ON THE ENUMERATION OF TREE-ROOTED MAPS

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1. Introduction. It is the purpose of this paper to show that many of the enumerative techniques available for counting rooted plane trees may be extended to tree-rooted maps, that is, rooted maps in which a spanning tree is distinguished as root tree. For example, tree-rooted maps are enumerated by partition, and the average number of trees in a rooted map with n edges is determined. An enumerative similarity between Hamiltonian rooted maps (that is, rooted maps with a distinguished Hamiltonian polygon) and tree-rooted maps is discussed. A 1–1 correspondence is established between tree-rooted maps with n edges and Hamiltonian rooted trivalent maps with $2n + 1$ vertices in which the root vertex is exceptional, being divalent, both of which are in 1–1 correspondence with non-separable Hamiltonian-rooted triangularized digons with n internal vertices, where both the latter are as defined in **(2)**.

2. A *map* is a finite, connected topological graph embedded in the closed Euclidean plane. Such a map is *rooted* if it is the vertex map or if an edge (distinguished as the root edge) is assigned a positive sense of description and right and left sides are specified for it. The negative end of the root edge is called the *root vertex*, and the face on the left of the root edge is called the *root face*. A rooted map is *tree-rooted* if a spanning tree in it is distinguished as root tree. The spanning tree may or may not contain the root edge. We refer to edges not in the spanning tree as being in the *co-tree*. Two rooted maps are equivalent if each can be transformed into the other by a homeomorphism of the plane onto itself which preserves incidence and rooting. *The term tree in this paper will refer exclusively to a plane tree.*

A rooted tree will be called *coloured* if some of its *non-root monovalent vertices* are distinguished as blue vertices. The remaining *non-root* vertices will be called ordinary.

3. LEMMA 1. *Let M be a tree-rooted map with $v = i + 1$ vertices and $f = j + 1$ faces. Then M determines a pair of rooted trees, one of which, T_1, is coloured, having i ordinary vertices and $2j$ blue vertices; the other, T_2, having f vertices. Conversely such a pair of trees determines a map.*

Proof. Suppose we are given a map M with v vertices and f faces. Let T^* be the co-tree of the root tree T in M. In M, on each edge of the co-tree distinguish two interior points as blue vertices. Each pair defines an open interval

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of their edge which contains no vertex of M. Delete this interval from each co-tree edge of M. The result is a plane tree T' which is to be rooted by the following convention: if the root edge E of M is an edge of T, it is retained as the root edge of T'; if E belongs to the co-tree, then the segment containing the negative end u of E (after the above deletion) is used as root edge of T' with u retained as negative end. T', thus rooted, is called T_1. T_1 has i ordinary vertices, namely those vertices of T_1 which correspond to the ordinary vertices of the map M. The root vertex of T_1 corresponds to the root vertex of M and is therefore colourless. By the Euler formula, there are j co-tree edges in M; each of these adds two blue vertices to T_1 which therefore has $2j$ blue vertices. However, T_1 does not give a complete description of M, since, given T_1, generally we do not know what pairs of blue vertices to identify to obtain M. Before completing our store of information about M in the form of a pair of trees, let us further consider M and its dual $\bar{M}$.

It is well known that if T is the spanning tree of the map M, the correspondents of the co-tree edges relative to T in M form, when taken with the vertices of $\bar{M}$, a spanning tree in $\bar{M}$. Let us denote this tree of $\bar{M}$ by the symbol $\bar{T}$.

Returning to the problem of obtaining a complete description of M by means of a pair of trees, let us root the tree $\bar{T}$ by the following device. There are two cases to consider.

Case 1. If M is itself a tree, then $T = M$, and the tree $\bar{T}$ of $\bar{M}$ consists of a single vertex, which is to be considered here as a rooted tree by definition, having a root vertex but no root edge.

Case 2. If T is a proper submap of M, (i.e. $T \neq M$), let us use the following procedure.

The edges incident with the root face of M can be listed in order of occurrence as they surround the root face beginning with the left side of the root edge and proceeding around the face in the direction indicated by the orientation of the root edge from negative to positive. Some edges may be incident with the root face twice, in which case they are listed twice. Evidently there will be at least one edge of the co-tree of T in this list; otherwise M would be itself a tree. Let E' be the first edge of the co-tree to occur in the above list. Now let us consider the image of the co-tree as a spanning tree $\bar{T}$ of the dual map $\bar{M}$. Let us consider the image of the edge E' as being the root edge of $\bar{T}$ with the vertex of $\bar{M}$ corresponding to the root face of M being taken as negative end. The tree $\bar{T}$, thus rooted, is called T_2. The pair of trees (T_1, T_2) is the pair required for the lemma. (Left and right sides are assigned to the above root edges consistently with the assignment of left and right to the root edge of M.)

Conversely given the pair (T_1, T_2) satisfying the conditions of the lemma, one lists the edges of T_2 as they occur beginning on the right side of the root edge and proceeding away from the root vertex. In such a list every edge will occur twice as the entire tree is traversed; thus the list contains $2j$ members.

The list of edges is transferred to the $2j$ blue vertices as they occur about T_1, beginning on the left side of the root edge and proceeding away from its root vertex, the first member of the list being assigned to the first vertex encountered, etc. Vertices labelled with the same edge E are then joined by an arc such that the obvious planarity constraints are not violated. That this can be done can be proved in a straightforward manner. After reducing the blue vertices to the status of internal points of their arcs, we have the required map M.

4. The number of coloured trees. The reader is reminded that the term coloured, as used here, refers only to rooted trees. Let us denote by c_{ik} the number of coloured trees with i ordinary vertices and k blue vertices. Let us define $C(x, y)$ by the formal power series,

$$C(x, y) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} c_{ik} x^i y^k. \tag{4.1}$$

Let us divide all coloured trees into three disjoint classes, the vertex tree, which consists of a single vertex (the root) and has no edges, the class of trees in which the positive end of the root edge is an ordinary vertex, and the class of trees in which the positive end of the root edge is blue.

Examining the first class, recalling that the root vertex is neither ordinary nor blue, we find $c_{00} = 1$.

Any tree of the second class may be obtained by joining the roots of two coloured trees by a directed edge, then specifying that the positive end of this edge be considered ordinary, using some convention such as that the root edges (if they exist) in the parent trees follow immediately after the adjoined edge when a clockwise sense of direction is taken about their respective root vertices. Thus trees in this class are counted by

$$xC^2(x, y). \tag{4.2}$$

Recalling that only monovalent vertices may be blue, we find in a similar fashion that trees in the third class are enumerated by

$$yC(x, y). \tag{4.3}$$

Thus

$$C(x, y) = 1 + yC(x, y) + xC^2(x, y), \tag{4.4}$$

which may be written as

$$C(x, y) = (1 - y)^{-1} + (1 - y)^{-1}xC^2(x, y),$$

and hence can be expanded by the Lagrange power series method to yield

$$C(x, y) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2i + k)!}{i!\,(i + 1)!\,k!} x^i y^k. \tag{4.5}$$

The first few trees counted by this series are shown in Figure 1.

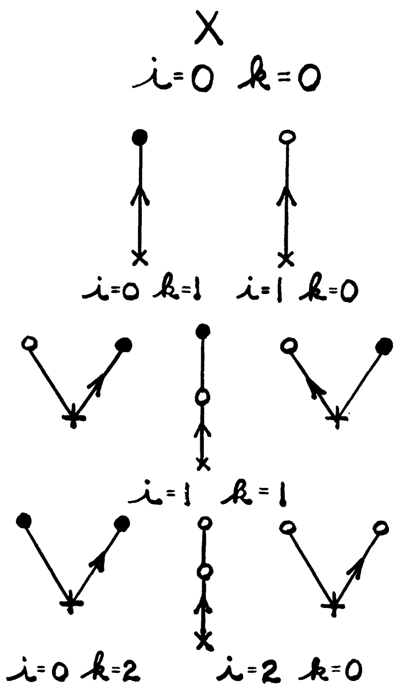

FIGURE 1. Rooted coloured plane trees with i ordinary and k blue vertices: ○, ordinary vertex; ●, blue vertex; ×, rooted vertex.

As a corollary, employing Lemma 1, the number of tree-rooted maps with $i + 1$ vertices and $j + 1$ faces is

$$c_{i,2j}\, c_{j,0} = \frac{(2i + 2j)!}{i!\,(i + 1)!\,j!\,(j + 1)!}. \tag{4.6}$$

By the Euler polyhedron formula, the number of edges in such a map is $i + j$. Therefore the number of tree-rooted maps with n edges is

$$\sum_{v=1}^{n+1} \frac{2n!}{v!\,(v - 1)!\,(n + 2 - v)!\,(n + 1 - v)!} = \frac{(2n)!\,(2n + 2)!}{n!\,[(n + 1)!]^2\,(n + 2)!}. \tag{4.7}$$

It has been shown by Tutte (**4**, p. 254) that the number of rooted maps with n edges is

$$\frac{2\,(2n)!\,3^n}{n!\,(n + 2)!}. \tag{4.8}$$

Thus we find that the average number of trees per map is

$$\frac{1}{3^n}\frac{(2n + 1)!}{n!\,(n + 1)!}. \tag{4.9}$$

This average is asymptotically, as $n \to \infty$,

$$\frac{2}{\sqrt{(\pi n)}}\left(\frac{4}{3}\right)^n. \tag{4.10}$$

It is shown in **(2)** that the number of almost trivalent Hamiltonian rooted maps with divalent root vertex and $2n$ other vertices is also given by the right side of (4.7), as is the number of non-separable Hamiltonian-rooted triangularized digons with n internal vertices. A direct proof of the 1–1 correspondence between the tree-rooted maps and almost trivalent Hamiltonian rooted maps is given later in this paper.

5. Tree-rooted maps by partition. The degree of a vertex in a map is the number of edges incident upon the vertex, loops being counted twice. For our purposes the *partition* of a tree-rooted map is given by the vector

$$V = (v^*; v_1, v_2, \ldots) \tag{5.1}$$

where v^* is the degree of the root vertex and v_i is the number of non-root vertices of valence i. Formally this vector has infinitely many components; but only a finite number of them are non-zero.

Knowing the number of rooted plane trees with a given partition, it is possible to deduce the number of tree-rooted maps of a given partition by means of Lemma 1. Tutte has enumerated planted plane trees by partition in **(5)**. (A planted tree is a tree in which one monovalent vertex is distinguished as root vertex.) Using the fact that a rooted plane tree with root degree k is equivalent to an ordered set of k planted trees, we use this result to find that the number $r(k, V)$ of rooted plane trees with partition

$$(k; v_1, v_2, \ldots, v_m, \ldots)$$

is zero unless there is an integer n such that

$$n - k = \sum_{m=1}^{\infty} (m-1)v_m \quad \text{and} \quad n = \sum_{m=1}^{\infty} v_m,$$

in which case the number is

$$k(n-1)! \Big/ \prod_{m=1}^{\infty} (v_m)!. \tag{5.3}$$

The integer n is, of course, the number of non-root vertices of the tree. This result may be obtained as follows.

Explicitly Tutte shows that $r(1, V)$ is the coefficient of

$$\pi(V) = \prod_{m=1}^{\infty} \left\{\frac{1}{(m-1)!}\left(\frac{d}{dx}\right)^{m-1} f(x)\right\}^{v_m}$$

in the expansion of the function Q in terms of derivatives of an arbitrary, infinitely differentiable function $f(x)$ where

$$Q = f(x + Q).$$

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Since π obeys the law $\pi(V + W) = \pi(V)\cdot\pi(W)$, and since every rooted plane tree with root degree k is equivalent to an ordered set of k planted trees, $r(k, V)$ is the coefficient of $\pi(V)$ in the expansion of Q^k. But, by Lagrange's theorem (**6**, pp. 151, 153)

$$Q^k = \sum_{n=1}^{\infty} \frac{1}{n!}\left[\left(\frac{d}{da}\right)^{n-1} f^n(x + a)ka^{k-1}\right]_{a=0} = \sum_{n=1}^{\infty} \frac{k(n-1)!}{n!\,(n-k)!}\left(\frac{d}{da}\right)^{n-k}[f^n(x+a)]_{a=0}$$

which, by arguments analogous to those of Tutte (**5**, p. 274), produces (5.3) when expanded.

Since all coloured trees with $2j$ blue vertices and with partition vector $(v_1, v_2, \ldots, v_m)$ can be obtained from a rooted plane tree whose ordinary vertices have partition vector $(v_1 + 2j, v_2, v_3, \ldots, v_m, \ldots)$ simply by selecting $2j$ of the $v_1 + 2j$ monovalent ordinary vertices of the latter, we find that the number of coloured rooted plane trees with $2j$ blue vertices, root degree v^*, and in which the ordinary vertices have partition $(v_1, v_2, \ldots, v_m \ldots)$ is

$$(5.5) \qquad \binom{v_1 + 2j}{2j} v^*(i + 2j - 1)!\Big/\left[(v_1 + 2j)!\prod_{m=2}^{\infty}(v_m)!\right]$$

where i is the number of ordinary vertices. This reduces to

$$(5.6) \qquad v^*(i + 2j - 1)!\Big/\left[(2j)!\prod_{m=1}^{\infty}(v_m)!\right].$$

But the total number of vertices in a map with partition $(v^*; v_1, v_2, \ldots, v_m, \ldots)$ is

$$e = \frac{1}{2}\left[v^* + \sum_{m=1}^{\infty} m v_m\right],$$

and by the Euler polyhedron formula, the number of faces is

$$f = e + 2 - v = e + 1 - \sum_{m=1}^{\infty} v_m.$$

Also by the Euler polyhedron formula, and the definition of i and j, we have $i + 2j - 1 = e + f - 2$, and (5.6) may be written as

$$(5.7) \qquad v^*(e + f - 2)!\Big/\left[(2j)!\prod_{m=1}^{\infty}(V_m)!\right].$$

However there are $(2j)!/j!\,(j+1)! = (2j)!/f!\,(f-1)!$ plane trees which may be used as the second component of the pair of trees used to determine the map. Thus the number of tree-rooted maps of partition

$$(v^*; v_1, v_2, \ldots, v_m, \ldots)$$

is

$$(5.8) \qquad v^*(e + f - 2)!\Big/\left[f!\,(f-1)!\prod_{m=1}^{\infty}(v_m)!\right].$$

6. Polygons. In this section we shall exhibit a relationship between tree-rooted and Hamiltonian (polygon) rooted maps, and show how the methods of the preceding sections can be applied to the enumeration of certain classes of Hamiltonian rooted maps. In particular we discuss the perm, a generalization of the k-perm which occurs in **(2)**, and show that perms are equivalent to coloured trees.

A plane polygon J is said to be *dissected* if one of its residual domains, henceforth called its interior, is dissected into simply connected domains by a set of disjoint open arcs whose ends are vertices of J; see Figure 2. Such a

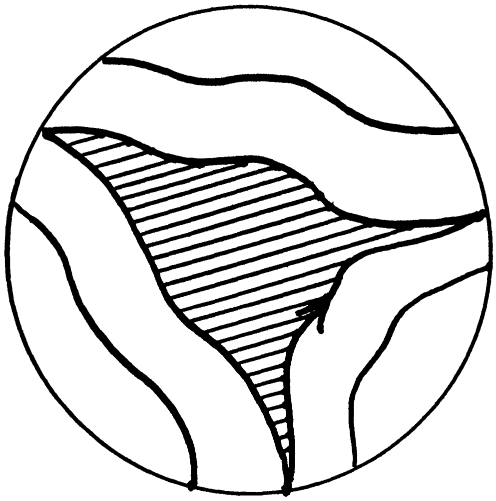

FIGURE 2

dissected polygon will be called a *perm* if one of the interior simply connected domains and an incident edge are distinguished as root face and root edge, respectively. The root edge may be considered to be oriented such that the root face is on the left as one proceeds from negative to positive. The negative end of the root edge (necessarily on the bounding polygon J) is distinguished as root vertex. A perm is said to be of type (n, k) if the polygon J contains n edges (vertices) and there are k edges bounding the root face.

LEMMA 2. *The set of* (n, k)*-perms is in* 1–1 *correspondence with the set of coloured trees with root degree* k *and* n *blue vertices.*

Proof. Let D be an (n, k)-perm. Take its dual $\bar{D}$ and root it by orienting the edge corresponding to the root edge of D away from the vertex corresponding to the root face of D. Let μ be that vertex of $\bar{D}$ corresponding to the exterior face of D. Interior to every edge E_i of $\bar{D}$ which is incident with the vertex μ, distinguish a blue vertex μ_i and delete the vertex μ and the arcs (μ, μ_i). The result is a rooted coloured tree as required. Indeed, the resultant plane graph D^* is connected since the exterior face of D is non-singular; thus μ was not a cut vertex. Also if D had m interior edges, it had $m + 1$ interior faces, so D^* has $n + m + 1$ vertices and $m + n$ edges; thus D^* is a tree. Since the root face of D was internal, the root vertex of D^* is colourless. Thus D^* is a coloured tree with root degree k and n blue vertices. The construction can be reversed. Thus the lemma.

COROLLARY. *The number of perms with j external edges and i non-root interior faces equals the number of rooted coloured plane trees with i ordinary and j blue vertices.*

This result is immediate from the preceding construction.

LEMMA 3. *The set of Hamiltonian rooted maps with n vertices in which the root face is incident with k edges is in* 1–1 *correspondence with the set of ordered pairs* (P_1, P_2) *where* P_1 *is an* (n, k)*-perm and* P_2 *is a dissected, rooted polygon with n vertices.*

The reader is reminded that a Hamiltonian rooted map is a rooted map in which a Hamiltonian polygon is distinguished as root polygon.

Proof. Let H be a Hamiltonian rooted map as above. Let J be its distinguished Hamiltonian polygon. J separates the map into two residual domains R_1 and R_2, R_1 being the domain containing the root face of H. In J, there are two edges incident with μ, the root vertex of H. One of these, directed away from μ, is such that the domain R_2 is on its right. This edge, with this orientation, serves to root $J \cup R_2$, which is P_2 of the lemma. $J \cup R_1$ is taken as P_1. The construction can be reversed.

Similar arguments may be used to prove

LEMMA 4. *There is a* 1–1 *correspondence between the class of dissected rooted n-gons with i internal faces and the class of planted coloured trees with i ordinary and* $n - 1$ *blue vertices.*

These lemmas, with suitable modification, have numerous applications in the enumeration of Hamiltonian rooted maps. For example, let $h_{f,v}$ denote the number of Hamiltonian rooted maps with f faces and v vertices. By the construction of Lemma 3, we may consider any such Hamiltonian rooted map as an ordered pair (P_1, P_2) where P_2 is a dissected rooted v-gon. Suppose P_2 has m internal faces; then P_1 is a perm with v external edges and $f - m - 1$ non-root interior faces. By Lemma 4, the number of dissected polygons suiting the description of P_2 is the same as the number of planted coloured trees with m ordinary and $v - 1$ blue vertices. Let us denote the number of planted coloured trees with i ordinary and j blue vertices by the symbol $d_{i,j}$.

By the corollary of Lemma 2, the number of perms answering the description of P_1 is $C_{f-1-m,v}$ where $c_{i,j}$ is the number of rooted coloured plane trees as defined in §4. Thus

$$h_{f,v} = \sum_{m=1}^{f-1} c_{f-m-1,v}\, d_{m,v-1}.$$

However, applying the Cayley–Polya formula **(1)** to plane trees, we find that $d_{i,j} = c_{i-1,j}$, and after substitution and simplification,

$$h_{f,v} = \frac{1}{v!\,(v-1)!}\sum_{m=1}^{f-1} \frac{(2f-2m+v-2)!\,(2m+v-3)!}{(f-m-1)!\,(f-m)!\,(m-1)!\,m!}. \tag{6.1}$$

Also it is shown in **(3)** that the number of (n, k) perms in which the non-root interior faces are all $(r+2)$-gons is

$$\frac{k\{(x[r+1]+k-1)!\}}{x!\,n!}, \qquad x = \frac{n-k}{r}, \tag{6.2}$$

and that the corresponding number of rooted dissected polygons with n vertices in which every face is an $(r+2)$-gon is

$$\frac{(y[r+1])!}{y!\,(n-1)!}, \qquad y = \frac{n-2}{r}. \tag{6.3}$$

Therefore the number of Hamiltonian rooted maps in which the root face is a k-gon and the other faces are $(r+2)$-gons, and in which there are n vertices, is (by the construction of Lemma 3) the product of the above numbers, namely

$$\frac{k\{(x[r+1]+k-1)!\}(y[r+1])!}{n!\,x!\,y!\,(n-1)!}, \qquad x = \frac{n-k}{r}, y = \frac{n-2}{r}. \tag{6.4}$$

(Both x and y must be integers or the result is to be considered zero.)

7. A direct correspondence. As mentioned in § 4, the number of almost trivalent Hamiltonian rooted maps with divalent root vertex and $2n$ trivalent vertices is equal to the number of tree-rooted maps with n edges. The correspondence between the two classes can be demonstrated as follows. The Hamiltonian circuit H in a map M of the former class divides the plane into two regions. That on the left of the root edge will be called the outside or exterior, and the remaining, the inside or interior (H must contain the root edge since the root vertex is divalent).

Each of these regions is dissected into simply connected domains (faces) by edges of the map M which terminate in distinct points of H; see Figure 3. With each face f of the interior region associate an internal point of that face $v(f)$: $v(f_i)$ and $v(f_j)$, $i \neq j$, are joined by an edge if the frontiers of f_i and f_j share at least one edge; call these edges red. The red edges are to be open arcs drawn such that they do not intersect. The resulting figure is a plane tree T. Let E be an external edge of M; that is, E is an arc which crosses the external region. Its ends will be incident with the frontiers of the internal faces f_i and f_j

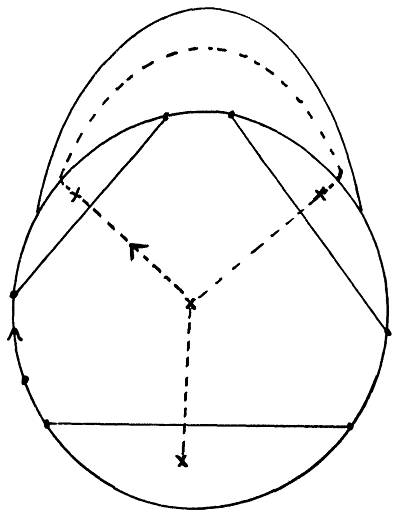

FIGURE 3

which need not be distinct. We join $v(f_i)$ and $v(f_j)$ by an edge, not violating planarity, and do this for every external edge. The resultant is a tree-rooted map M^* with n edges, in which the red tree is distinguished. To define the root edge of the map, recall that the root edge of the map M is an edge of H, since the root vertex is divalent. If the vertex w at its positive end is connected to an exterior edge, the correspondent of that in M^* is taken as root edge with w as negative end (an appropriate orientation modification is made in the convention if this edge is a loop). If that edge is internal, its correspondent is taken as root edge, with the vertex corresponding to the face incident with the root vertex of the Hamiltonian map as negative end. The construction can be reversed. The tree-rooted map dual to M is obtained by interchanging the roles of inside and outside regions.

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