BULL. AUSTRAL. MATH. SOC. 30D30
VOL. 38 (1988) [451–455]

MEROMORPHIC LIPSCHITZ FUNCTIONS

SHINJI YAMASHITA

Let f be a function meromorphic in $D = \{|z| < 1\}$ and let X be the chordal distance on the Riemann sphere. Then f satisfies the Lipschitz condition

$$X(f(z), f(w)) \leqq K|z-w|^\alpha \qquad (0 < \alpha \leqq 1)$$

in D if and only if $|f'(z)|/\left(1+|f(z)|^2\right) = O\left((1-|z|)^{\alpha-1}\right)$ and $|z| \to 1$.

The chordal distance $X(z, w)$ in the Riemann sphere $S = \{|z| \leqq \infty\}$ is defined by

$$X(z, w) = |z-w|\left(1+|z|^2\right)^{-1/2}\left(1+|w|^2\right)^{-1/2}$$

with the obvious change in case z or $w = \infty$. This is invariant,

$$X(T_a(z), T_a(w)) = X(z, w)$$

for the transformations $T_a(\zeta) = (\zeta - a)/(1+\overline{a}\zeta)$, ζ, $a \in S$ with $T_\infty(\zeta) = 1/\zeta$. Let $D = \{|z| < 1\}$ and let E be a nonempty subset of $D^* = \{|z| \leqq 1\}$. A map $f : E \to S$ is then said to satisfy the Lipschitz condition of order α, $0 < \alpha \leqq 1$, in notation: $f \in L(\alpha, E)$, if there exists a constant $K > 0$ such that

$$X(f(z), f(w)) \leqq K|z-w|^\alpha, \qquad z, w \in E. \tag{1}$$

In the case $E = D^*$, for example, we are able to add a restriction: $|z-w| < A$ to variables in (1) with a constant $A > 0$.

If $f : D \to S$ is meromorphic, then we set

$$f^\#(z) = \lim_{w \to z} X(f(w), f(z))/|w-z|.$$

Therefore, $f^\#(z) = |f'(z)|/\left(1+|f(z)|^2\right)$ if $f(z) \neq \infty$, and $f^\#(z) = \left|(1/f)'(z)\right|$ if $f(z) = \infty$.

Received 24 February 1988

THEOREM. *For f meromorphic in D, and for $0 < \alpha \leqq 1$, the following are equivalent:*

(I) $f \in L(\alpha, D)$.

(II) *f can be extended to D^*, and the resulting function f is in $L(\alpha, D^*)$.*

(III) $f^{\#}(z) = O\left((1-|z|)^{\alpha-1}\right)$ *as* $|z| \to 1$.

First, (II) $\Rightarrow$ (I) is trivial, and it is not difficult to prove directly (I) $\Rightarrow$ (II). We shall prove (I) $\Rightarrow$ (III) $\Rightarrow$ (II).

The present theorem is a meromorphic version of the known result for holomorphic functions; see [**1**, Theorems 3 and 4, pp. 411 and 413], for example. Our proof of the theorem is essentially different from the holomorphic case in some parts.

PROOF OF THE THEOREM: For the proof of (I) $\Rightarrow$ (III) we suppose (1) for $E = D$ and we choose then Q, $0 < Q < 1$, such that

$$K(1-|z|)^{\alpha} \leqq 1/2 \text{ for } Q < |z| < 1.$$

In order to verify that

$$f^{\#}(z) \leqq K'(1-|z|)^{\alpha-1} \text{ for } Q < |z| < 1, \tag{2}$$

where $K' = (2/\sqrt{3})K$, we fix z and we let $0 < r < 1 - |z|$. We observe that the function of w,

$$g(w) = T_{f(z)} \circ f(rw+z) = [f(rw+z) - f(z)]/[1 + \overline{f(z)} f(rw+z)]$$

is holomorphic and bounded on D^*. Actually, by (1) for $E = D$,

$$X(g(w), 0) = X(f(rw+z), f(z)) \leqq K(r|w|)^{\alpha} \leqq K(1-|z|)^{\alpha} \leqq 1/2, \tag{3}$$

whence $|g| \leqq 1/\sqrt{3}$ on D^*. Since $g(0) = 0$, there exists a holomorphic function h on D^* such that $g(w) = wh(w)$, $w \in D^*$. We thus obtain

$$\begin{aligned} rf^{\#}(z) = |g'(0)| = |h(0)| &\leqq (2\pi)^{-1} \int_0^{2\pi} \left|h(e^{it})\right| dt \\ &= (2\pi)^{-1} \int_0^{2\pi} \left|g(e^{it})\right| dt \leqq K'(1-|z|)^{\alpha}, \end{aligned}$$

because (3) yields

$$|g(w)| \leqq \left(1 + |g(w)|^2\right)^{1/2} K(1-|z|)^{\alpha} \leqq K'(1-|z|)^{\alpha},$$

for $w \in \partial D$. Letting $r \to 1 - |z|$ in $rf^{\#}(z) \leqq K'(1-|z|)^{\alpha}$, we now have (2).

For the proof of (III) $\Rightarrow$ (II) we suppose that

$$f^{\#}(z) \leqq K_1(1-|z|)^{\alpha-1} \text{ in } D, \tag{4}$$

and we begin with the existence of the radial limit

$$F(\zeta) = \lim_{r\to 1} f(r\zeta) \text{ at each } \zeta \in \partial D.$$

In view of (4) we have for $0 \leqq r < \rho < 1$ and $\zeta \in \partial D$,

$$\begin{aligned} X(f(r\zeta), f(\rho\zeta)) &\leqq \int_r^{\rho} f^{\#}(t\zeta)dt \leqq \alpha^{-1}K_1[(1-r)^{\alpha} - (1-\rho)^{\alpha}] \\ &\leqq K_2(\rho - r)^{\alpha} \end{aligned}$$

by $(A+B)^{\alpha} - A^{\alpha} \leqq B^{\alpha}$ for $A, B \geqq 0$; hereafter K_j, $1 < j \leqq 7$, are constants depending on f. Therefore, f satisfies the Lipschitz condition on each radius of D, so that $F(\zeta)$ exists at each $\zeta \in \partial D$.

Suppose henceforth that

$$\begin{aligned} f(z) &= F(z) \qquad \text{if} \quad z \in \partial D; \\ &= f(z) \qquad \text{if} \quad z \in D. \end{aligned}$$

We then obtain after the "limiting" procedure that

$$X(f(r\zeta), f(\rho\zeta)) \leqq K_2\,|r-\rho|^{\alpha} \tag{5}$$

for $\zeta \in \partial D$, $0 \leqq r \leqq 1$, and $0 \leqq \rho \leqq 1$.

We shall show that there exists A, $0 < A < 1$, such that

$$X(f(z), f(w)) \leqq K_3\,|z-w|^{\alpha} \tag{i}$$

for $z, w \in D^*$ and $|z-w| < A$.

We can find b, $0 < b < 1$, such that $X(f(bz), f(0)) \leqq 1/2$ for $z \in D$ by continuity. The desired A is then given by $A = b/4$. We first prove

$$X(f(z), f(w)) \leqq K_4\,|z-w|^{\alpha}, \qquad z, w \in B, \tag{ii}$$

where $B = \{|z| < b\}$; note that we impose no restriction $|z-w| < A$ in this case.

For $H(z) = T_{f(0)} \circ f(bz)$ we have $|H(z)| \leqq 1/\sqrt{3}$ in D by

$$X(H(z), 0) = X(f(bz), f(0)) \leqq 1/2.$$

Consequently, for $z \in D$,

$$\begin{aligned}(3/4)(1-|z|)^{1-\alpha}|H'(z)| &\leqq (1-|z|)^{1-\alpha}H^{\#}(z)\\ &= (1-|z|)^{1-\alpha}bf^{\#}(bz) \leqq bK_1[(1-|z|)/(1-b|z|)]^{1-\alpha}\\ &\leqq bK_1.\end{aligned}$$

Therefore, H satisfies the (Euclidean) Lipschitz condition:

$$|H(z)-H(w)| \leqq K_5|z-w|^{\alpha}, \qquad z, w \in D.$$

We thus have, for $z, w \in B$,

$$\begin{aligned}X(f(z), f(w)) &= X\big(H(b^{-1}z), H(b^{-1}w)\big)\\ &\leqq \big|H(b^{-1}z)-H(b^{-1}w)\big| \leqq b^{-\alpha}K_5|z-w|^{\alpha},\end{aligned}$$

which completes the proof of (ii).

To complete the proof of (i), now, it suffices to show that

(iii) $$X(f(z), f(w)) \leqq K_6|z-w|^{\alpha}$$

for $z, w \in B^* = \{b/2 \leqq |z| \leqq 1\}$ with $|z-w| < A$.

We begin with the specified case $z = re^{i\theta}$ and $w = re^{i\phi}$ with $z \neq w$ in (iii). Then,

(6) $$\mu = |\theta-\phi| < (\pi/2)r^{-1}|z-w| < (\pi/b)|z-w| < 1.$$

Set $R = 1-\mu$. If $r \leqq R$, then $1-r \geqq 1-R = \mu$, and

$$\begin{aligned}X(f(z), f(w)) &\leqq \left|\int_{\theta}^{\phi} f^{\#}(re^{it})r\,dt\right| \leqq K_1(1-r)^{\alpha-1}r\mu\\ &\leqq K_1\mu^{\alpha} \leqq K_1(\pi/b)^{\alpha}|z-w|^{\alpha}.\end{aligned}$$

On the other hand, if $r > R$, then it follows from (5), together with $r-R \leqq 1-R = \mu$, that

(7) $$X\big(f(z), f(Re^{i\theta})\big) \leqq K_2(r-R)^{\alpha} \leqq K_2(\pi/b)^{\alpha}|z-w|^{\alpha},$$

and similarly,

(8) $$X\big(f(Re^{i\phi}), f(w)\big) \leqq K_2(\pi/b)^{\alpha}|z-w|^{\alpha}.$$

Furthermore,

(9) $$X\big(f(Re^{i\theta}), f(Re^{i\phi})\big) \leqq K_1(1-R)^{\alpha-1}R\mu \leqq K_1(\pi/b)^{\alpha}|z-w|^{\alpha}.$$

The triangle inequalities, together with (7), (8), and (9), now yield

$$X(f(z),\, f(w)) \leqq K_7\, |z-w|^{\alpha}. \tag{10}$$

For the general case in (iii), we may assume that $r < \rho$ for $z = re^{i\theta}$, $w = \rho e^{i\phi}$. Then, (5) yields that

$$X\big(f(w),\, f\big(re^{i\phi}\big)\big) \leqq K_2(\rho - r)^{\alpha} \leqq K_2\, |z-w|^{\alpha}.$$

On the other hand, by (10) for the pair $re^{i\phi}$, z, we have

$$X\big(f\big(re^{i\phi}\big),\, f(z)\big) \leqq K_7\, |z-w|^{\alpha}.$$

The triangle inequality now yields (iii). The proof of (iii) is now complete with $K_6 = K_2 + K_7$. ∎

Remark 1. In contrast to the holomorphic case (see [**1**, Theorem 4, p. 413]) it is open to prove (IV) $\Rightarrow$ (II), where

(IV) *f can be extended continuously to D^*, and $f \in L(\alpha, \partial D)$.*

Remark 2. A result on hyperbolic Lipschitz functions may be found in [**2**].

References

[**1**] G.M. Goluzin, *Geometric theory of functions of a complex variable* (Trans. Math. Monographs 26, Amer. Math. Soc., Providence, 1969).

[**2**] S. Yamashita, 'Smoothness of the boundary values of functions bounded and holomorphic in the disk', *Trans. Amer. Math. Soc.* **272** (1982), 539–544.

Department of Mathematics
Tokyo Metropolitan University
Fukasawa, Setagaya
Tokyo 158, Japan