dm\alpha}.$$

By the definition of the BPS counts and the uniqueness statement, we conclude the $\widetilde{r}_{g,m,m^2(h-1)+1}$ does not depend upon the divisibility. □

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As a corollary of Proposition 39, we obtain basic properties of $\widetilde{r}_{g,m,h}$ required in Section 3.

COROLLARY 40. *We have* $\widetilde{r}_{g,m,h\leqslant 0} = 0$ *except for the case*

$$\widetilde{r}_{0,1,0} = 1.$$

Proof. By Proposition 39, we need only consider the $m = 1$ case. If $h < 0$, the vanishing follows from (7.7). If $h = 0$, the result is the consequence of the stable pair calculation of the conifold [40]. □

COROLLARY 41. *We have* $\widetilde{r}_{g<0,m,h} = 0$.

Proof. After reducing to the $m = 1$ case, the result is (7.8). □

8. P/NL correspondence

8.1. Overview. Our goal here is to prove the Pairs/Noether–Lefschetz correspondence of Theorem 8 in Section 3.5. The main tool needed is Theorem 33.

Following the definitions of Sections 1.2 and 2.1, let

$$(\pi_3 : \mathsf{X} \to \mathbb{P}^1, L_1, L_2, E)$$

be the 1-parameter family of Λ-polarized,

$$\Lambda = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$K3$ surfaces obtained from a very general anticanonical Calabi–Yau hypersurface,

$$\mathsf{X} \subset \widetilde{\mathbb{P}^2 \times \mathbb{P}^1} \times \mathbb{P}^1.$$

(Very general here is the complement of a countable set.) For a very general fiber of the base $\xi \in \mathbb{P}^1$,

$$\mathrm{Pic}(\mathsf{X}_\xi) \cong \Lambda. \tag{8.1}$$

We also assume, for each nodal fiber X_ξ, the $K3$ resolution $\widetilde{\mathsf{X}}_\xi$ satisfies

$$\mathrm{Pic}(\widetilde{\mathsf{X}}_\xi) \cong \Lambda \oplus \mathbb{Z}[\widetilde{E}] \tag{8.2}$$

where $\widetilde{E} \subset \widetilde{\mathsf{X}}_\xi$ is the exceptional -2 curve. (The nodal fibers have exactly 1 node.) Both (8.1) and (8.2) can be satisfied since Λ is the Picard lattice of

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a very general point of $\mathcal{M}_\Lambda$ and $\Lambda \oplus \mathbb{Z}[\widetilde{E}]$ is the Picard lattice of a very general point of the nodal locus in $\mathcal{M}_\Lambda$. (We leave these standard facts about $K3$ surfaces of type Λ to the reader. Huybrechts [19] is an excellent source for the study of $K3$ surfaces.)

The stable pairs potential $\widetilde{F}^{\mathsf{X}}$ for nonzero vertical classes is the series

$$\widetilde{F}^{\mathsf{X}} = \log\left(1 + \sum_{0 \neq \gamma \in H_2(\mathsf{X},\mathbb{Z})^{\pi_3}} \mathsf{Z}_{\mathsf{P}}(\mathsf{X}; q)_\gamma v^\gamma\right) = \sum_{n \in \mathbb{Z}} \sum_{0 \neq \gamma \in H_2(\mathsf{X},\mathbb{Z})^{\pi_3}} \widetilde{N}^{\mathsf{X}}_{n,\gamma} q^n v^\gamma.$$

Here, v is the curve class variable, and the second equality defines the connected stable pairs invariants $\widetilde{N}^{\mathsf{X}}_{n,\gamma}$. The stable pairs BPS counts $\widetilde{n}^{\mathsf{X}}_{g,\gamma}$ are then defined by

$$\widetilde{F}^{\mathsf{X}} = \sum_{g \in \mathbb{Z}} \sum_{0 \neq \gamma \in H_2(\mathsf{X},\mathbb{Z})^{\pi_3}} \widetilde{n}^{\mathsf{X}}_{g,\gamma} \sum_{d>0} \frac{(-1)^{g-1}}{d}((-q)^d - 2 + (-q)^{-d})^{g-1} v^{d\gamma}.$$

Let $\widetilde{n}^{\mathsf{X}}_{g,(d_1,d_2,d_3)}$ denote the stable pairs BPS invariant of X in genus g for π_3-vertical curve classes of degrees d_1, d_2, d_3 with respect to the line bundles L_1, L_2, $E \in \Lambda$, respectively. Let $\widetilde{r}_{g,m,h}$ be the stable pairs BPS counts associated to $K3$ surfaces in Section 3.4. The Pairs/Noether–Lefschetz correspondence is the following result.

THEOREM 4. *For degrees (d_1, d_2, d_3) positive with respect to the quasipolarization,*

$$\widetilde{n}^{\mathsf{X}}_{g,(d_1,d_2,d_3)} = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} \widetilde{r}_{g,m,h} \cdot NL^{\pi_3}_{m,h,(d_1,d_2,d_3)}.$$

8.2. Strategy of proof. Since the formulas relating the BPS counts to stable pairs invariants are the same for X and the $K3$ surface, Theorem 8 is equivalent to the analogous stable pairs statement:

$$\widetilde{N}^{\mathsf{X}}_{n,(d_1,d_2,d_3)} = \sum_{h} \sum_{m=1}^{\infty} \widetilde{R}_{n,m,h} \cdot NL^{\pi_3}_{m,h,(d_1,d_2,d_3)} \tag{8.3}$$

for degrees (d_1, d_2, d_3) positive with respect to the quasipolarization of X.

We denote the base of the fibration π_3 by $C \cong \mathbb{P}^1$. Let $C^\circ \subset C$ be the locus over which

$$\pi_3 : \mathsf{X} \to C$$

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is smooth. By condition (i) of Section 1.2 for a 1-parameter family of Λ-polarized $K3$ surfaces, the complement of C° consists of finitely many points over which each fiber of π_3 has a single ordinary node. For each $\xi \in C^\circ$, let

$$\mathcal{V}_\xi = H^2(\mathsf{X}_\xi, \mathbb{Z}).$$

As $\xi \in C^\circ$ varies, the fibers $\mathcal{V}_\xi$ determine a local system

$$\mathcal{V}^\circ \to C^\circ.$$

We denote the effective divisor classes on X_ξ by

$$\mathrm{Eff}_\xi = \{\beta \in \mathcal{V}_\xi \mid \beta \in \mathrm{Pic}(\mathsf{X}_\xi) \text{ and } \beta \text{ effective}\}.$$

For $\xi \in C \setminus C^\circ$, we denote the $K3$ resolution of singularities of the node of X_ξ by

$$\rho : \widetilde{\mathsf{X}}_\xi \to \mathsf{X}_\xi,$$

and we define

$$\mathcal{V}_\xi = H^2(\widetilde{\mathsf{X}}_\xi, \mathbb{Z}).$$

As before, let

$$\mathrm{Eff}_\xi = \{\beta \in \mathcal{V}_\xi \mid \beta \in \mathrm{Pic}(\widetilde{\mathsf{X}}_\xi) \text{ and } \beta \text{ effective}\}.$$

The push-forward of a divisor class on $\widetilde{\mathsf{X}}_\xi$ to X_ξ can be considered in $H_2(\mathsf{X}_\xi, \mathbb{Z})$ and, by Poincaré duality (for the quotient singularity), in

$$\tfrac{1}{2}H^2(\mathsf{X}_\xi, \mathbb{Z}) \subset H^2(\mathsf{X}_\xi, \mathbb{Q}).$$

We view the push-forward by ρ of the effective divisors classes as:

$$\rho_* : \mathrm{Eff}_\xi \to \tfrac{1}{2}H^2(\mathsf{X}_\xi, \mathbb{Z}). \tag{8.4}$$

We will study the contributions of the classes Eff_ξ to both sides of (8.3). Certainly, only effective curves contribute to the left side of (8.3). Suppose $\xi \in C$ lies on the Noether–Lefschetz divisor

$$D_{m,h,(d_1,d_2,d_3)} \subset \mathcal{M}_\Lambda.$$

(We follow the notation of Section 1.2.1.) Then there exists $\beta \in \mathrm{Pic}(\mathsf{X}_\xi)$ if $\xi \in C^\circ$ (or $\beta \in \mathrm{Pic}(\widetilde{\mathsf{X}}_\xi)$ if $\xi \in C \setminus C^\circ$) of divisibility m,

$$2h - 2 = \langle \beta, \beta \rangle,$$

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and degree (d_1, d_2, d_3) positive with respect to the quasipolarization. Let $\alpha = (1/m)\beta$ be the corresponding primitive class. If β is not effective on X_ξ (or $\widetilde{\mathsf{X}}_\xi$ if $\xi \in C \setminus C^\circ$), then α is also not effective. Since α is positive with respect to the quasipolarization, Riemann–Roch implies $h_\alpha < 0$, where

$$2h_\alpha - 2 = \langle \alpha, \alpha \rangle.$$

By Theorem 38 and the vanishing (7.7),

$$\widetilde{R}_{n,m,h} = 0.$$

Ineffective classes β therefore do not contribute to the right side of (8.3).

8.3. Isolated contributions. We consider first the simplest contributions. Let $\xi \in C^\circ$. A nonzero effective class $\beta \in \mathrm{Pic}(\mathsf{X}_\xi)$ is *completely isolated* on X if the following property holds:

($\star\star$) for every *effective* decomposition

$$\beta = \sum_{i=1}^{l} \gamma_i \in \mathrm{Pic}(\mathsf{X}_\xi),$$

the local Noether–Lefschetz locus $\mathrm{NL}(\gamma_i) \subset C$ corresponding to each class $\gamma_i \in \mathrm{Pic}(\mathsf{X}_\xi)$ contains ξ as an isolated point. (Nonreduced structure is allowed at ξ.)

Let $\gamma \in \mathrm{Pic}(\mathsf{X}_\xi)$ be an effective summand of β which occurs in condition ($\star\star$). The stable pairs with set-theoretic support on X_ξ form an open and closed component,

$$P_n^\xi(\mathsf{X}, \gamma) \subset P_n(\mathsf{X}, \gamma)$$

of the moduli space of stable pairs for every n. (As usual, we denote the push-forward of γ to $H_2(\mathsf{X}, \mathbb{Z})$ also by γ.)

We consider now the contribution of X_ξ for $\xi \in C^\circ$ to the stable pairs series $\widetilde{F}^{\mathsf{X}}$ of a nonzero effective and completely isolated class $\beta \in H^2(\mathsf{X}_\xi, \mathbb{Z})$ satisfying

$$\mathrm{div}(\beta) = m, \quad \langle \beta, \beta \rangle = 2h - 2$$

and of degree $\mathbf{d}$ with respect to Λ. More precisely, let $\mathrm{Cont}(\mathsf{X}_\xi, \beta, \widetilde{N}^{\mathsf{X}}_{n,\mathbf{d}})$ be the contribution corresponding to β of all the moduli of stable pairs with set-theoretic support on X_ξ:

$$\mathrm{Cont}(\mathsf{X}_\xi, \beta, \widetilde{N}^{\mathsf{X}}_{n,\mathbf{d}}) = \mathrm{Coeff}_{q^n v^\beta}\left[\log\left(1 + \sum_{n\in\mathbb{Z}} \sum_{\gamma \in \mathrm{Eff}(\beta)} q^n v^\gamma \int_{[P_n^\xi(\mathsf{X},\gamma)]^{\mathrm{vir}}} 1\right)\right], \tag{8.5}$$

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where $\mathrm{Eff}_\xi(\beta) \subset \mathrm{Pic}(\mathsf{X}_\xi)$ is the subset of effective summands of β. By condition ($\star\star$), the contribution is *well defined*.

Let ℓ_β be the length of the local Noether–Lefschetz locus $\mathrm{NL}(\beta) \subset C$ at $\xi \in C$. We define

$$\mathrm{Cont}(\mathsf{X}_\xi, \beta, NL^{\pi_3}_{m,h,\mathbf{d}}) = \ell_\beta,$$

the local intersection contribution to the Noether–Lefschetz number.

PROPOSITION 42. *For a completely isolated effective class $\beta \in \mathrm{Pic}(\mathsf{X}_\xi)$ with $\xi \in C^\circ$, we have*

$$\mathrm{Cont}(\mathsf{X}_\xi, \beta, \widetilde{N}^{\mathsf{X}}_{n,\mathbf{d}}) = \widetilde{R}_{n,m,h} \cdot \mathrm{Cont}(\mathsf{X}_\xi, \beta, NL^{\pi_3}_{m,h,\mathbf{d}}).$$

Proof. We perturb the family C locally near ξ to be transverse to *all* the local Noether–Lefschetz loci corresponding to effective summands of β on X_ξ. In order to perturb in algebraic geometry, we first approximate C by C' near $[\mathsf{X}_\xi]$ to sufficiently high order by a moving family of curves in the moduli space of Λ-polarized $K3$ surfaces. (The order should be high enough to obstruct all the deformations away from X_ξ of the stable pairs occurring on the right side of (8.5).) Then, we perturb the resulting moving curve C' to C'' to achieve the desired transversality. Since transversality is a generic condition, we may take the perturbation C'' to be as small as necessary. Let

$$\pi : \mathsf{X}'' \to C''$$

be the family of Λ-polarized $K3$ surfaces determined by C''.

Near $[\mathsf{X}_\xi]$ in the moduli of Λ-polarized $K3$ surfaces, the local system of second cohomologies is trivial. In a contractible neighborhood U of $[\mathsf{X}_\xi]$, we have a canonical isomorphism

$$H^2(S, \mathbb{Z}) \cong H^2(\mathsf{X}_\xi, \mathbb{Z}) \tag{8.6}$$

for all $[S] \in U$. We will use the identification (8.6) when discussing $H^2(\mathsf{X}_\xi, \mathbb{Z})$.

By deformation invariance of the stable pairs theory, the contribution (8.5) can be calculated after perturbation. We assume all the intersections of C'' with the above local Noether–Lefschetz loci which have limits tending to $[\mathsf{X}_\xi]$ (as C'' tends to C) lie in U.

For every $\gamma \in \mathrm{Eff}_\xi(\beta)$, the curve C'' intersects the local Noether–Lefschetz loci associated to γ transversely at a finite set of reduced points

$$I_\gamma \subset C''$$

near $[\mathsf{X}_\xi]$. The local intersection number at $\xi \in C$ of C with the local Noether–Lefschetz locus corresponding to γ is $|I_\gamma|$. Let $\xi_\gamma \in I_\gamma$ be one such point of intersection. Let m_γ be the divisibility of γ.

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On the $K3$ surface $\mathsf{X}''_{\xi_\gamma}$, the class $(1/m_\gamma)\gamma$ is primitive and positive since γ is positive on X_ξ. Moreover, the family C'' is m_γ-rigid for $(1/m_\gamma)\gamma$ on $\mathsf{X}''_{\xi_\gamma}$ since the effective summands of γ on $\mathsf{X}''_{\xi_\gamma}$ all lie in $\mathrm{Eff}_\xi(\beta)$. By upper semicontinuity, no more effective summands of β can appear near $[\mathsf{X}_\xi]$ in the moduli of $K3$ surfaces.

By Corollary 34 to Theorem 33, we have a formula for the stable pairs contribution of γ at $\mathsf{X}''_{\xi_\gamma}$,

$$P^{\star,\xi_\gamma}_{\gamma}(\mathsf{X}'') = \mathrm{Coeff}_{v^\gamma}\left[\exp\left(\sum_{\widehat{\gamma}\in \mathrm{Eff}_{\xi_\gamma}(\gamma)} v^{\widehat{\gamma}} R^{\mathrm{red}}_{\widehat{\gamma}}(\mathsf{X}''_{\xi_\gamma}\times \mathcal{R})\right)\right].$$

Here, $\mathrm{Eff}_{\xi_\gamma}(\gamma) \subset \mathrm{Pic}(\mathsf{X}''_{\xi_\gamma})$ is the subset of effective summands of γ (which is empty if γ is not effective).

Finally, consider the original contribution $\mathrm{Cont}(\mathsf{X}_\xi, \beta, \widetilde{N}^{\mathsf{X}}_{n,\mathbf{d}})$. Let

$$I \subset C''$$

be the union of all the I_γ for $\gamma \in \mathrm{Eff}_\xi(\beta)$. For each $\widehat{\xi} \in I$, let

$$\mathrm{Alg}_{\widehat{\xi},\xi}(\beta) \subset \mathrm{Eff}_\xi(\beta)$$

be the subset of classes of $\mathrm{Eff}_\xi(\beta)$ which are algebraic on $\mathsf{X}''_{\widehat{\xi}}$. All elements of $\mathrm{Alg}_{\widehat{\xi},\xi}(\beta)$ are positive. By semicontinuity

$$\mathrm{Eff}_{\widehat{\xi}}(\gamma) \subset \mathrm{Alg}_{\widehat{\xi},\xi}(\beta)$$

for all $\gamma \in \mathrm{Eff}_\xi(\beta)$. After perturbation, our formula for (8.5) is

$$\mathrm{Coeff}_{q^n v^\beta}\left[\log\left(\prod_{\widehat{\xi}\in I}\exp\left(\sum_{\gamma\in \mathrm{Alg}_{\widehat{\xi},\xi}(\beta)} v^{\gamma} R^{\mathrm{red}}_{\gamma}(\mathsf{X}''_{\widehat{\xi}}\times \mathcal{R})\right)\right)\right].$$

After taking the logarithm of the exponential, we have

$$\mathrm{Cont}(\mathsf{X}_\xi, \beta, \widetilde{N}^{\mathsf{X}}_{n,\mathbf{d}}) = \mathrm{Coeff}_{q^n v^\beta}\left[\sum_{\widehat{\xi}\in I}\ \sum_{\gamma\in \mathrm{Alg}_{\widehat{\xi},\xi}(\beta)} v^{\gamma} R^{\mathrm{red}}_{\gamma}(\mathsf{X}''_{\widehat{\xi}}\times \mathcal{R})\right].$$

Hence, we see only the $\xi \in I$ for which $\beta \in \mathrm{Alg}_{\widehat{\xi},\xi}(\beta)$ contribute. We conclude

$$\begin{aligned}\mathrm{Cont}(\mathsf{X}_\xi, \beta, \widetilde{N}^{\mathsf{X}}_{n,\mathbf{d}}) &= \sum_{\widehat{\xi}\in I_\beta}\mathrm{Coeff}_{q^n}\left[R^{\mathrm{red}}_{\beta}(\mathsf{X}''_{\widehat{\xi}}\times \mathcal{R})\right]\\ &= \widetilde{R}_{n,m,h}\cdot |I_\beta|,\end{aligned}$$

where the second equality uses Proposition 35.

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Since $\mathrm{Cont}(\mathsf{X}_\xi, \beta, NL^{\pi}_{m,h,\mathbf{d}}) = |I_\beta|$ is the local contribution of β to the Noether–Lefschetz number, the Proposition is established. $\square$

We have proven Proposition 42 for our original family π_3 of Λ-polarized $K3$ surfaces. Definition (⋆⋆) of a *completely isolated* class is valid for any family

$$\pi : X \to C \tag{8.7}$$

of lattice polarized $K3$ surfaces. By the proof given, Proposition 42 is valid for the contributions of every completely isolated class

$$\beta \in H^2(X_\xi, \mathbb{Z}), \quad \xi \in C^\circ$$

for any family (8.7). For different lattices $\widehat{\Lambda}$, the degree index $\mathbf{d}$ in Proposition 42 is replaced by the degree with respect to a basis of $\widehat{\Lambda}$.

8.4. Sublattice $\widehat{\Lambda} \subset \Lambda$.

For $\ell > 0$, we define

$$\mathrm{Eff}_{\mathsf{X}}(\lambda^\pi, \ell) \in H_2(\mathsf{X}, \mathbb{Z})^\pi$$

to be the set of classes of degree at most ℓ (with respect to the quasipolarization λ^π) which are represented by algebraic curves on X. By the boundedness of the Chow variety of curves of X of degree at most ℓ, $\mathrm{Eff}_{\mathsf{X}}(\lambda^\pi, \ell)$ is a *finite* set.

For any quasipolarization δ given by an ample class of X in Λ, let the set of effective classes of degree at most ℓ (with respect to δ) be

$$\mathrm{Eff}_{\mathsf{X}}(\delta, \ell) \in H_2(\mathsf{X}, \mathbb{Z})^\pi.$$

Similarly, let

$$\mathrm{Eff}_\xi(\delta, \ell) \subset \mathrm{Eff}_\xi$$

be the effective curve classes of δ-degree at most ℓ over $\xi \in C$. (We follow the notation of Section 8.2.)

We select a primitive sublattice $\widehat{\Lambda} = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \subset \Lambda$ satisfying the following properties:

(i) δ_1 is a quasipolarization with

$$L = \mathrm{Max}\{\langle \delta_1, \gamma \rangle \mid \gamma \in \mathrm{Eff}_{\mathsf{X}}(\lambda^\pi, \ell)\};$$

(ii) $\widehat{\Lambda} \cap \mathrm{Eff}_\xi(\delta_1, L) = \emptyset$ for all $\xi \in C^\circ$;

(iii) $\frac{1}{2}\widehat{\Lambda} \cap \rho_* \mathrm{Eff}_\xi(\delta_1, L) = \emptyset$ for all $\xi \in C \setminus C^\circ$;

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(iv) the intersection maps

$$\begin{aligned} \mathrm{Eff}_X(\delta_1, L) &\to \mathrm{Hom}(\Lambda, \mathbb{Z}), \quad \gamma \mapsto \langle \star, \gamma \rangle, \\ \mathrm{Eff}_X(\delta_1, L) &\to \mathrm{Hom}(\widehat{\Lambda}, \mathbb{Z}), \quad \gamma \mapsto \langle \star, \gamma \rangle \end{aligned}$$

have images of the same cardinality.

We will find such $\delta_1, \delta_2 \in \Lambda$ by a method explained below: we will first select δ_1 and then δ_2.

Consider the quasipolarization $\widehat{\delta} = k\lambda^{\pi_3}$ for $k > 2\ell$. Then $\frac{1}{2}\widehat{\delta}$ has $\widehat{\delta}$-degree $(k^2/2)\langle \lambda^{\pi_3}, \lambda^{\pi_3} \rangle$ on the $K3$ fibers while

$$\mathrm{Max}\{\langle \widehat{\delta}, \gamma \rangle \mid \gamma \in \mathrm{Eff}_X(\lambda^{\pi}, \ell)\} \leqslant k\ell < \frac{k^2}{2} \leqslant \frac{k^2}{2}\langle \lambda^{\pi_3}, \lambda^{\pi_3} \rangle.$$

We choose δ_1 to be a small shift of $k\lambda^{\pi_3}$ in the lattice Λ to ensure primitivity. Hence, we have met the conditions

$$\mathbb{Z}\delta_1 \cap \mathrm{Eff}_\xi(\delta_1, L) = \emptyset, \quad \tfrac{1}{2}\mathbb{Z}\delta_1 \cap \rho_* \mathrm{Eff}_\xi(\delta_1, L) = \emptyset \tag{8.8}$$

for $\xi \in C^\circ$ and $\xi \in C \setminus C^\circ$, respectively. Conditions (8.8) are required for (ii) and (iii).

Again by the boundedness of the Chow variety of curves in X, the following subset of $\frac{1}{2}\Lambda$ is a finite set:

$$\bigcup_{\xi \in C^\circ} \Lambda \cap \mathrm{Eff}_\xi(\delta_1, L) \cup \bigcup_{\xi \in C \setminus C^\circ} \frac{1}{2}\Lambda \cap \rho_* \mathrm{Eff}_\xi(\delta_1, L) \subset \frac{1}{2}\Lambda.$$

Since the rank of Λ is 3, we can easily find $\delta_2 \in \Lambda$ such that

$$\mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \subset \Lambda$$

is primitive and conditions (ii) and (iii) are satisfied.

Finally, since $\mathrm{Eff}_X(\delta_1, L)$ is a finite set, condition (iv) is easily satisfied when δ_2 is selected as above. Since

$$\mathrm{Eff}_X(\delta_1, L) \to \mathrm{Hom}(\Lambda, \mathbb{Z}) \to \mathrm{Hom}(\widehat{\Lambda}, \mathbb{Z}),$$

condition (iv) states there is no loss of information for a class in $\mathrm{Eff}_X(\delta_1, L)$ in taking degrees in $\widehat{\Lambda}$ instead of Λ.

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8.5. Nonisolated contributions: smoothing. Let $\widehat{\Lambda} \subset \Lambda$ be the rank 2 lattice selected in Section 8.4. We consider here the moduli space of $\widehat{\Lambda}$-polarized $K3$ surfaces which is 1 dimension larger than the moduli of Λ-polarized $K3$ surfaces,

$$\mathcal{M}_\Lambda \subset \mathcal{M}_{\widehat{\Lambda}}.$$

To the curve $C \subset \mathcal{M}_\Lambda$, we can attach a nonsingular complete curve $C' \subset \mathcal{M}_{\widehat{\Lambda}}$ satisfying the following properties:

(1) $C \cup C'$ is a connected nodal curve with nodes occurring at very general points of C;

(2) C' does not lie in any of the finitely many Noether–Lefschetz divisors of $\mathcal{M}_{\widehat{\Lambda}}$ determined by

$$\Lambda \cap \mathrm{Eff}_\xi(\delta_1, L) \quad \text{for } \xi \in C,$$

and C' is transverse to the Noether–Lefschetz divisor of nodal $K3$ surfaces;

(3) $C \cup C'$ smooths in $\mathcal{M}_{\widehat{\Lambda}}$ to a nonsingular curve

$$C'' \subset \mathcal{M}_{\widehat{\Lambda}}$$

which also does not lie in any of the finitely many Noether–Lefschetz divisors listed in (2) and is also transverse to the nodal divisor.

Let $\xi \in C$ and let $\beta \in \Lambda \cap \mathrm{Eff}_\xi(\delta_1, L)$. By the construction of $\widehat{\Lambda}$ in Section 8.4, $\beta \notin \widehat{\Lambda}$. Let

$$e_1 = \langle \delta_1, \beta \rangle, \quad e_2 = \langle \delta_2, \beta \rangle, \quad 2h - 2 = \langle \beta, \beta \rangle.$$

By the Hodge index theorem, the intersection form on the lattice generated by $\widehat{\Lambda}$ and β is of signature $(1, 2)$. In particular, the discriminant

$$\Delta(\widehat{\Lambda} \oplus \mathbb{Z}\beta) = (-1)^2 \det \begin{pmatrix} \langle \delta_1, \delta_1 \rangle & \langle \delta_1, \delta_2 \rangle & e_1 \\ \langle \delta_2, \delta_1 \rangle & \langle \delta_2, \delta_2 \rangle & e_2 \\ e_1 & e_2 & 2h - 2 \end{pmatrix}$$

is positive. Hence, by the construction of Section 1.2.1, the Noether–Lefschetz divisor

$$D_{m,h,(e_1,e_2)} \subset \mathcal{M}_{\widehat{\Lambda}}$$

is of pure codimension 1. Conditions (2) and (3) require C' and C'', respectively, not to lie in the Noether–Lefschetz divisors obtained from

$$\Lambda \cap \mathrm{Eff}_\xi(\delta_1, L), \quad \text{for } \xi \in C$$

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and the nodal Noether–Lefschetz locus $D_{1,0,(0,0)}$. Together, these are finitely many proper divisors in $\mathcal{M}_{\widehat{\Lambda}}$.

There is no difficulty in finding C' and the smoothing C'' since the moduli space $\mathcal{M}_{\widehat{\Lambda}}$ has a Satake compactification as a projective variety of dimension 18 with boundary of dimension 1. We first find a nonsingular projective surface

$$Y \subset \mathcal{M}_{\widehat{\Lambda}} \tag{8.9}$$

which contains C and does *not* lie in any of the Noether–Lefschetz divisors listed in (2). Then C' is chosen to be a very ample section of Y whose union with C smooths to a very ample section C'' of Y. As divisor classes

$$[C] + [C'] = [C''] \in \mathrm{Pic}(Y),$$

and the smoothing occurs as a pencil of divisors in the linear series on Y.

8.6. Nonisolated contributions: degeneration. Let X, X', and X'' denote the families of $\widehat{\Lambda}$-polarized $K3$ surfaces

$$\pi : X \to C, \quad \pi' : X' \to C', \quad \pi'' : X'' \to C''$$

obtained from the curves $C, C', C'' \subset \mathcal{M}_{\widehat{\Lambda}}$. By the transversality conditions in (2) and (3) of Section 8.5 with respect to the nodal Noether–Lefschetz divisor, the families X' and X'' are nonsingular and have only finitely many nodal fibers. Of course, π is just

$$\pi_3 : \mathsf{X} \to C$$

viewed with a different lattice polarization. As C'' degenerates to the curve $C \cup C'$, we obtain a degeneration of 3-folds

$$X'' \rightsquigarrow X \cup X'$$

with nonsingular total space.

Consider the degeneration formula for stable pairs invariants of X'' in fiber classes. The degeneration formula expresses such stable pairs invariants of X'' in terms of the relative stable pairs invariants of

$$X/\pi^{-1}(C \cap C') \quad \text{and} \quad X'/(\pi')^{-1}(C \cap C')$$

in fiber classes. Degeneration to the normal cone of $\pi^{-1}(C \cap C') \subset X$ (together with usual $K3$ vanishing via the reduced theory) shows the relative invariants of $X/\pi^{-1}(C \cap C')$ equal the standard stable pairs invariants of X in fiber classes. Similarly, the relative invariants of $X'/(\pi')^{-1}(C \cap C')$ equal the standard stable pairs invariants of X'.

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We index the fiber class invariants of X, X', and X'' by the degrees measured against δ_1 and δ_2. The stable pairs partition functions are:

$$\mathsf{Z}_{\mathsf{P}}(X;q) = 1 + \sum_{(e_1,e_2)\neq(0,0)} \mathsf{Z}_{\mathsf{P}}(X;q)_{(e_1,e_2)} v_1^{e_1} v_2^{e_2},$$
$$\mathsf{Z}_{\mathsf{P}}(X';q) = 1 + \sum_{(e_1,e_2)\neq(0,0)} \mathsf{Z}_{\mathsf{P}}(X';q)_{(e_1,e_2)} v_1^{e_1} v_2^{e_2},$$
$$\mathsf{Z}_{\mathsf{P}}(X'';q) = 1 + \sum_{(e_1,e_2)\neq(0,0)} \mathsf{Z}_{\mathsf{P}}(X'';q)_{(e_1,e_2)} v_1^{e_1} v_2^{e_2}.$$

Since δ_1 is ample, only terms with $e_1 > 0$ can occur in the above sums. The degeneration formula yields the following result.

PROPOSITION 43. *We have*

$$\mathsf{Z}_{\mathsf{P}}(X'';q) = \mathsf{Z}_{\mathsf{P}}(X;q) \cdot \mathsf{Z}_{\mathsf{P}}(X';q).$$

Let $\widetilde{F}^X$, $\widetilde{F}^{X'}$, and $\widetilde{F}^{X''}$ denote the logarithms of the partition functions of X, X', and X'', respectively. Proposition 43 yields the relation:

$$\widetilde{F}^{X''} = \widetilde{F}^X + \widetilde{F}^{X'}. \tag{8.10}$$

We now restrict ourselves to fiber classes of δ_1-degree bounded by L (as specified in the construction of $\widehat{\Lambda}$ in Section 8.4). For such classes on X', we will divide $\widetilde{F}^{X'}$ into two summands.

LEMMA 44. *There are no curves of X' in class $\beta \in H_2(X',\mathbb{Z})^{\pi'}$ of δ_1-degree bounded by L which move in families dominating X'.*

Proof. If a such a family of curves were to dominate X', then every fiber $X'_{\xi'}$ as ξ' varies in C' would contain an effective curve class of the family. In particular, the fibers over $\xi' \in C \cap C'$ would contain such effective curves. By construction, $\xi' \in C \cap C'$ is a very general point of C. Therefore,

$$\mathrm{Pic}(X'_{\xi'}) = \mathrm{Pic}(X_{\xi'}) \cong \Lambda.$$

Since C' was chosen *not* to lie in any Noether–Lefschetz divisors associated to effective curves on $X_{\xi'}$ in Λ of δ_1-degree bounded by L, we have a contradiction. □

By Lemma 44, we can separate the contributions of the components of $P_n(X', \beta)$ by the points $\xi' \in C'$ over which they lie:

$$\widetilde{F}^{X',L} = \sum_{\xi' \in C \cap C'} \widetilde{F}^{X',L}_{\xi'} + \widetilde{F}^{X',L}_{C' \setminus (C \cap C')}. \tag{8.11}$$

Here, $\widetilde{F}^{X',L}$ is the δ_1-degree L truncation of $\widetilde{F}^{X'}$. For each $\xi' \in C \cap C'$, $\widetilde{F}^{X',L}_{\xi'}$ is the δ_1-degree L truncation of $\log(\mathsf{Z}^L_{P,\xi'}(X'; q))$, the logarithm of the truncated stable pairs partition functions of moduli component contributions over ξ'. Finally, $\widetilde{F}^{X',L}_{C' \setminus (C \cap C')}$ is the δ_1-degree L truncation of $\log(\mathsf{Z}^L_{P,C' \setminus (C \cap C')}(X'; q))$, the logarithm of the truncated stable pairs partition functions of contributions over $C' \setminus (C \cap C')$.

LEMMA 45. *For $\xi' \in C \cap C'$, we have*

$$\widetilde{F}^{X',L}_{\xi'} = \sum_{\beta \in \mathrm{Eff}_{\xi'}(\delta_1, L)} \sum_{n \in \mathbb{Z}} q^n v_1^{e_1^\beta} v_2^{e_2^\beta} \widetilde{R}_{n,m_\beta,h_\beta} \cdot \mathrm{Cont}\big(X'_{\xi'}, \beta, NL^{\pi'}_{m_\beta,h_\beta,(e_1^\beta,e_2^\beta)}\big).$$

Here, m_β denotes the divisibility of β, and

$$2h_\beta - 2 = \langle \beta, \beta \rangle, \quad e_1^\beta = \langle \delta_1, \beta \rangle, \quad e_2^\beta = \langle \delta_2, \beta \rangle.$$

Proof. By Lemma 44, every class $\beta \in \mathrm{Eff}_{\xi'}(\delta_1, L)$ is completely isolated with respect to the family π'. Hence, we may apply Proposition 42 to the contributions of β to the $\widehat{\Lambda}$-polarized family π'. □

8.7. Nonisolated contributions: analysis of C''. By the construction of C'' in Section 8.5,

$$C, C', C'' \subset Y \subset \mathcal{M}_{\widehat{\Lambda}}$$

where Y is the nonsingular projective surface (8.9) not contained in any of the Noether–Lefschetz divisors listed in condition (2). Both C' and C'' were constructed as very ample divisors on Y which also do not lie in the Noether–Lefschetz loci listed in (2). Since

$$[C] + [C'] = [C''] \in \mathrm{Pic}(Y),$$

we have

$$\langle [C], [C] \rangle_Y + \langle [C], [C'] \rangle_Y = \langle [C], [C''] \rangle_Y > 0.$$

Let $r = \langle [C], [C''] \rangle_Y$, and let $\zeta_1, \ldots, \zeta_r$ be the r distinct intersection points of C with C''. We can choose C'' so the ζ_i are very general points of C. In particular,

$$\zeta_i \notin C \cap C'. \tag{8.12}$$

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The degeneration of C'' to $C \cup C'$ occurs in the pencil on Y spanned by C'' and $C \cup C'$.

Let $D_{m,h,(e_1,e_2)}$ occur in the finite list of Noether–Lefschetz divisors given in condition (2) of Section 8.5. Since Y does *not* lie in $D_{m,h,(e_1,e_2)}$, the intersection

$$D_{m,h,(e_1,e_2)} \cap Y \subset Y$$

is a proper divisor. We write

$$D_{m,h,(e_1,e_2)} \cap Y = w[C] + \sum_j w_j[T_j], \quad w, w_j \geqslant 0$$

where w is the multiplicity of C and the $T_j \subset Y$ are curves not containing C. By the genericity hypotheses,

$$C \cap C' \cap T_j = C \cap C'' \cap T_j = \emptyset. \tag{8.13}$$

LEMMA 46. *There are no curves of X'' in class $\beta \in H_2(X'', \mathbb{Z})^{\pi''}$ of δ_1-degree bounded by L which move in families dominating C''.*

Proof. If a such a family of curves were to dominate C'', then every fiber $X''_{\xi''}$ as ξ'' varies in C'' would contain an effective curve class of the family. In particular, the fibers over $\zeta \in C \cap C''$ would contain such effective curves. By construction, $\zeta \in C \cap C''$ is a very general point of C. Therefore,

$$\mathrm{Pic}(X''_\zeta) = \mathrm{Pic}(X_\zeta) \cong \Lambda.$$

Since C'' was chosen *not* to lie in any Noether–Lefschetz divisors associated to effective curves on X_ζ in Λ of δ_1-degree bounded by L, we have a contradiction. □

We now consider the family of nonsingular curves C''_t in the pencil as C'' degenerates to $C \cup C'$. Here t varies in Δ, the base of the pencil. The total space of the pencil is

$$\mathcal{C}'' \to \Delta$$

with special fiber

$$C''_0 = C \cup C' \quad \text{over } 0 \in \Delta.$$

For fixed $t \neq 0$, the stable pairs theory in δ_1-degree bounded by L for each C''_t can be separated, by Lemma 46, into contributions over isolated points of C''_t. The union of these support points for the stable pairs theory of C''_t defines an algebraic curve

$$\mathrm{Supp} \subset \mathcal{C}'' \setminus C''_0 \to \Delta_{t \neq 0}$$

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in the total space of the pencil. Precisely, Supp equals the set

$$\{(t, p) \mid t \neq 0,\ p \in C''_t,\ X''_{(t,p)} \text{ carries an effective curve of degree} \leqslant L\}.$$

LEMMA 47. *The closure* $\overline{\mathrm{Supp}} \subset \mathcal{C}''$ *contains no components which intersect the special fiber* $C''_0 = C \cup C'$ *in* $C \cap C'$.

Proof. If a component $Z \subset \overline{\mathrm{Supp}}$ meets $\xi' \in C \cap C'$, then there must be an effective curve class $\beta \in \mathrm{Pic}(X_{\xi'})$ which remains effective (and algebraic) on all of Z. By construction,

$$\mathrm{Pic}(X_{\xi'}) = \Lambda.$$

Hence, Z must be contained in the Noether–Lefschetz divisor at ξ' corresponding to β. The latter Noether–Lefschetz divisor is on the list specified in condition (2) of Section 8.5 and hence takes the form

$$D_{m,h,(e_1,e_2)} = w[C] + \sum_j w_j[T_j].$$

The intersection of $C''_{t\neq 0}$ with C is always $\{\zeta_1, \ldots, \zeta_r\}$ since C''_t is a pencil. By condition (8.12), ξ' is not a limit. The intersection of $C''_{t\neq 0}$ with T_j can not have limit ξ' by (8.13). □

The subvariety $\overline{\mathrm{Supp}} \subset \mathcal{C}''$ is proper (by Lemma 47) and therefore consists of finitely many curves and points. Let

$$\mathrm{Supp}_1 \subset \overline{\mathrm{Supp}}$$

denote the union of the 1-dimensional components of $\overline{\mathrm{Supp}}$. By Lemmas 46 and 47 no such component lies in the fibers of the pencil

$$\mathcal{C}'' \to \Delta.$$

Hence, after a base change $\epsilon : \widetilde{\Delta} \to \Delta$ possibly ramified over $0 \in \Delta$, the pull-back of $\widetilde{\mathrm{Supp}}_1$ to the pulled-back family

$$\widetilde{\mathcal{C}}'' \to \widetilde{\Delta}$$

is a *union of sections*. We divide the sections into two types

$$\widetilde{\mathrm{Supp}}_1 = \bigcup_i A_i \cup \bigcup_j B_j$$

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by the values of the sections over $0 \in \widetilde{\Delta}$:

$$A_i(0) \in C \subset C_0'', \quad B_j(0) \in C' \subset C_0''.$$

By Lemma 46, no section meets $C \cap C'$ over $0 \in \widetilde{\Delta}$.

The stable pairs partition functions of every 3-fold

$$X_t'' \to \widetilde{C}_t'', \quad t \neq 0$$

factors into into contributions over the sections $A_i(t)$ and $B_j(t)$,

$$\mathsf{Z}_P^L(X_t'', q) = \left[\prod_i \mathsf{Z}_{P,A_i(t)}^L(X_t'', q) \cdot \prod_j \mathsf{Z}_{P,B_j(t)}^L(X_t'', q)\right]_{\leqslant L}. \tag{8.14}$$

As before, the partition functions are all δ_1-degree L truncations. The finitely many points of $\mathrm{Supp} \setminus \mathrm{Supp}_1$ are easily seen not to contribute to (8.14) by deformation invariance of the virtual class. We can take the logarithm,

$$\widetilde{F}^{X_t'',L} = \sum_i \widetilde{F}^{X_t'',L}_{A_i(t)} + \sum_j \widetilde{F}^{X_t'',L}_{B_j(t)}.$$

Since the sections B_j all meet $C' \setminus C \cap C'$ over $0 \in \widetilde{\Delta}$, the moduli space of stable pairs supported over

$$\widetilde{C}'' \setminus \left(C \cup \bigcup_i A_i\right)$$

is proper. Again, by deformation invariance of the virtual class,

$$\sum_j \widetilde{F}^{X_t'',L}_{B_j(t)} = \widetilde{F}^{X',L}_{C' \setminus C \cap C'}.$$

Then relations (8.10) and (8.11) imply

$$\widetilde{F}^{X,L} = -\sum_{\xi' \in C \cap C'} \widetilde{F}^{X',L}_{\xi'} + \sum_i \widetilde{F}^{X_t'',L}_{A_i(t)} \tag{8.15}$$

for every $0 \neq t \in \widetilde{\Delta}$.

LEMMA 48. *For general $t \in \widetilde{\Delta}$, the section $A_i(t)$ does* not *lie in the nodal Noether–Lefschetz locus in $\widetilde{C}_t''$.*

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Proof. Suppose $A_i(t)$ is always contained in the nodal Noether–Lefschetz locus in $\widetilde{C}''_t$. We can assume by construction that Y meets the nodal Noether–Lefschetz divisor $D_{1,0,(0,0)}$ at very general points of the latter divisor in $\mathcal{M}_{\widehat{\Lambda}}$. Hence, for very general t, $X''_{t,A_i(t)}$ is a nodal $K3$ surface with $K3$ resolution

$$\rho : \widetilde{X}''_{t,A_i(t)} \longrightarrow X''_{t,A_i(t)}$$

with Picard lattice

$$\mathrm{Pic}(\widetilde{X}''_{t,A_i(t)}) \cong \widehat{\Lambda} \oplus \mathbb{Z}[\widetilde{E}] \tag{8.16}$$

where E is the exceptional -2-curve. (The lattice $\widehat{\Lambda} \oplus \mathbb{Z}[\widetilde{E}]$ is primitive in $\mathrm{Pic}(\widetilde{X}''_{t,A_i(t)})$ since $\widehat{\Lambda} \subset \Lambda$ is primitive and (8.2) holds. The isomorphism (8.16) is then immediate at a very general point of the nodal Noether–Lefschetz divisor of $\mathcal{M}_{\widehat{\Lambda}}$.)

By the definition of $A_i(t)$, there must exist an effective curve

$$Q \subset X''_{t,A_i(t)}$$

of δ_1-degree bounded by L on $X''_{t,A_i(t)}$. Since $X''_{t,A_i(t)}$ is nodal, Q may not be a Cartier divisor. However, $2Q$ is Cartier. By pulling back via ρ, using the identification of the Picard lattice (8.16), and pushing forward by ρ,

$$2Q \in \widehat{\Lambda}. \tag{8.17}$$

The effective curve Q moves with the $K3$ surfaces $X''_{t,A_i(t)}$ as t goes to 0. The condition (8.17) holds for very general t and hence for *all* t. In particular the condition (8.17) hold at $t = 0$. The $K3$ surface $X''_{0,A_i(0)}$ is a nodal fiber of

$$X \to C.$$

The existence of an effective curve $Q \in \frac{1}{2}\widehat{\Lambda}$ directly contradicts condition (iii) of Section 8.4 of $\widehat{\Lambda}$. □

Let $\widetilde{C}''_t$ be a general curve of the pencil $\widetilde{\Delta}$. By Lemma 48, the sections $A_i(t) \in \widetilde{C}''_t$ are disjoint from the nodal Noether–Lefschetz divisor on $\widetilde{C}''_t$. We would like to apply the contribution relation of Proposition 42 to the fiber of

$$X''_t \to \widetilde{C}''_t$$

over $A_i(t)$. We therefore need the following result.

LEMMA 49. *For general $t \in \widetilde{\Delta}$, the curve $\widetilde{C}''_t$ does not lie in the Noether–Lefschetz divisor associated to any effective curve*

$$\beta \in \mathrm{Pic}(X''_{t,A_i(t)})$$

of δ_1-degree bounded by L.

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Proof. If the assertion of the Lemma were false, there would exist a moving family of effective curves

$$\beta \in \mathrm{Pic}(X''_{t,A_i(t)})$$

such that $\widetilde{C}''_t$ always lies in the Noether–Lefschetz divisor corresponding to β. Then all of Y would lie in the Noether–Lefschetz divisor corresponding to β. The limit of β is an effective class in the $K3$ fiber over $A_i(0) \in C$ of δ_1-degree bounded by L. Hence, the Noether–Lefschetz divisor corresponding to β is listed in condition (2) of Section 8.5. But $C'' \subset Y$ was constructed to not lie in any of the Noether–Lefschetz divisors of the list (2), a contradiction. $\square$

We may now apply Proposition 42 to the fiber of

$$X''_t \to \widetilde{C}''_t$$

over $A_i(t)$ for a general curve $\widetilde{C}''_t$ of the pencil $\widetilde{\Lambda}$. Just as in Lemma 45, we obtain

$$\widetilde{F}^{X''_t,L}_{A_i(t)} = \sum_{\beta \in \mathrm{Eff}_{A_i(t)}(\delta_1, L)} \sum_{n \in \mathbb{Z}} q^n v_1^{e_1^\beta} v_2^{e_2^\beta} \widetilde{R}_{n,m_\beta,h_\beta} \cdot \mathrm{Cont}\Big(X''_{A_i(t)}, \beta, NL^{\pi''}_{m_\beta,h_\beta,(e_1^\beta,e_2^\beta)}\Big). \tag{8.18}$$

8.8. Proof of Theorem 8. We now complete the proof of Theorem 8 by proving the relation

$$\widetilde{N}^{\mathsf{X}}_{n,(d_1,d_2,d_3)} = \sum_h \sum_{m=1}^{\infty} \widetilde{R}_{n,m,h} \cdot NL^{\pi_3}_{m,h,(d_1,d_2,d_3)} \tag{8.19}$$

for degrees (d_1, d_2, d_3) positive with respect to the quasipolarization in Λ. The Noether–Lefschetz divisors lie in the moduli space $\mathcal{M}_\Lambda$.

The degrees (d_1, d_2, d_3) of a class in $H_2(\mathsf{X}, \mathbb{Z})^{\pi_3}$ with respect to Λ determine the degrees (e_1, e_2) with respect to $\widehat{\Lambda}$. We first show relation (8.19) is equivalent for classes of δ_1-degree bounded by L to the relation

$$\widetilde{N}^{\mathsf{X}}_{n,(e_1,e_2)} = \sum_h \sum_{m=1}^{\infty} \widetilde{R}_{n,m,h} \cdot NL^{\pi_3}_{m,h,(e_1,e_2)}. \tag{8.20}$$

The Noether–Lefschetz theory in (8.20) occurs in the moduli space $\mathcal{M}_{\widehat{\Lambda}}$.

The equivalence of (8.19) and (8.20) for classes of δ_1-degree bounded by L is a consequence of condition (4) in Section 8.4 for $\widehat{\Lambda} \subset \Lambda$. For effective classes of δ_1-degree bounded by L, condition (4) says the (e_1, e_2) degrees *determine* the (d_1, d_2, d_3) degrees. The left sides of (8.19) and (8.20) then match since the stable pairs invariants only involve effective classes. As shown in Section 8.2, only effective classes contribute to the right sides as well. So the right sides of (8.19) and (8.20) also match.

We prove (8.20) by the result obtains in Sections 8.6 and 8.7. By Equations (8.15) and (8.18) and Lemma 45, we have

$$\begin{aligned}\widetilde{F}^{X,L} = &-\sum_{\xi' \in C\cap C'} \sum_{\beta \in \mathrm{Eff}_{\xi'}(\delta_1,L)} \sum_{n\in\mathbb{Z}} q^n v_1^{e_1^\beta} v_2^{e_2^\beta} \widetilde{R}_{n,m_\beta,h_\beta} \cdot \mathrm{Cont}\big(X'_{\xi'}, \beta, NL^{\pi'}_{m_\beta,h_\beta,(e_1^\beta,e_2^\beta)}\big) \\ &+\sum_i \sum_{\beta \in \mathrm{Eff}_{A_i(t)}(\delta_1,L)} \sum_{n\in\mathbb{Z}} q^n v_1^{e_1^\beta} v_2^{e_2^\beta} \widetilde{R}_{n,m_\beta,h_\beta} \cdot \mathrm{Cont}\big(X''_{A_i(t)}, \beta, NL^{\pi''}_{m_\beta,h_\beta,(e_1^\beta,e_2^\beta)}\big)\end{aligned} \tag{8.21}$$

for a general $t \in \widetilde{\Delta}$.

By definition, the $q^n v_1^{e_1} v_2^{e_2}$ coefficient of the left side of (8.21) is $\widetilde{N}^{\mathsf{X}}_{n,(e_1,e_2)}$. The $q^n v_1^{e_1} v_2^{e_2}$ coefficients of the right side of (8.21) correspond to intersections with the Noether–Lefschetz divisors $D_{m,h,(e_1,e_2)}$. As in Section 8.7, we write

$$D_{m,h,(e_1,e_2)} \cap Y = w[C] + \sum_j w_j[T_j], \qquad w, w_j \geqslant 0, \tag{8.22}$$

where w is the multiplicity of C and the $T_j \subset Y$ are curves not containing C.

The first sum on the right side of (8.21) concerns $C \cap C'$. The contribution of $D_{m,h,(e_1,e_2)}$ to the $q^n v_1^{e_1} v_2^{e_2}$ coefficient of the first sum is

$$-w\langle [C], [C']\rangle_Y \widetilde{R}_{n,m,h} \tag{8.23}$$

if all the instances of $\beta \in \mathrm{Pic}(X'_{\xi'})$ associated to the Noether–Lefschetz divisor $D_{m,h,(e_1,e_2)}$ are effective. As we have seen in Section 8.2, if *any* such instance of β is not effective, then $\widetilde{R}_{n,m,h} = 0$ and the entire Noether–Lefschetz divisor $D_{m,h,(e_1,e_2)}$ contributes 0 to the right sides of both (8.20) and (8.21).

Next, we study the intersection of $D_{m,h,(e_1,e_2)}$ with $\widetilde{C}''_t$. The intersection with C,

$$C \cap \widetilde{C}''_t = \{\zeta_1, \ldots, \zeta_r\}, \qquad r = \langle C, C''\rangle_Y,$$

is independent of t by the construction of the pencil. The contribution to the $q^n v_1^{e_1} v_2^{e_2}$ coefficient of the full intersection $C \cap \widetilde{C}''_t$ is

$$w\langle [C], [C'']\rangle_Y \widetilde{R}_{n,m,h} \tag{8.24}$$

if all the instances of

$$\beta \in \mathrm{Pic}(X''_{t,\zeta_k}) = \mathrm{Pic}(X_{\zeta_k})$$

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associated to the Noether–Lefschetz divisor $D_{m,h,(e_1,e_2)}$ are effective. Such an effective β implies the point $\zeta_k \in \widetilde{C}''_s$ is always in Supp and hence corresponds to a section A_i. In the effective case, the contributions (8.24) all occur in the second sum of (8.21).

On the other hand, if *any* class $\beta \in \mathrm{Pic}(X''_{t,\zeta_k})$, for any $\zeta_k \in C \cap C'$, associated to the Noether–Lefschetz divisor $D_{m,h,(e_1,e_2)}$ is not effective, then $\widetilde{R}_{n,m,h} = 0$ and the entire Noether–Lefschetz divisor $D_{m,h,(e_1,e_2)}$ contributes 0 to the right sides of (8.20) and (8.21).

Finally, we consider the intersection of T_j with $\widetilde{C}''_t$. By construction, $T_j \cap \widetilde{C}''_t \subset \widetilde{C}''_t$ is a finite collection of points. We divide the intersection

$$T_j \cap \widetilde{C}''_t = I_{j,t} \cup I'_{j,t} \tag{8.25}$$

into disjoint subset with the following properties:

- as $t \to 0$, the points of $I_{j,t}$ have limit in C;
- as $t \to 0$, the points of $I'_{j,t}$ have limit in C'.

Since T_j does not intersect $C \cap C'$, the disjoint union is well defined and unique (8.25) for t sufficiently near 0. Moreover, the sum of the local intersection numbers of $T_j \cap \widetilde{C}''_t$ over $I_{j,t}$ is $\langle C, T_j \rangle_Y$. The points of $I'_{j,t}$, related to the sections $B(t)$ in the analysis of Section 8.7, do not play a role in the analysis of (8.21). (The contributions of the sections $B(t)$ is canceled in (8.15).)

If a single instance of a class

$$\beta \in \mathrm{Pic}(X''_{t,\xi''}) \quad \text{for } \xi'' \in I_{j,t}$$

associated to the Noether–Lefschetz divisor $D_{m,h,(e_1,e_2)}$ is ineffective for t sufficiently near 0, then $\widetilde{R}_{n,m,h} = 0$ and the entire Noether–Lefschetz divisor $D_{m,h,(e_1,e_2)}$ contributes 0 to the right sides of (8.20) and (8.21). Otherwise, every instance of such a β is effective for all t sufficiently near 0. Then

$$I_{j,t} \subset \bigcup_i A_i(t),$$

and the contribution to the $q^n v_1^{e_1} v_2^{e_2}$ coefficient of the right side of (8.21) of the full intersection $I_{j,t}$ is

$$w_j \langle [C], T_j \rangle_Y \widetilde{R}_{n,m,h}. \tag{8.26}$$

Summing all the contributions (8.23), (8.24), and (8.26) to the right side of (8.21) associated to $D_{m,h,(e_1,e_2)}$ yields

$$\left(-w\langle [C], [C'] \rangle_Y + w\langle [C], [C''] \rangle_Y + \sum_j w_j \langle [C], T_j \rangle_Y \right) \cdot \widetilde{R}_{n,m,h}. \tag{8.27}$$

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Using the relation $-[C']+[C''] = [C]$ and (8.22), the sum (8.27) exactly matches contribution

$$\widetilde{R}_{n,m,h} \cdot NL^{\pi_3}_{m,h,(e_1,e_2)} = \widetilde{R}_{n,m,h} \cdot \int_C [D_{m,h,(e_1,e_2)}]$$

of $D_{m,h,(e_1,e_2)}$ to the right side of (8.20). The proofs of (8.20) and of Theorem 8 are complete.

9. Katz–Klemm–Vafa conjecture

The proof of the P/NL correspondence of Theorem 8 was the last step in the proof of Proposition 10:

$$r_{g,m,h} = \widetilde{r}_{g,m,h} \quad \text{for all } g \in \mathbb{Z},\ m > 0,\ h \in \mathbb{Z}.$$

The proof of Theorem 4 is also now complete.

In Section 7.6, several properties of the stable pairs invariants $\widetilde{r}_{g,m,h}$ were established (and in fact were used in the proofs of Theorem 8 and Proposition 10). The most important property of $\widetilde{r}_{g,m,h}$ is independence of divisibility established in Proposition 39,

$$\widetilde{r}_{g,\beta} \quad \text{depends only upon } g \text{ and } \langle \beta, \beta \rangle.$$

Also proven in Section 7.6 were the basic vanishing results

$$\widetilde{r}_{g<0,m,h} = 0, \quad \widetilde{r}_{g,m,h<0} = 0.$$

The independence of $\widetilde{r}_{g,\beta}$ upon the divisibility of β reduces the Katz–Klemm–Vafa conjecture to the primitive case.

The stable pairs BPS counts in the primitive case are determined by Proposition 35, relation (7.5), and the interpretation of the Kawai–Yoshioka results presented in Section 5.7. Taken together, we prove the Katz–Klemm–Vafa conjecture in the primitive case and hence in all cases. The proof of Theorem 3 is complete.

Following the notation of Section 1.3, let

$$\pi : X \to C$$

be a 1-parameter family of Λ-polarized $K3$ surfaces with respect to a rank r lattice Λ. Using the independence of divisibility, Theorem 5 in Gromov–Witten theory takes a much simpler form.

THEOREM 50. *For degrees* $(d_1, \ldots, d_r)$ *positive with respect to the quasipolarization* λ^π,

$$n^X_{g,(d_1,\ldots,d_r)} = \sum_{h=0}^{\infty} r_{g,h} \cdot NL^{\pi}_{h,(d_1,\ldots,d_r)}.$$

Theorems 3 and 50 together give closed form solutions for the BPS states in fiber classes in term of the Noether–Lefschetz numbers (which are expressed in terms of modular forms by Borcherds' results). A classical example is given in the next section.

10. Quartic $K3$ surfaces

We provide a complete calculation of the Noether–Lefschetz numbers and BPS counts in fiber classes for the family of $K3$ surfaces determined by a Lefschetz pencil of quartics in $\mathbb{P}^3$:

$$\pi : X \to \mathbb{P}^1, \quad X \subset \mathbb{P}^3 \times \mathbb{P}^1 \text{ of type } (4, 1).$$

Let A and B be modular forms of weight $1/2$ and level 8,

$$A = \sum_{n\in\mathbb{Z}} q^{n^2/8}, \quad B = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/8}.$$

Let Θ be the modular form of weight $21/2$ and level 8 defined by

$$\begin{aligned}2^{22}\Theta = {} & 3A^{21} - 81A^{19}B^2 - 627A^{18}B^3 - 14436A^{17}B^4 \\ & - 20007A^{16}B^5 - 169092A^{15}B^6 - 120636A^{14}B^7 \\ & - 621558A^{13}B^8 - 292796A^{12}B^9 - 1038366A^{11}B^{10} \\ & - 346122A^{10}B^{11} - 878388A^9B^{12} - 207186A^8B^{13} \\ & - 361908A^7B^{14} - 56364A^6B^{15} - 60021A^5B^{16} \\ & - 4812A^4B^{17} - 1881A^3B^{18} - 27A^2B^{19} + B^{21}.\end{aligned}$$

We can expand Θ as a series in $q^{1/8}$,

$$\Theta = -1 + 108q + 320q^{9/8} + 50016q^{3/2} + 76950q^2 \ldots.$$

Let $\Theta[m]$ denote the coefficient of q^m in Θ.

The modular form Θ first appeared in calculations of [27]. The following result was proven in [36]: *the Noether–Lefschetz numbers of the quartic pencil π are coefficients of Θ,*

$$NL^{\pi}_{h,d} = \Theta\left[\frac{\triangle_4(h,d)}{8}\right],$$

where the discriminant is defined by

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$$\triangle_4(h,d) = -\det\begin{pmatrix} 4 & d \\ d & 2h-2 \end{pmatrix} = d^2 - 8h + 8.$$

By Theorem 50, we obtain

$$n^X_{g,d} = \sum_{h=0}^{\infty} r_{g,h} \cdot \Theta\left[\frac{\triangle_4(h,d)}{8}\right],$$

as predicted in [27]. Similar closed form solutions can be found for all the classical families of $K3$-fibrations, see [36].

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The paper was completed in April 2014. Further developments in the Gromov–Witten and stable pairs theories of $K3$ geometries can be found in [23], where the motivic invariants are considered, and [38], where the 3-fold $K3 \times E$ is studied.

Appendix A. Invariants

We include here a short table of the various invariants associated to a $K3$ surface S and a class $\beta \in \mathrm{Pic}(S)$.

$R_{g,\beta}(S)$	Reduced GW invariants of S	Section 0.1
$r_{g,\beta}$	BPS counts for $K3$ surfaces in GW theory	Section 0.3
$\widetilde{R}_{n,\beta}(S)$	Stable pair invariants of S parallel to $R_{g,\beta}$	Sections 0.6, 6.2
$\widetilde{r}_{n,\beta}$	BPS counts for $K3$ surfaces via stable pairs	Sections 3.4, 7.6
$R^{\mathrm{red}}_{n,\beta}(S \times \mathcal{R})$	Reduced stable pair invariants of the rubber	Section 6.6
$\langle 1 \rangle^{\mathrm{red}}_{Y,\beta}$	Reduced stable pair residues of $Y = S \times \mathbb{C}$	Section 5.6
I_h	$I_h = \langle 1 \rangle^{\mathrm{red}}_{Y,\alpha}$ for α primitive with $\langle \alpha, \alpha \rangle = 2h-2$	Section 5.6

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Associated to $K3$ fibrations over a curve

$$X \longrightarrow C,$$

there are several more invariants. Here, $\beta \in H_2(X, \mathbb{Z})$ is a fiber class.

$N^X_{g,\beta}$	Connected GW invariants of the $K3$-fibration X	Section 2.2
$n^X_{g,\beta}$	BPS counts for X in GW theory	Section 2.2
$\widetilde{N}^X_{n,\beta}$	Connected stable pairs invariants of the $K3$-fibration X	Section 8.1
$\widetilde{n}^X_{g,\beta}$	BPS counts for X via stable pairs	Sections 3.5, 8.1

When S is a nonsingular $K3$ fiber of $X \to C$ and $\beta \in \mathrm{Pic}(S)$ is a class for which no effective summand on S deforms over C, we have two invariants.

$P^\star_{n,\beta}(X)$	Contribution of stable pairs supported on S to the stable pairs invariant of X	Section 6.2
$P^\star_{n,\beta}(X/S)$	Contribution of stable pairs over S to the relative stable pairs invariant of X/S	Section 6.5

Appendix B. Degenerations

Let $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$ be the blow-up of $\mathbb{P}^2 \times \mathbb{P}^1$ at a point. Consider the toric 4-fold

$$\mathsf{Y} = \widetilde{\mathbb{P}^2 \times \mathbb{P}^1} \times \mathbb{P}^1$$

of Picard group of rank 4,

$$\mathrm{Pic}(\mathsf{Y}) \cong \mathbb{Z}L_1 \oplus \mathbb{Z}L_2 \oplus \mathbb{Z}E \oplus \mathbb{Z}L_3.$$

Here, L_1, L_2, E are the pull-backs of divisors from $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$ and L_3 is the pull-back of $\mathcal{O}(1)$ from the last $\mathbb{P}^1$. (We follow the notation of Section 2.) The divisors L_1, L_2, and L_3 are certainly base point free on Y. Since $L_1 + L_2 - E$ arises on $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$ via the projection from a point of the $(1, 1)$-Segre embedding

$$\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5,$$

the divisor $L_1 + L_2 - E$ determines a map to the quadric $Q \subset \mathbb{P}^4$,

$$\widetilde{\mathbb{P}^2 \times \mathbb{P}^1} \to Q \subset \mathbb{P}^4.$$

Hence, $L_1 + L_2 - E$ is base point free on both $\widetilde{\mathbb{P}^2 \times \mathbb{P}^1}$ and Y.

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The anticanonical series $3L_1 + 2L_2 - 2E + 2L_3$ is base point free on Y since L_1, L_2, L_3, and $L_1 + L_2 - E$ are all base point free. Let

$$\mathsf{X} \subset \mathsf{Y}$$

be a general anticanonical divisor (nonsingular by Bertini). In [39], the Gromov–Witten/Pairs correspondence is proven for Calabi–Yau 3-fold which admit appropriate degenerations. To find such degenerations for X, we simply factor equations.

Let $\mathsf{X}_{a,b,c,d} \subset \mathsf{Y}$ denote a general divisor of class $aL_1 + bL_2 + cE + dL_3$. We first degenerate $\mathsf{X} = \mathsf{X}_{3,2,-2,2}$ via the product

$$\mathsf{X}_{2,1,-1,1} \cdot \mathsf{X}_{1,1,-1,1}.$$

For such a degeneration to be used in the scheme of [39], all of the following varieties must be nonsingular:

$$\mathsf{X}_{3,2,-2,2}, \quad \mathsf{X}_{2,1,-1,1}, \quad \mathsf{X}_{1,1,-1,1},$$
$$\mathsf{X}_{2,1,-1,1} \cap \mathsf{X}_{1,1,-1,1}, \quad \mathsf{X}_{3,2,-2,2} \cap \mathsf{X}_{2,1,-1,1} \cap \mathsf{X}_{1,1,-1,1}.$$

Since all three divisor classes $\mathsf{X}_{3,2,-2,2}$, $\mathsf{X}_{2,1,-1,1}$, $\mathsf{X}_{1,1,-1,1}$ are base point free, the required nonsingularity follows from Bertini. Next, we degenerate $\mathsf{X}_{2,1,-1,1}$ via the product

$$\mathsf{X}_{1,1,-1,1} \cdot \mathsf{X}_{1,0,0,0}.$$

The nonsingularity of the various intersections is again immediate by Bertini. Since $\mathsf{X}_{1,0,0,0}$ is a toric 3-fold, no further action must be taken for $\mathsf{X}_{1,0,0,0}$.

We are left with the divisor $\mathsf{X}_{1,1,-1,1}$ which we degenerate via the product

$$\mathsf{X}_{1,0,0,1} \cdot \mathsf{X}_{0,1,-1,0}.$$

While the divisor classes of $X_{1,1,-1,1}$ and $\mathsf{X}_{1,0,0,1}$ are base point free, the class $\mathsf{X}_{0,1,-1,0}$ is not. There is unique effective divisor

$$\mathsf{X}_{0,1,-1,0} \subset \mathsf{Y}.$$

Fortunately $\mathsf{X}_{0,1,-1,0}$ is a nonsingular toric 3-fold isomorphic to $\widetilde{\mathbb{P}}^2 \times \mathbb{P}^1$ where $\widetilde{\mathbb{P}}^2$ is the blow-up of $\mathbb{P}^2$ in a point. The nonsingularity of $\mathsf{X}_{0,1,-1,0}$ is sufficient to guarantee the nonsingularity of

$$\mathsf{X}_{1,0,0,1} \cap \mathsf{X}_{0,1,-1,0}, \quad \mathsf{X}_{1,1,-1,1} \cap \mathsf{X}_{1,0,0,1} \cap \mathsf{X}_{0,1,-1,0}$$

since $\mathsf{X}_{1,1,-1,1}$ and $\mathsf{X}_{1,0,0,1}$ are both base point free.

The result of [39] reduces the GW/P correspondence for X to the toric cases

$$\mathsf{X}_{1,0,0,0}, \quad \mathsf{X}_{0,1,-1,0}, \quad \mathsf{X}_{0,0,0,1}$$

and the geometries of the various $K3$ and rational surfaces and higher genus curves which occur as intersections in the degenerations. The GW/P correspondences for all these end states have been established in [39] and earlier work. Hence, the GW/P correspondence holds for X.

Appendix C. Cones and virtual classes

C.1. Fulton Chern class. Let X be a scheme of dimension d. (The constructions are also valid for a Deligne–Mumford stack which admits embeddings into nonsingular Deligne–Mumford stacks.) Let

$$X \subset M$$

be a closed embedding in a nonsingular ambient M of dimension $m \geqslant d$. Of course, we also have an embedding

$$X \subset M \times \mathbb{C} = \widetilde{M}$$

where X lies over $0 \in \mathbb{C}$. The normal cones $C_X M$ and $C_X \widetilde{M}$ of X in M and $\widetilde{M}$ are of pure dimensions m and $m+1$, respectively. Moreover,

$$C_X \widetilde{M} = C_X M \oplus 1,$$

following the notation of [15]. Let q be the structure morphism of the projective cone

$$q : \mathbb{P}(C_X \widetilde{M}) \to X,$$

and let $[\mathbb{P}(C_X \widetilde{M})]$ be the fundamental class of pure dimension m. The Segre class $s(X, M)$ is defined by

$$\begin{aligned} s(X, M) &= q_* \left(\sum_{i=0}^{\infty} c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C_X M \oplus 1)] \right) \\ &= q_* \left(\sum_{i=0}^{\infty} c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C_X \widetilde{M})] \right). \end{aligned}$$

The Fulton total Chern class,

$$\begin{aligned} c_F(X) &= c(T_{M|X}) \cap s(X, M) \\ &= c(T_{\widetilde{M}|X}) \cap \; q_* \left(\sum_{i=0}^{\infty} c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C_X \widetilde{M})] \right), \end{aligned} \tag{C.1}$$

is independent of the embedding M; see [15, 4.2.6].

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Let $E^\bullet = [E^{-1} \to E^0]$, together with a morphism to the cotangent complex $L^\bullet$, be a perfect obstruction theory on X. The virtual class associated to $E^\bullet$ can be expressed in terms of Chern classes of $E^\bullet$ and the Fulton total Chern class of X:

$$\begin{aligned}[X]^{\mathrm{vir}} &= \left[s\big((E^\bullet)^\vee\big) \cap c_F(X)\right]_{\mathrm{virdim}} \\ &= \left[\frac{c(E_1)}{c(E_0)} \cap c_F(X)\right]_{\mathrm{virdim}},\end{aligned}$$

where $E_1 = (E^{-1})^*$ and $E_0 = (E^0)^*$. The above formula occurs in [44] and earlier in the excess intersection theory of [15]. As a consequence, the virtual class depends only upon the K-theory class of $E^\bullet$.

C.2. The curvilinear condition. Let $Y \subset X$ be a subscheme satisfying the curvilinear lifting property:

$$\text{every map } \operatorname{Spec} \mathbb{C}[x]/(x^k) \to X \text{ factors through } Y.$$

By the $k = 1$ case, the curvilinear lifting property implies $Y \subset X$ is a bijection on closed points.

We view the embedding $X \subset M \times \mathbb{C} = \widetilde{M}$ also as an embedding of

$$Y \subset X \subset \widetilde{M}.$$

Let $I_X \subset I_Y$ be the ideal sheaves of X and Y in $\widetilde{M}$. There is a canonical rational map over $\widetilde{M}$,

$$f : \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} I_Y^i\right) \dashrightarrow \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} I_X\right)$$

associated to the morphism of graded algebras

$$\bigoplus_{i=0}^{\infty} I_X^i \to \bigoplus_{i=0}^{\infty} I_Y^i, \qquad I_X^0 = I_Y^0 = \mathcal{O}_{\widetilde{M}}.$$

By definition, the source and target of f are the blow-ups of $\widetilde{M}$ along Y and X, respectively,

$$\begin{aligned}\mathrm{Bl}_Y(\widetilde{M}) &= \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} I_Y^i\right) \xrightarrow{\pi_Y} \widetilde{M}, \\ \mathrm{Bl}_X(\widetilde{M}) &= \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} I_X^i\right) \xrightarrow{\pi_Y} \widetilde{M},\end{aligned}$$

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with exceptional divisors

$$\pi_Y^{-1}(Y) = \mathbb{P}(C_Y \widetilde{M}), \quad \pi_X^{-1}(X) = \mathbb{P}(C_X \widetilde{M}).$$

PROPOSITION 51. *The rational map has empty base locus and thus yields a projective morphism*

$$f : \mathrm{Bl}_Y(\widetilde{M}) \longrightarrow \mathrm{Bl}_X(\widetilde{M}).$$

Moreover, as Cartier divisors on $\mathrm{Bl}_Y(\widetilde{M})$,

$$f^*(\mathbb{P}(C_X \widetilde{M})) = \mathbb{P}(C_Y \widetilde{M}).$$

Proof. Away from the exception divisor $\pi_Y^{-1}(Y)$, f is certainly a morphism. We need only study the base locus on the exceptional divisor

$$\mathbb{P}(C_Y \widetilde{M}) \subset \mathrm{Bl}_Y \widetilde{M}.$$

We can reach any closed point $q \in \mathbb{P}(C_Y \widetilde{M})$ by the strict transform to $\mathrm{Bl}_Y \widetilde{M}$ of map of a nonsingular quasiprojective curve

$$g : (\Delta, p) \to \widetilde{M}$$

with $g^{-1}(Y)$ supported at p. In other words, the strict transform

$$g_Y : (\Delta, p) \to \mathrm{Bl}_Y \widetilde{M}$$

satisfies $g_Y(p) = q$. Since $Y \subset X$ is a bijection on closed points, $g^{-1}(X)$ is also supported at p.

We work locally on an open affine $U = \mathrm{Spec}(A) \subset \widetilde{M}$ containing $g(p) \in \widetilde{M}$. Let

$$a_1, \ldots, a_r \in I_X, \quad a_1, \ldots, a_r, b_1, \ldots, b_s \in I_Y$$

be generators of the ideals $I_X \subset I_Y \subset A$. By definition of the blow-up,

$$\pi_Y^{-1}(U) \subset U \times \mathbb{P}^{r+s-1}, \quad \pi_X^{-1}(U) \subset U \times \mathbb{P}^{r-1}.$$

Let t be the local parameter of Δ at p with $t(p) = 0$. From the map g, we obtain functions

$$a_i(t) = a_i(g(t)), \quad b_j(t) = b_j(g(t))$$

in the local parameter t which are regular at 0. Since $g(p) \in Y \subset X$, we have $a_i(0) = 0$ and $b_j(0) = 0$ for all i and j. Let ℓ_X be the *lowest* valuation of t among all the functions $a_i(t)$. Since the $a_i(t)$ can not all vanish identically, $\ell_x > 0$. The limit

$$v = \lim_{t \to 0} \left(\frac{a_1(t)}{t^{\ell_X}}, \ldots, \frac{a_r(t)}{t^{\ell_X}} \right)$$

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is a well-defined *nonzero* vector v. By definition of the blow-up,

$$g_X(p) = (g(p), [v]) \in U \times \mathbb{P}^{r-1}.$$

Similarly, let ℓ_Y be the lowest valuation of t among all the functions $a_i(t)$ *and* $b_j(t)$. Then, the limit

$$w = \lim_{t\to 0}\left(\frac{a_1(t)}{t^{\ell_Y}}, \dots, \frac{a_r(t)}{t^{\ell_Y}}, \frac{b_1(t)}{t^{\ell_Y}}, \dots, \frac{b_s(t)}{t^{\ell_Y}}\right)$$

is a well-defined nonzero vector w, and

$$g_Y(p) = (g(p), [w]) \in U \times \mathbb{P}^{r+s-1}.$$

Certainly, $\ell_Y \leqslant \ell_X$ since ℓ_Y is a minimum over a larger set. If $\ell_Y < \ell_X$, then there is a $b_j(t)$ with lower valuation than all the $a_i(t)$. Such a situation directly contradicts the curvilinear lifting property for the map

$$\operatorname{Spec}(\mathcal{O}_D/t^{\ell_X}) \subset D \xrightarrow{g} X.$$

Hence, $\ell_Y = \ell_X$.

The equality of ℓ_Y and ℓ_X has the following consequence: *the first r coordinates of w are not all 0*. As a result, the rational map

$$f : \mathrm{Bl}_Y \widetilde{M} \dashrightarrow \mathrm{Bl}_X \widetilde{M}$$

defined on $U \times \mathbb{P}^{r+s-1}$ by projection

$$f(q) = f((g(p), w)) = (g(p), (w_1, \dots, w_r)) = (g(p), v)$$

has no base locus at q. Since q was arbitrary, f has no base locus on $\mathrm{Bl}_Y(\widetilde{M})$.

The exceptional divisors $\mathbb{P}(C_Y\widetilde{M})$ and $\mathbb{P}(C_X\widetilde{M})$ are $\mathcal{O}(-1)$ on $\mathrm{Bl}_Y(\widetilde{M})$ and $\mathrm{Bl}_X(\widetilde{M})$, respectively. Since the morphism f respects $\mathcal{O}(-1)$, the relation

$$f^*(\mathbb{P}(C_X\widetilde{M})) = \mathbb{P}(C_Y\widetilde{M})$$

holds as Cartier divisors. □

By Proposition 51 and the push–pull formula for the degree 1 morphism f, we find

$$\begin{aligned} f_*[\mathbb{P}(C_Y\widetilde{M})] &= f_*[f^*\mathbb{P}(C_X\widetilde{M})] \\ &= \deg(f)\cdot[\mathbb{P}(C_X\widetilde{M})] \\ &= [\mathbb{P}(C_X\widetilde{M})], \end{aligned} \tag{C.2}$$

where $[D]$ denotes the fundamental cycle of a Cartier divisor D. The restriction of f to the exceptional divisors yields a morphism

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$$f : \mathbb{P}(C_Y\widetilde{M}) \to \mathbb{P}(C_X\widetilde{M})$$

which covers $\iota : Y \to X$. Since the morphism on projective cones respects $\mathcal{O}(1)$, relation (C.2) and definition (C.1) together imply

$$\iota_* c_F(Y) = c_F(X).$$

In other words, the Fulton total Chern class is the same for embeddings satisfying the curvilinear lifting property.

C.3. The divisor D_{n_1,β_1}. Following the notation of Section 6.9, we have

$$D_{n_1,\beta_1} \subset W(n,\beta),$$

and we would like to compare the Fulton total Chern classes of these two moduli spaces. The subspace D_{n_1,β_1} is the pull-back to $W(n,\beta)$ of a nonsingular divisor in the Artin stack $\mathscr{B}^p_{n,\beta}$. Hence, D_{n_1,β_1} is locally defined by a single equation.

There is no obstruction to smoothing the crease for any stable pair parameterized by D_{n_1,β_1}. An elementary argument via vector fields moving points in $\mathbb{P}^1$ shows $W(n,\beta)$ to be étale locally a trivial product of D_{n_1,β_1} with the smoothing parameter in $\mathbb{C}$. (Consider the versal deformation of the degeneration of $\mathbb{P}^1$ to the chain $\mathbb{P}^1\cup\mathbb{P}^1$. Given any finite collection of nonsingular points of the special fiber $\mathbb{P}^1\cup\mathbb{P}^1$, an open set U of the special fiber can be found containing the points together with a vector field which translates U over the base of the deformation. In the case the degeneration is a longer chain, such a vector field can be found for each node.) Hence, the equation of D_{n_1,β_1} is nowhere a zero divisor. Given an embedding $W(n,\beta) \subset M$ in a nonsingular ambient space, we consider

$$W(n,\beta) \subset M \times \mathbb{C} = \widetilde{M}$$

with $W(n,\beta)$ lying over $0 \in \mathbb{C}$. There is an exact sequence of cones on D_{n_1,β_1},

$$0 \longrightarrow N \longrightarrow C_{D_{n_1,\beta_1}}\widetilde{M} \longrightarrow C_{W(n,\beta)}\widetilde{M}|_{D_{n_1,\beta_1}} \longrightarrow 0,$$

where $N = \mathcal{O}_{D_{n_1,\beta_1}}(D_{n_1,\beta_1})$. As a consequence,

$$s(W(n,\beta), M)|_{D_{n_1,\beta_1}} = s(D_{n_1,\beta_1}, M)c(\mathcal{O}_{D_{n_1,\beta_1}}(D_{n_1,\beta_1})).$$

By the definition of the Fulton total Chern class

$$\begin{aligned} c_F(W(n,\beta))|_{D_{n_1,\beta_1}} &= c(T_M|_{D_{n_1,\beta_1}}) \cap s(W(n,\beta), M)|_{D_{n_1,\beta_1}} \\ &= c(T_M|_{D_{n_1,\beta_1}}) \cap s(D_{n_1,\beta_1}, M)c(\mathcal{O}_{D_{n_1,\beta_1}}(D_{n_1,\beta_1})) \\ &= c_F(D_{n_1,\beta_1})c(\mathcal{O}_{D_{n_1,\beta_1}}(D_{n_1,\beta_1})), \end{aligned}$$

which is (6.79) of Section 6.9.

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