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GENERAL POSITION PROPERTIES THAT CHARACTERIZE 3-MANIFOLDS

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ABSTRACT. This paper defines three simplicial approximation properties for maps of 2-cells and 2-spheres into spaces, each providing homotopical tameness conditions on the approximating images. These are the general position properties used in the two main results. The first shows that a resolvable generalized 3-manifold is a genuine 3-manifold if and only if it has the weakest of these approximation properties as well as a mild 3-dimensional disjoint disks condition known as the Light Map Separation Property. The second shows a resolvable generalized 3-manifold to be a 3-manifold if and only if it satisfies the strongest of these approximation properties.

In proposing the manifold recognition problem, J. W. Cannon [10] asked for a short list of relatively simple topological properties, reasonably easy to check, that characterize topological manifolds among topological spaces. He conjectured that manifolds might be characterized as generalized manifolds satisfying a minimal amount of general position. Here we address the 3-dimensional version of this recognition problem.

Though the focus will rest on one particular dimension, contrast with results obtained for other cases provides useful insight as well as motivation. In low dimensions ($n = 1, 2$) the problem has long been solved. It is convenient to explain this now in terms of *generalized n-manifolds* X^n, namely, locally compact, locally contractible, finite dimensional metric spaces with the local relative homology of $\mathbb{R}^n$ (i.e., $H_*(X^n, X^n \setminus x; \mathbf{Z})$ is isomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus 0; \mathbf{Z})$ for all $x \in X^n$). (In this paper manifolds and generalized manifolds will be assumed to have no boundary, unless otherwise specified.) For $n \leq 2$ the n-manifolds coincide with the generalized n-manifolds [33], but for $n > 2$ the situation is much more complex. Upon making the obvious observations that n-manifolds are generalized n-manifolds and that the latter are defined in terms of elementary properties, one sees why the goal in Cannon's conjecture is to recognize genuine manifolds among the generalized ones.

If $f: M \longrightarrow X$ is a proper, cell-like surjective map defined on an n-manifold and $\dim X < \infty$, then X is a generalized n-manifold, but examples like R. H. Bing's famous dogbone space [4] reveal that X need not be a topological manifold. Cell-like maps like this one form the primary source for non-manifold examples. With that in mind, one calls a generalized n-manifold X *resolvable* if there exists a proper, cell-like, surjective map

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$f: M \longrightarrow X$ defined on some n-manifold M, in which case the map f is called a (cell-like) *resolution* of X. Except for certain 3-dimensional examples whose existence depends on the hypothetical failure of the 3-dimensional Poincaré Conjecture, the generalized manifolds explicitly described in the literature are all known to be resolvable. According to F. S. Quinn [26], the existence of a resolution for a given generalized n-manifold X, $n \geq 4$, reduces to an integer-valued algebraic obstruction (to local surgery) problem. This obstruction has the intriguing feature of being locally defined and locally constant. Consequently, if X is connected and the obstruction vanishes on some open subset of X (for instance, if some open subset is a manifold), then X is resolvable.

QUINN'S RESOLUTION THEOREM [26]. *A generalized n-manifold X, $n \geq 4$, has a resolution if and only if a certain local surgery obstruction $i(X)$ equals 1.*

No example of a generalized n-manifold X with $i(X) \neq 1$ is known to exist. Note that no such connected example could have a manifold neighborhood at any point. If the 3-dimensional Poincaré Conjecture is false, however, there is a nonresolvable generalized 3-manifold X^3; moreover, X^3 contains a point x_0 such that $X^3 \setminus \{x_0\}$ is a 3-manifold [33]. M. G. Brin and D. R. McMillan, Jr. [8] even have such an example where $X^3 \setminus \{x_0\}$ is a monotone union of cubes with handles. The central 3-dimensional resolution problem asks: under the assumption that the Poincaré conjecture is true, do all generalized 3-manifolds have resolutions? T. L. Thickstun [31] has supplied an affirmative answer for generalized 3-manifolds with 0-dimensional nonmanifold set. See D. Repovš' survey article [27] for additional background information concerning the existence of resolutions.

In dimensions greater than four the work of R. D. Edwards [16] provides a means for detecting genuine manifolds among the resolvable generalized ones, in terms of the following optimal general position property. A metric space X is said to have the *Disjoint Disks Property* (abbreviated as DDP) if every pair of maps of I^2 to X can be approximated, arbitrarily closely, by a pair of maps with disjoint images.

EDWARDS' CELL-LIKE APPROXIMATION THEOREM [16]. *Let $p: M \longrightarrow X$ be a cell-like resolution of a generalized n-manifold X, $n \geq 5$. Then p is a near-homeomorphism if and only if X has the* DDP.

(A *near homeomorphism* is a map $X \longrightarrow Y$ onto a metric space that is the uniform limit of surjective homeomorphisms.) This paper presents some 3-dimensional adaptations of Edwards' theorem. Part of the difficulty has been to produce appropriate general position properties. The DDP is clearly inappropriate, being possessed by neither 3-manifolds nor 4-manifolds, so some alternative must be set forth.

No general position property found in the literature is entirely satisfactory. The first obvious analog to the DDP in dimension three—the disjoint arcs property—is useless since every generalized 3-manifold satisfies it [12]. The Map Separation Property and Dehn's Lemma Property introduced in [20] and further investigated in [29] function most effectively for those generalized 3-manifolds whose nonmanifold sets are known to have dimension at most zero. M. Starbird's two Disjoint Disks Properties [30], like the Inessentially Spanning Property of [12, Sec. 19], have the liability from this perspective of pertaining to the domains of given cell-like resolutions instead of intrinsically depending on the target spaces.

We investigate several alternatives here, chiefly described in terms of what we call simplicial approximation properties. A space B is said to have the *Weak Simplicial Approximation Property* (WSAP) if for each map $\mu: I^2 \longrightarrow B$ and each $\epsilon > 0$, there exists a map $\psi: I^2 \longrightarrow B$ such that $\mathrm{dist}_B(\mu, \psi) < \epsilon$ and $\psi(I^2)$ is contained in a finite union of 2-cells $D_i \subset B$, each 1-LCC embedded in B (see § 1 for the definition of 1-LCC). Moreover, B is said to have the *Simplicial Approximation Property* (SAP) if for each $\mu: I^2 \longrightarrow B$ and each $\epsilon > 0$, there exist a map $\psi: I^2 \longrightarrow B$ and a finite topological 2-complex $K_\psi \subset B$ such that (1) $\mathrm{dist}_B(\psi, \mu) < \epsilon$, (2) $\psi(I^2) \subset K_\psi$, and (3) $B \setminus K_\psi$ is 1-FLG in B (a term also defined in § 1). Finally, B is said to have the *Spherical Simplicial Approximation Property* (SSAP) if the same holds when I^2 is replaced by S^2 throughout. The 1-FLG condition is known to characterize tamely embedded 2-complexes K_ψ in 3-manifolds B [23]. There are three elementary but significant observations to make: first, that for a 2-complex $K_\psi \subset B$ having no local cut points, where B is a generalized 3-manifold, $B \setminus K_\psi$ is 1-FLG in B if and only if each 2-simplex in K_ψ is 1-LCC embedded in B; second, that SSAP implies SAP and SAP in turn implies WSAP; and third, that manifolds of dimension $n \geq 3$ have all of these approximation properties. Our main results are two recognition theorems for 3-manifolds, which depend on these terms. The first of them, Theorem 2.4, shows that a resolvable generalized 3-manifold X is a topological 3-manifold if and only if X possesses both the WSAP and a feature introduced and studied in [14] called the *Light Map Separation Property*. The second one, Theorem 3.1, demonstrates that a resolvable generalized 3-manifold is a topological 3-manifold if and only if it possesses the SSAP. As a corollary, a resolvable generalized 3-manifold with nowhere dense nonmanifold set is 3-manifold if and only if it has the SAP.

In a similar vein we also derive answers in terms of two hybrid properties, hybrid in the sense of being defined with the aid of given resolution even though ultimately the properties are measured in the range of the resolution map. These involve approximations to restrictions of the resolution either by embeddings or by maps achieving some disjointness. The most useful hybrids are two *Resolution Disk Embedding Properties*, abbreviated as RDEP and RDEP*: a cell-like map $p: M \longrightarrow B$ defined on a manifold M is said to have the RDEP if for every disk D tamely embedded (equivalently, 1-LCC embedded) in M and for every $\epsilon > 0$, there exists an embedding $\lambda: D \longrightarrow B$ satisfying $\mathrm{dist}_B(\lambda, p|D) < \epsilon$; if, more strongly, λ can be taken to be a 1-LCC embedding, then p is said to have the RDEP*. In § 4 we prove that every cellular resolution $p: M \longrightarrow X$ of a generalized 3-manifold X with RDEP* is a near-homeomorphism (and consequently X is a 3-manifold); capitalizing on our earlier work [14], we also find that a cellular resolution $p: M \longrightarrow B$ with RDEP is a near-homeomorphism whenever p has nondegeneracy set of embedding dimension ≤ 1 in M.

Another hybrid, called the *Resolution Disjoint Disks Property* (RDDP_ω), played a major role in [14]. § 5 puts forward a relative version of it powerful enough to imply that all cellular resolutions of generalized 3-manifolds with this relative RDDP_ω are near-homeomorphisms.

§6 offers an application to generalized 4-manifolds by demonstrating, for any such space X satisfying a 4-dimensional variation to the WSAP, how arbitrary resolutions of X can be altered to ones for which the image of the nondegeneracy set is 1-dimensional.

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1. **Preliminaries.** Given a map $f: X \longrightarrow Y$, we say that f is 1-1 *over* $A \subset Y$ if $f|f^{-1}(A)$ is 1-1, and we define the *nondegeneracy set* of f as $N_f = \cup\{f^{-1}(y) : f^{-1}(y)$ contains more than one point$\}$.

By the *singular set* $S(X)$ of a generalized n-manifold X we mean

$$\{x \in X : x \text{ has no Euclidean neighborhood in } X\}.$$

PROPOSITION 1.1 (APPROXIMATION PRINCIPLE FOR COUNTABLE UNIONS). *Suppose $f: M \longrightarrow X$ is a proper, cellular mapping defined on an n-manifold M and $\{A_j\}$ is a sequence of closed subsets of X such that, for each j, f can be approximated, arbitrarily closely, by a proper, cellular map $f_j: M \longrightarrow X$ that is 1-1 over A_j. Then f can be approximated, arbitrarily closely, by a cellular map $F: M \longrightarrow X$ that is 1-1 over $\cup A_j$, where F is a uniform limit of maps fh_i determined by self-homeomorphisms h_i of M.*

Edwards made strong and repeated used of this Approximation Principle in his Cell-like Approximation Theorem. For discussions of the argument, see [16, pp. 120–122] or [12, pp. 175–176].

A subset C of a space X is said to be *locally k-coconnected* (abbreviated as k-LCC) if every neighborhood $U \subset X$ of an arbitrary point $x \in X$ contains another neighborhood V of x such that all maps $\partial I^{k+1} \longrightarrow V \setminus C$ extend to maps $I^{k+1} \longrightarrow U \setminus C$. We list without proof some elementary information suggesting how certain k-LCC properties will be employed: C is (-1)-LCC in X iff $X \setminus C$ is dense; C is 0-LCC in X iff $U \setminus C$ is 0-connected, for all 0-connected open sets $U \subset X$; and C is k-LCC for $k \in \{-1, 0, 1\}$ iff the collection of all maps $\{f: I^2 \longrightarrow X \setminus C\}$ is dense in the space of maps $\{g: I^2 \longrightarrow X\}$ (with the sup-norm metric).

Somewhat similarly, for C homeomorphic to a finite complex $X \setminus C$ is said to have *free local fundamental group at $c \in C$* (abbreviated as: $X \setminus C$ is 1-FLG at c) if for each sufficiently small neighborhood U of c there exists another neighborhood V, with $c \in V \subset U$, and if W is any open connected set with $c \in W \subset V$, then for each non-empty component W' of $W \setminus C$ the (inclusion-induced) image of $\pi_1(W') \longrightarrow \pi_1(U')$ is a free group on $m - 1$ generators, where U' is the component of $U \setminus C$ containing W' and m is the number of components of $st(p) \setminus p$ that meet $C\ell(W')$. In standard fashion, $X \setminus C$ is 1-FLG in X if $X \setminus C$ is 1-FLG at each point of its frontier relative to X.

To better perceive the role of these local fundamental group conditions, consider a wedge W of two 2-simplexes in $\mathbb{R}^3$. The 1-LCC property cannot hold for W, as it always fails to be 1-LCC at the wedge point. Recall that $\mathbb{R}^3$ contains mildly wild arcs [17]; to be explicit, these are wild arcs expressed as a wedge of two tame ones. Any mildly wild arc can be blown up to a wild wedge W of two tame 2-simplexes in $\mathbb{R}^3$. What sets tame wedges apart from wild ones is the 1-FLG property, not the inapplicable 1-LCC property:

PROPOSITION 1.2. *Let K be a finite, connected 2-complex having no local separating points and $X \supset K$ be a generalized 3-manifold X. The following statements are equivalent.*

*1) $X \setminus K$ is 1-*FLG *in X;*

2) K is 1-LCC in X;

3) each 2-simplex of K is 1-LCC in X.

PROOF. Just as Nicholson observes in [23, Proposition 1.3], the equivalence of 1) and 2) follows routinely from the definitions. The implication 2) $\Rightarrow$ 3) relies on the determination of a homotopy between a given small loop in $X\setminus$ (2-simplex) and a product of small loops in $X \setminus K$, whereas the converse depends on little more than a local Seifert-vanKampen argument.

Next we point out why the simplicial approximation properties defined in the Introduction can be construed as reasonable 3-dimensional analogs of the DDP. Common to all three properties is their provision of a dense collection of maps into B with images covered by a finite union of 1-LCC embedded 2-cells.

PROPOSITION 1.3. *A generalized n-manifold X ($n \geq 5$) has* WSAP *(alternatively:* SAP*;* SSAP*) if and only if it has* DDP.

PROOF. Given maps $\mu_1, \mu_2 \colon I^2 \longrightarrow X$, where X has WSAP, we approximate μ_1 by a map $\varphi_1 \colon I^2 \longrightarrow X$ whose image is contained in a finite union of 2-cells 1-LCC embedded in X. Since these 2-cells have codimension at least 3 in X, $\varphi_1(I^2)$ is 1-LCC in X. Hence, we can approximate μ_2 by φ_2 where $\varphi_2(I^2) \cap \varphi_1(I^2) = \emptyset$.

The converse is even easier, for the DDP implies every map of a finite 2-complex into X can be approximated by a 1-LCC embedding [11, Theorem 2.1].

A *cellular resolution* of a generalized manifold X is a resolution $f \colon M \longrightarrow X$ for which all point preimages are cellular in the domain. Although certain resolvable generalized 3-manifolds have no cellular resolution (for example, the quotient space obtained by collapsing out some noncellular arc in S^3), they always have a resolution circumventing the Poincaré Conjecture locally, in the sense that each point preimage has a neighborhood embeddable in S^3. New terminology to describe such improved resolutions is not necessary for our purposes because among generalized 3-manifolds with WSAP the existence of a resolution implies the existence of a cellular one.

PROPOSITION 1.4. *Suppose X is a generalized 3-manifold with a (cell-like) resolution $f \colon M \longrightarrow X$. Then X has a cellular resolution $p \colon M' \longrightarrow X$ if and only if each $x \in X$ is 1-LCC in X.*

PROOF. By [13, Proposition 4.5], the resolution $f \colon M \longrightarrow X$ leads to another resolution $p \colon M' \longrightarrow X$ where all $p^{-1}(x)$ have neighborhoods in M' that embed in S^3. Then $x \in X$ being

1-LCC in X is equivalent to $p^{-1}(x)$ satisfying McMillan's Cellularity Criterion in M' (see [12, p. 145]) and, in this context, the latter is equivalent to $p^{-1}(x)$ being cellular in M' [21].

PROPOSITION 1.5. *If X is a generalized 3-manifold having* WSAP *(resp.,* SAP, SSAP*), then every $x \in X$ is 1-LCC in X.*

PROOF. Assume X has WSAP. From the space $\mathcal{F}$ of all maps $I^2 \longrightarrow X$ extract a countable dense subset $\{\psi_i\}$ where $\psi_i(I^2)$ is contained in a finite union of 1-LCC embedded 2-cells $D_{i,1}, \ldots, D_{i,k(i)}$. Given any neighborhood $U \subset X$ of $x \in \psi_i(I^2)$, choose some $D_{i,j}$ with $x \in D_{i,j}$. The 1-LCC condition provides a smaller neighborhood V of x in U such that any loop in $V \setminus D_{i,j}$ contracts in $U \setminus D_{i,j}$, and a Seifert-vanKampen argument shows $\pi_1(V \setminus \{x\}) \longrightarrow \pi_1(U \setminus \{x\})$ is trivial. For points of $X \setminus \cup \psi_i(I^2)$ the same 1-LCC property follows automatically from the denseness of $\{\psi_i\}$ in $\mathcal{F}$.

COROLLARY 1.6. *Every resolvable generalized 3-manifold with* WSAP *(resp.,* SAP, SSAP*) has a cellular resolution.*

Reference will be made to the shrinking theorem of [14], which involves another hybrid property. A resolution $p: M \longrightarrow X$ of a generalized 3-manifold X is said to have the *Resolution Disjoint Disks Property* (abbreviated here as "RDDP$_\omega$", although in [14] it was written simply as "RDDP") if for each $\epsilon > 0$, integer $k \geq 2$, and collection of k pairwise disjoint, tamely embedded disks E_i in M, there exist maps $g_i: E_i \longrightarrow X$ satisfying (i) $\text{dist}_X(g_i, p|E_i) < \epsilon$ and (ii) $g_i(E_i) \cap g_j(E_j) = \emptyset$ whenever $i \neq j$.

We will use a somewhat unfamiliar shrinking theorem in deriving one of our recognition theorems. The idea behind it has appeared previously in work of both Cannon [9] and E. P. Woodruff [34], [35]. We introduce new terminology, however, and say that a usc decomposition G of a metric space S is *locally semi-controlled shrinkable* if to every $g_0 \in G$ and neighborhood U_0 of g_0 there corresponds a neighborhood $W_0 \subset U_0$ of g_0 such that for every $\epsilon > 0$ and every homeomorphism $h: S \longrightarrow S$ there exists another homeomorphism $h': S \longrightarrow S$ satisfying:

1) h' and h coincide on $S \setminus U_0$,
2) $\text{diam}\, h'(g) < \epsilon$ for all $g \in G$ with $g \subset W_0$, and
3) $\text{diam}\, h'(g') < \epsilon + \text{diam}\, h(g')$ for all $g' \in G$.

This merely describes a partial control because the difference in the two motions is limited only by the original neighborhood U_0, not at all by the epsilon governing the sizes effected via the partial shrinking h'.

THEOREM 1.7. *If G is a locally semi-controlled shrinkable decomposition of a locally compact metric space S such that* $\dim(S/G) < \infty$, *then G is shrinkable (equivalently, the decomposition map $\pi: S \longrightarrow S/G$ is a near-homeomorphism).*

In effect, this was proved by Cannon [9, pp. 97–100]. Later Woodruff [35, Lemma 2.1] treated a closely related local shrinkability condition in the 3-manifold setting and developed a global shrinking result. Because of the result's significance and for the convenience of the readers, we sketch a proof for the compact case.

OUTLINE OF PROOF (S COMPACT). The definition of shrinkability requires that for every $\epsilon > 0$ there exists a homeomorphism $\Theta: S \longrightarrow S$ such that $\mathrm{dist}_{S/G}(\pi, \pi\Theta) < \epsilon$ and $\mathrm{diam}\,\Theta(g) < \epsilon$ for all $g \in G$.

Let $n = \dim(S/G)$. For fixed $g \in G$ specify a neighborhood $V_g \supset g$ with $\mathrm{diam}\,\pi(V_g) < \epsilon/(n+1)$. Restrict V_g using the upper semicontinuity of G so that $V_g = \pi^{-1}\pi(V_g)$. Find an open refinement Ω of the cover $\{\pi(V_g) \mid g \in G\}$ of S/G, where Ω is partitioned into $n+1$ subcollections $\Omega_0, \Omega_1, \ldots, \Omega_n$ and every Ω_j consists of pairwise disjoint sets. Now for each $g \in G$ choose an index $p(g)$ and $U_\alpha \in \Omega_{p(g)}$ such that $\pi(g) \in U_\alpha$; since G is locally semi-controlled shrinkable, g has a neighborhood $W_g \subset \pi^{-1}(U_\alpha)$ satisfying conditions 1–3 of the definition (with $\pi^{-1}(U_\alpha)$ playing the role of U_0). Note for future reference that $\mathrm{diam}\,U_\alpha < \epsilon/(n+1)$ for all α.

After again restricting to make $\pi^{-1}\pi(W_g) = W_g$, determine a finite subcover $W_1, \ldots, W_k$ of S (by compactness). Order them so that there are integers

$$0 = d(-1) < d(0) < d(1) < \cdots < d(n) = k$$

and $d(m-1) < s \leq d(m)$, $m \in \{0, 1, \ldots, n\}$, implies the existence of some $U_\alpha \in \Omega_m$ with $\pi^{-1}(U_\alpha) \supset W_s$. Relabel this index α as s, and set h_0 =Identity. Maintaining the specified ordering, use the definition of locally semi-controlled shrinkability to obtain successive homeomorphisms $h_1, \ldots, h_i, \ldots, h_k: S \longrightarrow S$ satisfying

1) h_{i-1} and h_i coincide on $S \setminus \pi^{-1}(U_i)$;
2) $\mathrm{diam}\,h_i(g) < \epsilon/2^i$ for all $g \in G$ in W_i; and
3) $\mathrm{diam}\,h_i(g') < \mathrm{diam}\,h_{i-1}(g') + \epsilon/2^i$ for all $g' \in G$.

Clearly the final homeomorphism h_k shrinks all elements of G to small size: for every $g \in G$ there exists an index $i = i(g)$ with $g \subset W_i$. Then $\mathrm{diam}\,h_i(g) < \epsilon/2^i$ by 2) above, and inductively we have by 3), for $j = 1, \ldots, k-i$,

$$\mathrm{diam}\,h_{i+j}(g) < \mathrm{diam}\,h_i(g) + \sum_{\ell=1}^{i} \frac{\epsilon}{2^{i+\ell}} < \sum_{\ell=0}^{j} \frac{\epsilon}{2^{i+\ell}} < \epsilon.$$

Moreover, size restrictions on the various $U_i \subset \pi(V_i)$ force

$$\mathrm{dist}_{S/G}(\pi h_{d(m-1)}, \pi h_{d(m)}) < \epsilon/(n+1) \text{ for } m \in \{0, 1, \ldots, n\},$$

because $\pi h_{d(m-1)}$, $\pi h_{d(m)}$ agree outside the union of the pairwise disjoint sets $\{U_{d(m-1)+1}, U_{d(m-1)+2}, \ldots, U_{d(m)}\}$ and because $\pi h_{d(m-1)}(\pi h_{d(m)})^{-1}(U_\ell) = U_\ell$ for all sets U_ℓ in this list. As a consequence, $\mathrm{dist}_{S/G}(\pi, \pi h_k) < \epsilon$.

2. **The First Recognition Theorem.** The foundation of this paper is an important recognition theorem first noted by J. W. Cannon. Using radically different terminology, Cannon stated our Theorem 2.1 as his 1-LC Taming Theorem [9, pp. 90, 102], only he did so for 2-spheres instead of 2-cells. (In the same paper Cannon also derived higher-dimensional versions of this theorem.) Although the details are formidable, the philosophical approach, which has since become standard in this subject, is straightforward: if X were a 3-manifold, the hypothesis would imply D is tame, and proofs for this known taming result can be modified to establish 2.1, even for 2-cell case at hand in place of 2-spheres.

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THEOREM 2.1 (CANNON). *Suppose X is a resolvable generalized 3-manifold such that $S(X) \subset D$, where D is a closed 2-cell 1-*LCC *embedded in X. Then X is a 3-manifold.*

COROLLARY 2.2. *If $p: M \longrightarrow X$ is a cellular resolution of a generalized 3-manifold X and if $D \subset X$ is a 1-*LCC *embedded closed 2-cell, then p can be approximated by a cellular map $q: M \longrightarrow X$ which is 1-1 over D.*

PROOF. Let G_D denote the (usc) decomposition of M induced by p over D; that is, G_D consists of all sets $p^{-1}(d)$, $d \in D$, plus the singletons from $M \setminus p^{-1}(D)$. By Theorem 2.1 and Armentrout's Cellular Approximation Theorem [3], the quotient map $\pi: M \longrightarrow M/G_D$ is a near-homeomorphism, and we define $q = p \circ \pi^{-1} \circ \theta$, where $\theta: M \longrightarrow M/G_D$ is a homeomorphic approximation to π. Obviously then

$$q^{-1}(d) = \theta^{-1}\pi p^{-1}(d) = \text{ point}$$

for all $d \in D$ while, for $x \in X \setminus D$, $q^{-1}(x) = \theta^{-1}\pi p^{-1}(x)$ is embedded in $M \setminus q^{-1}(D)$ just like $p^{-1}(x)$ in $M \setminus p^{-1}(D)$, implying q is cellular. (Remark: making θ sufficiently close to π causes q to be close to $p \circ \pi^{-1} \circ \pi = p$.)

LEMMA 2.3. *Suppose $p: M \longrightarrow X$ is a cellular resolution of a generalized 3-manifold X which satisfies* WSAP. *Then p can be approximated by a cellular resolution $q: M \longrightarrow X$ where $q(N_q)$ is 0-dimensional, q is a uniform limit of $\{ph_i\}$, and each $h_i: M \longrightarrow M$ is a homeomorphism.*

PROOF. Corollary 2.2 combines with the Countable Shrinking Principle (Proposition 1.1) to provide $q: M \longrightarrow X$ which is 1-1 over a countable dense subset of images $\psi_i(I^2)$ obtained from WSAP.

Since $q(N_q) \subset X \setminus \cup\, \psi_i(I^2)$ is σ-compact, it is enough to show every compact subset of $X \setminus \cup\, \psi_i(I^2)$ is 0-dimensional. Consider $A \subset U$, where A is compact and U is an orientable open subset of X [7] with $A \subset U \setminus \cup\, \psi_i(I^2)$. Clearly it suffices to prove $\dim A \leq 0$ and, to that end, it suffices to show A has (integral) cohomological dimension ≤ 0 (cf. [32]). Since X locally satisfies Alexander-Lefshetz duality (see [7] or [1]), $H^1(A, A') \cong H_2(U \setminus A', U \setminus A)$ for every compact $A' \subset A$. We will explain why the latter is trivial.

Strictly speaking, we first should prove that $\dim A \leq 1$, but we omit the details and treat this as a fact—its proof is similar to but easier than the forthcoming one that $\dim A \leq 0$. Given a relative singular 2-cycle with carrier $(C, \partial C) \subset (U \setminus A', U \setminus A)$, subdivide to make the singular 2-simplexes in this chain extremely small (all contained in small open sets $V \subset U \setminus A'$), and homotopically adjust, using the omitted step $\dim A \leq 1$, to make each boundary miss A. Let σ denote a typical singular 2-simplex from this subdivided cycle. For some open set V, σ can be approximated by a singular disk $\psi_k(I^2) \subset V \setminus A$ closely enough that there is a homotopy between $\partial\sigma$ and $\psi_k(\partial I^2)$ in $V \setminus A$, implying $\partial\sigma$ bounds a singular disk $D_\sigma \subset V \setminus A$. Had we insisted each V be contractible in $U \setminus A'$, a matter easily arranged, then σ and D_σ would be homologous (rel ∂) relative cycles in

$(U\setminus A', U\setminus A)$, and the combination over all 2-simplexes σ appearing in C would show $(C, \partial C)$ to be homologically trivial.

The next general position property appeared in [14] and now reappears here as part of our first recognition theorem. A generalized 3-manifold Y has the *Light Map Separation Property* (abbreviation: LMSP) if, for every positive integer k, every $\epsilon > 0$, and every map $f: B \longrightarrow Y$ defined on the disjoint union B of k 2-cells B_i such that (i) $N_f \subset \operatorname{Int} B$, (ii) $\dim(N_f) \leq 0$, and (iii) $\dim(Z_f) \leq 0$, where

$$Z_f = \{\, y \in Y \mid y \in f(B_i) \cap f(B_j) \text{ for some } i \neq j \,\},$$

there exists a map $F: B \longrightarrow Y$ satisfying

1) $\operatorname{dist}_Y(F, f) < \epsilon$,
2) $F|\partial B = f|\partial B$, and
3) the images $\{\, F(B_i) \mid i = 1, \ldots, k \}$ are pairwise disjoint.

RECOGNITION THEOREM 2.4. *A resolvable generalized 3-manifold X is a topological 3-manifold if and only if X possesses the* WSAP *and* LMSP.

PROOF. If X has WSAP, Corollary 1.6 ensures it has a cellular resolution $p: M \longrightarrow X$, and Lemma 2.3 attests that $p: M \longrightarrow X$ can be approximated by $q: M \longrightarrow X$, where $q(N_q)$ is 0-dimensional. When X also has LMSP, the Recognition Theorem (1.2) of [14] indicates X is a 3-manifold.

Conversely, as remarked in the Introduction, 3-manifolds clearly satisfy WSAP. Theorem 1.2 of [14] certifies that they also satisfy LMSP.

REMARKS. Still unsettled is the issue of whether in Theorem 2.4 the WSAP by itself implies X is a 3-manifold; the warning below spotlights a key difficulty. It can be said that WSAP is not enough, however, if the 1-FLG requirement is dropped, since the non-manifolds resulting from decompositions of $\mathbb{R}^3$ into points and straight line segments of Armentrout [2] or Bing [6] satisfy this less restrictive WSAP condition.

WARNING. Bing [6] (or see [12, Example 9.6]) has described a remarkably simple cellular decomposition of S^3 automatically leading to a resolution $p: S^3 \longrightarrow X$ of a non-manifold X where $p(N_p)$ is countable and all singular disks in X can be approximated by singular disks in $X\setminus p(N_p)$. Sharing key properties with the intermediate cell-like map $q: M \longrightarrow X$ which arises in the proof of Theorem 2.4, Bing's map makes it clear that, in order to approximate q by homeomorphisms without appealing to the LMSP hypothesis, one must rely on additional geometric features of singular disks $\psi(I^2)$ obtained from WSAP, not merely their avoidance of $q(N_q)$.

THEOREM 2.5. *If X is a generalized 3-manifold with* WSAP *and* $p: M \longrightarrow X$ *is a cellular resolution with* RDDP_ω, *then p is a near-homeomorphism.*

PROOF. Use Lemma 2.3 to approximate p by $q: M \longrightarrow X$, where $q(N_q)$ is 0-dimensional and q is a limit of a sequence $\{ph_i\}$ determined by homeomorphisms $h_i: M \longrightarrow M$. By [15] q can be further adjusted so N_q has embedding dimension ≤ 1

and q has RDDP_ω, by virtue of arising as a uniform limit of $\{ph_i\}$ as before. Then the Shrinking Theorem (1.1) of [14] yields the conclusion.

3. **The Second Recognition Theorem.**

RECOGNITION THEOREM 3.1. *A resolvable generalized 3-manifold is a topological 3-manifold if and only if it possesses the* SSAP.

This theorem is an immediate consequence of Corollary 1.6 and the subsequent Proposition 3.2.

PROPOSITION 3.2. *A cellular resolution* $p: M^3 \longrightarrow X$ *is a near-homeomorphism if and only if* X *has* SSAP.

PROOF. The forward implication is trivial. For the reverse, specify countable collections $\{\psi_i: S^2 \longrightarrow X\}$ with $\{\psi_i\}$ dense in the space of all maps $S^2 \longrightarrow X$ and 2-complexes $\{K_{\psi(i)} \supset \psi_i(S^2)\}$ satisfying the conditions of SSAP. Use Proposition 1.1 and Corollary 2.2 to approximate p so it is 1-1 over $\cup K_{\psi(i)}$.

It will suffice to show that this approximation (still called p) is a near-homeomorphism. To that end, we shall prove that the decomposition $G = \{p^{-1}(x) : x \in X\}$ induced by p is locally semi-controlled shrinkable and apply Theorem 1.7. To be completely rigorous, we should insert a homeomorphism $h: M \longrightarrow M$ and study $\{hp^{-1}(x) : x \in X\}$, but this would affect only the notation, not the proof, so we suppress h.

Fix $g_0 \in G$ and a neighborhood U_0 of g_0 in M^3, and restrict U_0, if necessary, to an open 3-cell neighborhood. Choose a neighborhood $V \subset X$ of $x_0 = p(g_0)$ with $p^{-1}(V) \subset U_0$. Find a map $\psi_i: S^2 \longrightarrow X$ from the collection above giving a singular 2-sphere $S = \psi_i(S^2) \subset V \setminus \{x_0\}$ such that ψ_i is null-homotopic in V but not in $V \setminus \{x_0\}$ and S separates x_0 from $X \setminus V$. Let W denote the component of $X \setminus S$ containing x_0, and let $W_0 = p^{-1}(W)$.

The 2-complex $K_{\psi(i)} \supset S$ promised by the SSAP hypothesis can be trimmed back, if necessary, to a 2-complex K with $S \subset K \subset V \cap K_{\psi(i)}$ and $X \setminus K$ 1-FLG in X. Since p is 1-1 over $K_{\psi(i)} \supset K$, approximate lifting properties (cell-like maps naturally induce π_1-isomorphisms) routinely disclose that, for $K' = p^{-1}(K)$, $M \setminus K'$ is 1-FLG in M. As a result, K' is tame in M [23].

For simplicity we first discuss the case in which K' has no local separating points.

Find an open set U_1 with $K' \subset U_1 \subset U_0$ such that $p^{-1}(x) \cap U_1 \neq \emptyset$ implies $\operatorname{diam} p^{-1}(x) < \epsilon/6$. We construct an auxiliary map χ of M to itself fixed outside U_1, collapsing some topological regular neighborhood $N(K')$ onto K', moving points less than $\epsilon/6$, being 1-1 over $M \setminus K'$. This is arranged so that if z belongs to the interior of an i-cell of K', then $\chi^{-1}(z)$ is a tame $(3-i)$-cell. (Express χ as $\chi_2\chi_1\chi_0$, where χ_0 collapses a 3-cell neighborhood of each vertex v to v while sending both K' and its 1-skeleton to themselves, where χ_1 squeezes a relative regular neighborhood of each edge β in $K'^{(1)} = \chi_0(K'^{(1)})$ to β while sending K' to itself, and where χ_2 behaves similarly for 2-simplexes of K'.) In particular, we do this carefully to make certain that the decomposition

$$\Gamma = \{(p\chi)^{-1}(x) \mid x \in X \setminus K\} \cup \big\{\{y\} \mid y \in N(K')\big\}$$

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is upper semicontinuous, which comes about by modifying χ so all components of $\partial\chi^{-1}(x)\setminus \operatorname{Int} N(K')$ become singletons. Since each of these components is cell-like (here is where the hypothesis concerning no local separating is invoked), we can apply a theorem of R. L. Moore [22] about cell-like decompositions of 2-manifolds to obtain a map $\theta : M \longrightarrow M$ which sends $N(K')$ onto itself, sends the components of $\partial\chi^{-1}(x)\setminus \operatorname{Int} N(K')$ to distinct points, and is 1-1 over $M\setminus \partial N(K')$. We define the desired modification as $\chi\theta^{-1}$. Note that $p^{-1}(x)\cap U_1 \neq \emptyset$ implies $\operatorname{diam}(p\chi)^{-1}(x) < \epsilon/2$.

The Sphere Theorem [24] ensures the existence of a PL embedded 2-sphere Σ in $\operatorname{Int} N(K') \subset U_1$ separating $(p\chi)^{-1}(x_0)$ from $M\setminus \chi^{-1}(W_0)$, so Σ bounds a 3-cell C with

$$(p\chi)^{-1}(x_0) \subset \chi^{-1}(W_0) \subset \operatorname{Int} C \subset C \subset U_0.$$

There exists a homeomorphism $h: M \longrightarrow M$ fixed outside C such that

$$\operatorname{diam} h(p\chi)^{-1}(x) < \epsilon \text{ for all } x \in p\chi(C)\setminus K.$$

All that remains is to find a map $f: M \longrightarrow M$ shrinking out all the nontrivial point preimages of $h\chi^{-1}$ without allowing the sizes of $h(p\chi)^{-1}(x)$ to increase very much. This can be accomplished by shrinking the tame complex $h\big(N(K')\big)$ to a copy of K' via a map supported in $C\cup U_1$ and expressed as a finite sequence of adjustments each moving points less than $\epsilon/4$—any nondegenerate element $h(g')$, $g' \in \Gamma$ and $g' \subset C\cup U_1$, which expands to dangerously large size (i.e., $\epsilon/2 < \operatorname{diam}\Phi h(g') < \epsilon$) after the composition Φ of finitely many adjustments is left alone by the remainder—and the guiding principles require a homeomorphism $\lambda : K' \longrightarrow f(K')$ such that $\lambda\chi|K' = fh|K'$. Then, being well-defined, the rule $f\circ h\circ \chi^{-1}$ is a self-homeomorphism of M, and one can easily verify it has the desired effect on G.

Now we describe the modification needed for the general case in which K is allowed to have local separating points. The major change appears with the map $\theta : M \longrightarrow M$, which now will take $N(K')$ into, not onto, itself, while sending components of $\partial\chi^{-1}(z)\setminus \operatorname{Int} N(K')$ to distinct points and being 1-1 over $M\setminus \theta\left(N(K')\right)$. When x is a vertex with disconnected link in K, for each non-simply connected component κ of $\left(\partial\chi^{-1}(z)\setminus \operatorname{Int} N(K')\right)$, we produce a compact set $T_\kappa \subset \chi^{-1}(x)$ such that $\kappa\cup T_\kappa$ is cellular in M and $T_\kappa\setminus\kappa$ is a finite union of open disks in $\operatorname{Int}\chi^{-1}(x)$. Standard general position adjustments allow us to make the sets T_κ associated with the various components κ be pairwise disjoint. If $\Sigma\cap T_\kappa \neq \emptyset$, we trade subdisks of Σ for those parallel to disks in $T_\kappa\setminus\kappa$ (since $\Sigma\cap T_\kappa \subset T_\kappa\setminus\kappa$) to form a new PL 2-sphere Σ' in $N(K')\setminus \cup\{T_\kappa\}$ bounding another 3-cell C' with

$$(p\chi)^{-1}(x_0) \subset \chi^{-1}(W_0) \subset \operatorname{Int} C' \subset C' \subset U_0.$$

Then θ will crush the distinct sets $T_\kappa\cup\kappa$ to distince points, keeping Σ' pointwise fixed. Similarly, for points x belonging to the interior of a 1-simplex β, where $\operatorname{Int}\beta$ is an open subset of K, we can alter Σ' to make $\Sigma'\cap\chi^{-1}(x) = \emptyset$ and then require that $\theta\chi^{-1}(x)$ equals x. The rest of the program proceeds as before.

COROLLARY 3.3. *Let X be a resolvable generalized 3-manifold. The following statements are equivalent:*

(i) X has the WSAP *and the* LMSP*;*
(ii) X has the SSAP*;*
(iii) X is a 3-manifold.

The proof of Theorem 3.1 actually establishes the slightly stronger result below.

THEOREM 3.4. *A resolvable generalized 3-manifold X is a 3-manifold if each $x \in X$ is 1-*LCC *and has arbitrarily small neighborhoods U such that there exists a map $\mu: S^2 \longrightarrow U\setminus\{x\}$ such that μ is null-homotopic in U but not in $U\setminus\{x\}$ as well as a 2-complex K satisfying properties 2) and 3) of* SSAP.

Our formal application of Theorem 3.4 is the following corollary, which is also an immediate consequence of [9, Theorem 62]. The argument here, which generalizes Cannon's in [9], is less efficient than his due to complications surrounding the blow-up procedure required in Proposition 3.2. These complications emerge specifically for treating singular 2-spheres but can be easily circumvented in deriving (3.5) directly.

COROLLARY 3.5 (CANNON [9]). *A cellular resolution $p: M \longrightarrow X$ is a near-homeomorphism if and only if, for every 2-sphere S tamely embedded in M, $p|S$ can be approximated, arbitrarily closely, by 1-*LCC *embeddings $\lambda: S \longrightarrow X$.*

THEOREM 3.6. *Suppose X is a resolvable generalized 3-manifold such that* $\dim S(X) \leq 2$. *Then X is a 3-manifold if and only if X has* SAP.

PROOF. This is another application of Theorem 3.4. Name a cellular resolution $p: M \longrightarrow X$ (Corollary 1.6) and consider any neighborhood U of $x \in X$. Use cellularity to find a pair of open 3-cells W, W' in M and a pair of connected open subsets V, V' in X with null-homotopic inclusion $V' \longrightarrow V$ and

$$p^{-1}(x) \subset W' \subset p^{-1}(V') \subset p^{-1}(V) \subset W \subset p^{-1}(U).$$

Locate a tame 2-sphere $\Sigma \subset W'$ separating $p^{-1}(x)$ from $M\setminus W'$ (and thus from $M\setminus W$), and adjust Σ so $\Sigma \cap p^{-1}\big(X\setminus S(X)\big)$ contains a 2-cell B.

Choose a homeomorphism $e: I^2 \longrightarrow \Sigma\setminus \operatorname{Int} B$ and invoke SAP to obtain a map $\psi: I^2 \longrightarrow V'\setminus\{x\}$ with $\psi(I^2)$ in the usual sort of nicely embedded 2-complex K and with ψ close enough to $p \circ e$ to make $\psi|\partial I^2$ null-homotopic in $V'\setminus S(X)$. Combine ψ and the null-homotopy as a map $\mu': \Sigma \longrightarrow V'\setminus\{x\}$ with $\mu'|\Sigma\setminus \operatorname{Int} B = \psi e^{-1}$ and $\mu'(B) \subset V'\setminus S(X)$. Since again $\psi(I^2) \cap \big(V'\setminus S(X)\big)$ is tame [23], general position techniques in the 3-manifold $V'\setminus S(X)$ yield an approximation μ to μ' such that μ and μ' coincide on $\Sigma\setminus B$ and $K \cup \mu(\Sigma\setminus B)$ is a 2-complex whose complement is 1-FLG in X (simply require $\mu(\Sigma\setminus B) \cup \big(K\setminus S(X)\big)$ to be tame in $V'\setminus S(X)$). All of these adjustments can be done so $p|\Sigma$ and μ are homotopic in $V'\setminus\{x\}$.

Clearly μ is null-homotopic in V, but it cannot be so in $V\setminus\{x\}$, for otherwise relative approximate lifting properties would show Σ is homotopically trivial in $p^{-1}(V\setminus\{x\}) \subset W\setminus p^{-1}(x)$, an impossibility since the latter retracts to Σ.

4. **Hybrid Properties.** This section examines the resolution disk embedding properties defined in the Introduction. Our primary goal here is to verify that all cellular resolutions with RDEP* are near-homeomorphisms; along the way, we obtain the same conclusion for cellular resolutions $p: M \longrightarrow X$ with RDEP, provided $\dim p(N_p) \leq 0$. Approximation techniques presented in § 2 make this quite effortless.

In a precise sense each of these resolution disk embedding properties, like the WSAP, is a natural analog of the DDP for resolvable generalized manifolds.

LEMMA 4.1. *Suppose $p: M \longrightarrow X$ is a resolution of a generalized n-manifold, $n \geq 5$. Then the following are equivalent:*

1) p has RDEP;

2) p has RDEP*;

3) X has DDP.

PROOF. Assume 1). Consider maps $\mu_1, \mu_2: I^2 \longrightarrow X$ and $\epsilon > 0$. Standard approximate lifting and general position methods give tame embeddings $e_i: I^2 \longrightarrow M$ $(i = 1, 2)$ with $\text{dist}_X(pe_i, \mu_i) < \epsilon / 2$. Connecting the two disks together with a tame band, we find a disk E with $e_i(I^2) \cup e_2(I^2) \subset E \subset M$ and apply RDEP to obtain an embedding $\lambda: E \longrightarrow X$ with $\text{dist}_X(\lambda, p|E) < \epsilon / 2$. The maps λe_1, λe_2 show 3) holds.

That 3) implies 2) stems from the same Cannon result [11, Theorem 2.1] used in Proposition 1.3. Of course, that 2) implies 1) is immediate.

LEMMA 4.2. *Every resolution $p: M \longrightarrow B$ with* RDEP *also has* RDDP_ω.

PROOF. Given a finite collection $\{E_i\}$ of tame disks in M, use multiple banding operations as in Lemma 4.1 to collect them in a single disk E, where $\cup E_i \subset E \subset M$, and apply RDEP. The restrictions $g_i = \lambda | E_i$ of the resulting embedding $\lambda: E \longrightarrow B$ establish RDDP_ω.

THEOREM 4.3. *If $p: M \longrightarrow B$ is a cellular resolution with* RDEP *and p has (domain) nondegeneracy set of embedding dimension ≤ 1, then p is a near-homeomorphism.*

PROOF. By Lemma 4.2 p has RDDP_ω, so the Shrinking Criterion (1.1) of [14] gives the conclusion.

LEMMA 4.4. *If $p: M \longrightarrow X$ had* RDEP *(*RDEP*) and $\{h_i\}$ is a sequence of homeomorphisms such that $\{ph_i\}$ converges uniformly to $q: M \longrightarrow B$, then q has* RDEP *(*RDEP**).*

PROOF. This is straightforward: choose i with ph_i close to q, and apply RDEP (RDEP*) for p to the disk $h_i(E) \subset M$.

THEOREM 4.5. *Every cellular resolution $f: M \longrightarrow X$ with* RDEP, *where* $\dim f(N_f) \leq 0$, *is a near-homeomorphism.*

PROOF. According to [15], f can be approximated by $F: M \longrightarrow X$, where there are self-homeomorphisms h_i of M such that $\{fh_i\}$ converges uniformly to F and the nondegeneracy set N_F of F has embedding dimension ≤ 1. Lemma 4.4 and Theorem 4.3 disclose F and f can be approximated by homeomorphisms.

THEOREM 4.6. *Every cellular resolution $p: M \longrightarrow X$ with* RDEP* *is a near-homeomorphism.*

PROOF. We identify a countable collection of 1-LCC embedded disks $D_i \subset X$, consisting of images promised by RDEP* of a dense subset of the space of all maps of the form $p\lambda$, where $\lambda : I^2 \longrightarrow M$ is an embedding. As in the proof of Lemma 2.3, we approximate p by a resolution $q = \lim(ph_i)$ which is 1-1 over $\cup D_i$. Once we verify $X \setminus \cup D_i$ is 0-dimensional, Lemma 4.4 and Theorem 4.5 will show q and p are near-homeomorphisms.

Consider any $x \in X \setminus \cup D_i$ and connected neighborhood U of $q^{-1}(x)$ in M. Select an index i such that $B = q^{-1}(D_i) \subset U$. Since B is 1-LCC and therefore locally flat in U [5], we can produce another 2-cell $B' \subset U$ such that $B \cup B'$ is a 2-sphere separating the cellular set $q^{-1}(x)$ from $M \setminus U$. Finally, by choosing another index j for which there exists a homotopy in $U \setminus q^{-1}(x)$ between $q^{-1}(D_j)$ and B', we can conclude $B \cup q^{-1}(D_j)$ also separates $q^{-1}(x)$ from $M \setminus U$, as required.

COROLLARY 4.7. *Let X be a generalized 3-manifold with a cellular resolution $p: M \longrightarrow X$. The following statements are equivalent.*

(i) p has RDEP *and X has* WSAP;

(ii) p has RDEP*;

(iii) X is a 3-manifold.

PROOF. Clearly (iii) implies both (i) and (ii). That (i) implies (iii) is an immediate consequence of Lemma 2.3 and Theorem 4.5, and Theorem 4.6 certifies that (ii) implies (iii).

5. **Another Hybrid Property.** Whether all cellular resolutions $p: M \longrightarrow X$ with RDDP_ω are near-homeomorphisms is an open problem. One natural and attractive approach involves attempting to show that, for all closed $A \subset X$, the map associated with the decomposition G_A induced by p over A (i.e., G_A consists of all $p^{-1}(a)$, $a \in A$, and singletons from $M \setminus p^{-1}(A)$) also has RDDP_ω, but to date this secondary problem has proved insurmountable. Here in §5 we look at a relative version of the RDDP_ω strong enough to allow consummation of this attack.

A cellular resolution $p: M \longrightarrow B$ has the *relative resolution disjoint disks property* (RRDDP_ω) if for every finite collection $\{E_i \mid i = 1, \ldots, k\}$ of pairwise disjoint disks tamely embedded in M, for every collection of 1-dimensional finite graphs $\{\Gamma_i \subset E_i\}$, and for every $\epsilon > 0$, there exist a homeomorphism $\theta: M \longrightarrow M$ moving points less than ϵ and maps $f_i: E_i \longrightarrow B$ satisfying: (1) $\{f_i(E_i)\}$ is a pairwise disjoint collection; (2) $f_i | \Gamma_i = p\theta | \Gamma_i$; and (3) $\text{dist}_B(f_i, p\theta | E_i) < \epsilon$.

LEMMA 5.1. *If $p: M \longrightarrow B$ has* RRDDP_ω *and A is a closed subset of B, then the natural map $\pi: M \longrightarrow M/G_A$ also has* RRDDP_ω.

PROOF. Fix $\epsilon > 0$ and a collection $\{E_i \mid i = 1, \ldots, k\}$ of pairwise disjoint disks tamely embedded in M. For each i we require a small mesh triangulation T_i of E_i (explicit restrictions on mesh size are provided in the next paragraph). Then we let Γ_i denote the 1-skeleton of E_i and P_i the union of all $\sigma \in T_i$ meeting $p^{-1}(A)$.

Find $\delta > 0$ and a compact neighborhood U of $N\big(A \cap p(\cup E_i); \delta\big)$ in X such that, for every $Y \subset U$ with $\operatorname{diam} Y < \delta$, $\operatorname{diam} \pi p^{-1}(Y) < \epsilon / 9$. Also find $\delta' > 0$ and a compact neighborhood $R \supset N\big(p^{-1}p(\cup E_i); 3\delta'\big)$ in M such that for all $Y' \subset R$ with $\operatorname{diam} Y' < 3\delta'$, $\operatorname{diam} \pi(Y') < \epsilon / 9$. Then find $\eta > 0$ such that $\operatorname{diam} p(K) < \delta / 7$ for $K \subset N(\cup E_i; \eta)$ with $\operatorname{diam} K < \eta$. Finally, measure the distance d between $p^{-1}(A)$ and $\cup(E_i \setminus P_i)$, assume $2\gamma < \min\{\delta, \delta', \eta, d\}$, and choose T_i of mesh less than γ.

The RRDDP_ω promises a γ-homeomorphism $h: M \longrightarrow M$ and maps $f_i: E_i \longrightarrow B$ such that (1) $\{f_i(E_i)\}$ is a pairwise disjoint collection; (2) $f_i|\Gamma_i = ph|\Gamma_i$, and (3) $\operatorname{dist}_B(f_i, ph|E_i) < \delta / 7$. Clearly $h(\sigma) \subset R$ and $\operatorname{diam} h(\sigma) < 3\delta'$ for all $\sigma \in T_i$, so $\operatorname{diam} \pi h(\sigma) < \epsilon / 9$. Restrictions on γ imply (4) $\operatorname{dist}_B(p|E_i, ph|E_i) < \delta / 7$. Relative approximate lifting properties of cell-like maps ensure the existence of maps $F_i: E_i \longrightarrow M$ such that (5) $\{pF_i(E_i)\}$ are pairwise disjoint; (6) $F_i|\Gamma_i = h|\Gamma_i$, (7) $\operatorname{dist}_B(pF_i, ph|E_i) < \delta / 7$. Now (3), (4), and (7) combine with the other size restrictions to yield: (8) for all $\sigma \in T_i$, $pF_i(\sigma)$ lives in the $(3\delta / 7)$-neighborhood of $p(\sigma)$. Hence, $\sigma \subset P_i$ implies $\operatorname{diam} p(\sigma) < \delta / 7$ and $pF_i(\sigma) \subset U$. For such σ choices of γ, δ give that

$$\operatorname{dist}_B(\pi|\sigma, \pi F_i|\sigma) = \operatorname{dist}_B(\pi p^{-1}p|\sigma, \pi p^{-1}pF_i|\sigma) < \epsilon / 9;$$

as a result, $\operatorname{diam} \pi F_i(\sigma) < \epsilon / 3$.

One could hope to define the required maps $E_i \longrightarrow M / G_A$ simply using $\pi F_i|P_i$ and $\pi h|E_i \setminus P_i$, but there remains a difficulty caused by potential intersections between $\pi F_i(P_i)$ and $\pi h(E_j \setminus P_j)$. All these intersections occur in $\pi h(\operatorname{Int} \sigma)$, where $\sigma \subset \operatorname{Cl}(E_j \setminus P_j)$ is a 2-simplex of T_j, and constraints on h reveal $h(\sigma) \cap p^{-1}(A) = \emptyset$, so $\pi h(\operatorname{Int} \sigma)$ resides in the 3-manifold $\pi\big(M \setminus p^{-1}(A)\big)$. Working successively with indices k for which some component of $\pi F_k(P_k)$ is contained in a disk in $\pi h(\sigma)$ whose interior meets no other $\pi F_\ell(P_\ell)$ (including $\pi F_j(P_j)$ among these possibilities here is imperative), we do (singular) disk trading to eliminate such intersections without creating new ones; the map redefinition occurs on disks in the interiors of 2-simplexes τ from T_i in P_i and entails adjustments limited by $2\epsilon / 3$, the diameter of $\pi F_i(\tau) \cup \pi h(\sigma)$. The outcome is a collection of maps $\nu_i: E_i \longrightarrow M / G_A$ close to $p|E_i$ and having mutually exclusive images, as required.

LEMMA 5.2. *If $p: M \longrightarrow B$ has* RRDDP_ω *and A is a closed subset of B, then the natural map $\pi: M \longrightarrow M / G_A$ is a near-homeomorphism.*

PROOF. By induction on $\dim A$. If $\dim A = 0$, this follows immediately from Lemma 5.1 and our earlier shrinking theorem from [14]. When $\dim A = k > 0$, elementary dimension theory enables us to find closed subsets A_j of A such that $\dim A_j < k$ and $\dim(A \setminus \cup A_j) = 0$. The inductive assumption applied to the various A_j's combines with the Countable Approximation Principle (Proposition 1.1) to provide a new CE map $\pi': M \longrightarrow M / G_A$, where $\pi' = \lim \pi h_i$ as $i \longrightarrow \infty$ (h_i a self-homeomorphism of M) and π' is 1-1 over $(\cup A_j) \cup \pi\big(M \setminus p^{-1}(A)\big)$. Thus $\dim \pi'(N_{\pi'}) \leq \dim \pi(N_\pi) \leq 0$ and π' has the

RDDP, since the RDDP is preserved by taking limits of the form πh_i [14]. Just as above, π' (and, thus, π itself) is a near-homeomorphism.

The case $A = B$ of Lemma 5.2 precipitates the desired theorem.

THEOREM 5.3. *Every cellular resolution* $p: M \longrightarrow B$ *with* RRDDP_ω *is a near-homeomorphism.*

It is noteworthy that a converse to Theorem 5.3 fails—the RRDDP_ω cannot hold if some point-preimage has non-empty interior.

6. **Applications to generalized 4-manifolds.** Here we briefly outline how the preceding techniques allow improvement to resolutions of generalized 4-manifolds satisfying the relevant variation to the WSAP.

Locally flat embeddings of 2-cells in 4-manifolds are not classified by the 1-LCC condition but, rather, by the following one. A closed subset C of a space B is *locally homotopically unknotted* in B if for each $c \in C$ and each neighborhood U of c there exists a smaller neighborhood V of c such that (1) for $k > 1$ every map $S^k \longrightarrow V \setminus C$ is null-homotopic in $U \setminus C$ and (2) every map $S^1 \longrightarrow V \setminus C$ which is null-homologous in $V \setminus C$ is null-homotopic in $U \setminus C$. The WSAP must be modified to accomodate this distinction. Accordingly, we say that a space B has the 4-*dimensional* WSAP if every map $I^2 \longrightarrow B$ can be approximated by one whose image is covered by a finite number of 2-cells locally homotopically unknotted in B.

LEMMA 6.1. *If X is a generalized 4-manifold satisfying the 4-dimensional* WSAP, *then every resolution $p: M \longrightarrow X$ is cellular.*

PROOF. Just as in Proposition 1.5, each $x \in X$ is 1-LCC embedded, implying $p^{-1}(x)$ satisfies McMillan's Cellularity Criterion in M. For cell-like subsets of 4-manifolds, the Cellularity Criterion ensures cellularity [18, Theorem 1.11]. (See [28] for an additional hypothesis required in the statement of Freedman's theorem.)

At the heart of this 4-dimensional application is the following strong adaptation of Cannon's Recognition Theorem (2.1) due to M. H. Freedman and F. S. Quinn [19] (see also [25]).

THEOREM 6.2. *Suppose X is a generalized 4-manifold (without boundary) and $D \subset X$ is a 2-cell locally homotopically unknotted in X such that $X \setminus D$ is a 4-manifold. Then X is a 4-manifold.*

THEOREM 6.3. *Suppose $p: M \longrightarrow X$ is a resolution of a generalized 4-manifold X satisfying the 4-dimensional* WSAP. *Then p can be approximated by a resolution $q: M \longrightarrow X$ such that* $\dim q(N_q) \leq 1$ *and q is a uniform limit of $\{ph_i\}$, where each h_i represents a self-homeomorphism of M.*

SKETCH. With Theorem 6.2 supplanting Theorem 2.1 and Quinn's Cell-like Approximation Theorem [25, Corollary 2.6.2] supplanting Armentrout's, the proof of Corollary 2.2 indicates that p can be approximated by a new resolution 1-1 over any predetermined locally homotopically unknotted 2-cell. The hypothesis provides a dense collection of maps $\psi_i: I^2 \longrightarrow X$, where each individual image is covered by finitely many embedded 2-cells locally homotopically unknotted in X, and application of the Countable Approximation Principle furnishes an approximation $q: M \longrightarrow X$ which is 1-1 over $\cup \psi_i(I^2)$. That $\dim q(N_q) \leq 1$ follows as in Lemma 2.3.

References

1. F. D. Ancel, *The locally flat approximation of cell-like embedding relations*, Ph. D Thesis, University of Wisconsin-Madison, (1976).

2. S. Armentrout, *Decompositions into straight arcs and singletons*, Diss. Mat. **73**(1970), 5–39.

3. ———, *Cellular decompositions of 3-manifolds that yield 3-manifolds*, Mem. Amer. Math. Soc., **107** (1971).

4. R. H. Bing, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3*, Ann. of Math. (2) **65**(1957), 484–500.

5. ———, *A surface is tame if its complement is 1-ULC*, Trans. Amer. Math. Soc. **101**(1961), 294–305.

6. ———, *Point-like decompositions of E^3*, Fund. Math. **50**(1962), 431–453.

7. G. E. Bredon, *Generalized manifolds, revisited.* In: Topology of Manifolds, (J. C. Cantrell and C. H. Edwards, Jr., eds.), Markham Publ. Co., Chicago, (1970), 461–469.

8. M. G. Brin and D. R. McMillan, Jr., *Generalized three-manifolds with zero-dimensional non-manifold set*, Pacific J. Math. **97**(1981), 29–58.

9. J. W. Cannon, *Taming cell-like embedding relations.* In: Geometric Topology, (L. C. Glaser and T. B. Rushing, eds.), Lecture Notes in Math., **438**, Springer-Verlag, Berlin, (1975), 66–118.

10. ———, *The recognition problem: what is a topological manifold?*, Bull. Amer. Math. Soc. **84**(1978), 832–866.

11. ———, *Shrinking cell-like decompositions of manifolds. Codimension three*, Ann. of Math (2) **110**(1979), 83–112.

12. R. J. Daverman, *Decompositions of Manifolds*, Academic Press, Orlando, FL, (1986).

13. ———, *1-dimensional phenomena in cell-like mappings on 3-manifolds*, Pacific J. Math. **135**(1988), 269–285.

14. R. J. Daverman and D. Repovš, *A new shrinking theorem for 3-manifolds*, Trans. Amer. Math. Soc. **315** (1989), 219–230.

15. R. J. Daverman and W. H. Row, *Cell-like 0-dimensional decompositions of S^3 are 4-manifold factors*, Trans. Amer. Math. Soc. **254**(1979), 217–236.

16. R. D. Edwards, *The topology of manifolds and cell-like maps.* In: Proc. Internat. Cong. Mathematicians, Helsinki, (1978) (O. Lehto, ed.), Acad. Sci. Fenn., Helsinki, (1980), 111–127.

17. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. (2) **49**(1948), 979–990.

18. M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry **17**(1982), 357–453.

19. M. H. Freedman and F. Quinn, *Topology of 4-Manifolds*, Princeton Univ. Press, Princeton, NJ, (1990).

20. H. W. Lambert and R. B. Sher, *Pointlike 0-dimensional decompositions of S^3*, Pacific J. Math. **24**(1968), 511–518.

21. D. R. McMillan, Jr., *A criterion for cellularity in a manifold*, Ann. of Math. (2) **79**(1964), 327–337.

22. R. L. Moore, *Concerning upper semicontinuous collections of continua*, Trans. Amer. Math. Soc. **27**(1925), 416–428.

23. V. A. Nicholson, *1-FLG complexes are tame in 3-manifolds*, Gen. Topology Appl. **2**(1972), 277–285.

24. C. D. Papakyriakopolous, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. (2) **66**(1957), 1–26.

25. F. Quinn, *Ends of maps III: dimensions 4 and 5*, J. Differential Geometry **17**(1982), 503–521.

26. ———, *An obstruction to the resolution of homology manifolds*, Michigan Math. J. **34**(1987), 285–291.

27. D. Repovš, *The recognition problem for topological manifolds.* In Geometric and Algebraic Topology, (H. Toruńczyk, S. Jackowski, and S. Spież, eds., Banach Center Publ., **18**, PWN, Warsaw, (1986), 77–108.

28. ———, *A criterion for cellularity in a topological 4-manifold*, Proc. Amer. Math. Soc. **100**(1987), 564–566.

29. D. Repovš and R. C. Lacher, *A disjoint disks property for 3-manifolds*, Topology Appl. **16**(1983), 161–170.

30. M. Starbird, *Cell-like 0-dimensional decompositions of E^3*, Trans. Amer. Math. Soc. **249**(1979), 203–215.

31. T. L. Thickstun, *Strongly acyclic maps and homology 3-manifolds with 0-dimensional singular set*, Proc. London Math. Soc. (3) **55**(1987), 378–432.

32. J. J. Walsh, *Dimension, cohomological dimension, and cell-like mappings.* In: Shape Theory and Geometric Topology, (S. Mardešić and J. Segal, eds.), Lecture Notes in Math., **870**, Springer-Verlag, Berlin, 1981, 105–118.

33. R. L. Wilder, *Topology of Manifolds*, Amer. Math. Soc. Colloq. Publ., **32**, Providence, R.I., (1963).

34. E. P. Woodruff, *Decomposition spaces having arbitrarily small neighborhoods with 2-sphere boundaries*, Trans. Amer. Math. Soc. **232**(1977), 195–204.

35. ______, *Decomposition spaces having arbitrarily small neighborhoods with 2-sphere boundaries II*, Fund. Math. **119**(1983), 185–204.

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