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DIMENSIONS OF SPACES OF SIEGEL MODULAR FORMS OF LOW WEIGHT IN DEGREE FOUR

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We calculate the dimensions of $M_4^{12}, M_4^8, S_4^{12}, S_4^8, S_4^6$ using Erokhin's work on Niemeier lattices and geometric methods involving the hyperelliptic locus.

0. INTRODUCTION.

In this note we calculate some dimensions of spaces of Siegel modular forms and of cusp forms. We obtain the results that $\dim M_4^{12} = 6$, $\dim M_4^8 = 2$, $\dim S_4^{12} = 2$, $\dim S_4^8 = 1$, and $\dim S_4^6 = 0$. Explicit generators for these spaces of cusp forms are also given. The dimensions of the M_g^k are known for $g \leqslant 3$ [**14**]; previously it was known [**6**, p.50] only that $S_4^k = 0$ for $1 \leqslant k \leqslant 5$. Calculations of Erokhin for the Niemeier lattices play an essential role in our arguments, as does a theorem of Igusa that elements of S_4^k for even $k \leqslant 8$ must vanish on the hyperelliptic locus. The admittedly special nature of these calculations in weights less than or equal to 12 has a natural origin. The Type II lattices in dimension 24 are 24 in number and have been classified by Niemeier, whereas the Type II lattices in dimension 32 number in excess of 80 million and will never be classified in the same detail. A complete classification is essential for applications to Siegel modular forms if Böcherer's result (Theorem 1.2) is to be applied. On the other hand, these dimensions are notoriously difficult to calculate by any means (see [**13**, pp.60–61]), and this note provides more data toward this famous problem.

The results on cusp forms may be summarised as follows. The theta series for the Niemeier lattices provide a basis for S_4^{12} and the study of S_4^8 and S_4^6 is reduced to S_4^{12} via the inclusions $M_4^4 S_4^8 \subseteq S_4^{12}$ and $S_4^6 S_4^6 \subseteq S_4^{12}$. Let $f_4 \in M_4^4$ be the theta series of the E_8 lattice and let $j_8 \in S_4^8$ be Schottky's modular form vanishing on the Jacobian locus. Then S_4^{12} has a basis $\{f_4 j_8, \psi_{12}\}$ such that ψ_{12} does not vanish on $\bigoplus^4 \mathcal{H}_1$. For any $e \in S_4^8$ and $f \in S_4^6$ we then have $f_4 e = a f_4 j_8 + b\psi_{12}$ and $f^2 = \alpha f_4 j_8 + \beta \psi_{12}$. Since e and f necessarily vanish on the hyperelliptic locus we can evaluate the coefficients a, b, α, β by restriction to $\bigoplus^4 \mathcal{H}_1$. We obtain $S_4^8 = \mathbb{C} j_8$ and $S_4^6 = 0$ by this procedure. This note is another example using geometric information to calculate the dimensions of spaces of cusp forms.

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1. Notation.

Let $\mathcal{H}_g$ be the Siegel upper half space of degree $g \geqslant 1$ [**12**, p.2] and let $\Gamma_g = \mathrm{Sp}_g(\mathbb{Z})$ denote the full Siegel modular group which acts on $\mathcal{H}_g$. Let M_g^k be the complex vector space of Siegel modular forms of weight k on $\mathcal{H}_g$ [**12**, p.43], $\Phi_g : M_g^k \to M_{g-1}^k$ be the Siegel map [**12**, p.54], and $S_g^k = \ker \Phi_g$ be the subspace of cusp forms.

To any integral lattice $\Lambda \subseteq \mathbb{R}^n$ we may define the *theta series* of Λ (the analytic class invariant) $\vartheta_\Lambda : \mathcal{H}_g \to \mathbb{C}$ as follows: for $\Omega \in \mathcal{H}_g$ let

$$\vartheta_\Lambda(\Omega) = \sum_{\ell_1,\dots,\ell_g \in \Lambda} \exp\Bigl(i\pi \sum_{j,k=1}^{g} \Omega_{jk}\langle \ell_j, \ell_k\rangle\Bigr). \tag{1.1}$$

A lattice is called "Type II" [**2**, p.48] if it is even and self dual. For Λ of Type II, we have $\vartheta_\Lambda \in M_g^{n/2}$ for each $g \geqslant 1$ [**6**, p.17]. We specify lattices in the notation of [**2**, p.119, 120, 407] and use the further designations:

$$f_4 = \vartheta_{E_8},$$
$$f_8 = \vartheta_{D_{16}^+},$$
$$j_8 = f_4^2 - f_8.$$

Also, for the 24 Niemeier lattices, we denote their theta series by

$$\vartheta_i \in M_g^{12} \text{ for } i \in \{0, 1, \dots, 23\},$$

where we index the Niemeier lattices by 0 through 23 in some manner with the Leech lattice being the 0^{th} one; that is, ϑ_0 is theta series of the Leech lattice. Let h_i be the Coxeter number [**2**, p.407] of the $i^{\underline{th}}$ Niemeier lattice and let $\tau_i = 24h_i$. Except for the Leech lattice, where $\tau_0 = 0$, τ_i is the kissing number of the $i^{\underline{th}}$ Niemeier lattice. The following Theorem is due to Böcherer [**1**, p.22, 44].

Theorem 1.2. (Böcherer) *For $k > 2g$ and $k \equiv 0 \mod 4$, the theta series of all Type II lattices span M_g^k.*

This Theorem assures us, for example, that the ϑ_i span M_g^{12} for $1 \leqslant g \leqslant 5$.

We now recall some geometrically defined subsets of $\mathcal{H}_4/\Gamma_4$ that we shall use. View $\mathcal{A}_g = \mathcal{H}_g/\Gamma_g$ as the moduli space of principally polarised Abelian varieties. The Torelli map sends a compact Riemann surface of genus g to its Jacobian's class in $\mathcal{A}_g$. The closure in $\mathcal{A}_g$ of the image of the Torelli map is called the *Jacobian locus*, Jac_g. Similarly, the closure of the image of the restriction of the Torelli map to hyperelliptic Riemann surfaces is called the *hyperelliptic locus*, h_g. We call the image of $\bigoplus^g \mathcal{H}_1$ in $\mathcal{A}_g$ the *diagonal locus*, Diag_g. The following inclusions hold:

$$\mathcal{A}_g \supseteq \mathrm{Jac}_g \supseteq h_g \supseteq \mathrm{Diag}_g.$$

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The final inclusion follows easily from the techniques for the degeneration of curves in [5]. The following theorems of Igusa relate the above loci to the ring of Siegel modular forms. We say that a Siegel modular form f vanishes on h_g if for all $\Omega \in \mathcal{H}_g$ such that $[\Omega] \in h_g$ we have $f(\Omega) = 0$.

THEOREM 1.3. (Igusa, [**9**, p.845]) *A cusp form in S_g^k of even weight $k < 8+4/g$ necessarily vanishes on the hyperelliptic locus h_g.*

THEOREM 1.4. (Igusa, [**10, 11**]) *The ideal of Siegel modular forms in $\bigoplus_{k=1}^{\infty} M_4^k$ that vanish on the Jacobian locus, Jac_4, is principal and is generated by the irreducible element j_8.*

2. Dimension Calculations.

We begin by constructing certain cusp forms of weight 12 for $1 \leqslant g \leqslant 4$ with simple behaviour on direct sums of $\mathcal{H}_1$. We let $\Delta \in S_1^{12}$ denote the usual generator of S_1^{12} [**2**, p.105]. We also use the following notation:

$$\det_{i,j,k,l}\left(\vartheta, \tau^2, \tau, 1\right) = \begin{vmatrix} \vartheta_i & \vartheta_j & \vartheta_k & \vartheta_l \\ \tau_i^2 & \tau_j^2 & \tau_k^2 & \tau_l^2 \\ \tau_i & \tau_j & \tau_k & \tau_l \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

LEMMA 2.1. *For all $i,j,k,l,m \in \{0,1,\ldots,23\}$ we have*

$$\det_{i,j}(\vartheta,1) \in S_1^{12} \text{ and } \det_{i,j}(\vartheta,1) = \det_{i,j}(\tau,1)\,\Delta,$$
$$\det_{i,j,k}(\vartheta,\tau,1) \in S_2^{12} \text{ and } \det_{i,j,k}(\vartheta,\tau,1)|_{\bigoplus^2 \mathcal{H}_1} = \det_{i,j,k}\left(\tau^2,\tau,1\right)\Delta\otimes\Delta,$$
$$\det_{i,j,k,l}\left(\vartheta,\tau^2,\tau,1\right) \in S_3^{12} \text{ and } \det_{i,j,k,l}\left(\vartheta,\tau^2,\tau,1\right)|_{\bigoplus^3 \mathcal{H}_1} = \det_{i,j,k,l}\left(\tau^3,\tau^2,\tau,1\right)\Delta\otimes\Delta\otimes\Delta,$$
$$\det_{i,j,k,l,m}\left(\vartheta,\tau^3,\tau^2,\tau,1\right) \in S_4^{12} \text{ and } \det_{i,j,k,l,m}\left(\vartheta,\tau^3,\tau^2,\tau,1\right)|_{\bigoplus^4 \mathcal{H}_1}$$
$$= \det_{i,j,k,l,m}\left(\tau^4,\tau^3,\tau^2,\tau,1\right)\Delta\otimes\Delta\otimes\Delta\otimes\Delta.$$

PROOF: Recall that for $\Omega \in \mathcal{H}_1$, the coefficient of $e^{2\pi i\Omega}$ in the theta series $\vartheta_i(\Omega)$ is τ_i, and the coefficient of $e^{2\pi i\Omega}$ in $\Delta(\Omega)$ is 1. Since $\vartheta_i - \vartheta_j \in S_1^{12}$ and Δ is a generator of S_1^{12}, we must have $\vartheta_i - \vartheta_j = (\tau_i - \tau_j)\Delta$ on $\mathcal{H}_1$. Another way of saying this is that $\det_{i,j}(\vartheta,1) = \vartheta_i - \vartheta_j = (\tau_i - \tau_j)\Delta = \det_{i,j}(\tau,1)\,\Delta$ on $\mathcal{H}_1$. Using $\tau_0 = 0$ we have $\vartheta_i = \vartheta_0 + \tau_i\Delta$ with ϑ_0 being the theta series of the Leech lattice. It follows that $\det_{i,j,k}(\vartheta,\tau,1)$ is identically zero on $\mathcal{H}_1$ because the first row is a linear combination of the second and third rows in the determinant; hence it is a cusp form

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on $\mathcal{H}_2$ because we have $\Phi_g(\vartheta_\Lambda \text{ on } \mathcal{H}_g) = \vartheta_\Lambda$ on $\mathcal{H}_{g-1}$. Restriction to $\bigoplus^2 \mathcal{H}_1$ gives $\vartheta_i = \vartheta_i|_{\mathcal{H}_1} \otimes \vartheta_i|_{\mathcal{H}_1} = \vartheta_0 \otimes \vartheta_0 + \tau_i(\vartheta_0 \otimes \Delta + \Delta \otimes \vartheta_0) + \tau_i^2 \Delta \otimes \Delta$ so that by subtracting multiples of the second and third rows from the first row in the determinant, we have $\det_{i,j,k}(\vartheta, \tau, 1)|_{\bigoplus^2 \mathcal{H}_1} = \det_{i,j,k}\left(\tau^2 \Delta \otimes \Delta, \tau, 1\right) = \det_{i,j,k}\left(\tau^2, \tau, 1\right) \Delta \otimes \Delta$ as claimed.

Without loss of generality and for convenience, we can number the Niemeier lattices so that the first five Coxeter numbers are distinct [**2**, p.407] and $\tau_0 = 0$ still. Then $\det_{0,1,2}\left(\tau^2, \tau, 1\right) \neq 0$, so that we may let $\phi = \det_{0,1,2}(\vartheta, \tau, 1) / \det_{0,1,2}\left(\tau^2, \tau, 1\right)$. Since $\det_{0,1,2}(\vartheta, \tau, 1) = \det_{0,1,2}\left(\tau^2, \tau, 1\right) \Delta \otimes \Delta$ on $\bigoplus^2 \mathcal{H}_1$, we have $\phi = \Delta \otimes \Delta$ on $\bigoplus^2 \mathcal{H}_1$. This implies ϕ is not identically zero on $\bigoplus^2 \mathcal{H}_1$, and hence $\phi \neq 0$ in S_1^{12}. Since $\dim S_2^{12} = 1$, ϕ must be a generator. So on $\mathcal{H}_2$, we must have

$$\det_{i,j,k}(\vartheta, \tau, 1) = s_{ijk}\phi, \tag{2.2}$$

for some constant s_{ijk}. We also know that $\det_{i,j,k}(\vartheta, \tau, 1) = \det_{i,j,k}\left(\tau^2, \tau, 1\right) \Delta \otimes \Delta$ on $\bigoplus^2 \mathcal{H}_1$. From (2.2), we also have that $\det_{i,j,k}(\vartheta, \tau, 1) = s_{ijk} \Delta \otimes \Delta$ on $\bigoplus^2 \mathcal{H}_1$. So we conclude that $s_{ijk} = \det_{i,j,k}\left(\tau^2, \tau, 1\right)$. So on $\mathcal{H}_2$, we have $\det_{i,j,k}(\vartheta, \tau, 1) = \det_{i,j,k}\left(\tau^2, \tau, 1\right) \phi$. Expanding the determinant by cofactors using the second row (τ^2), we have on $\mathcal{H}_2$,

$$\begin{aligned}
\det_{i_1, i_2, i_3, i_4}\left(\vartheta, \tau^2, \tau, 1\right) &= \sum_{k=1}^{4} (-1)^{k+1} \tau_{i_k}^2 \det_{\substack{i_1, \dots, i_4 \\ \text{no } i_k}} (\vartheta, \tau, 1) \\
&= \sum_{k=1}^{4} (-1)^{k+1} \tau_{i_k}^2 \det_{\substack{i_1, \dots, i_4 \\ \text{no } i_k}} \left(\tau^2, \tau, 1\right) \phi \\
&= \det_{i_1, i_2, i_3, i_4}\left(\tau^2, \tau^2, \tau, 1\right) \phi \\
&= 0.
\end{aligned}$$

Therefore, $\det_{i,j,k,l}\left(\vartheta, \tau^2, \tau, 1\right)$ is a cusp form on $\mathcal{H}_3$. Restriction to $\bigoplus^3 \mathcal{H}_1$ gives $\vartheta_i = \vartheta_i \otimes \vartheta_i \otimes \vartheta_i = \vartheta_0 \otimes \vartheta_0 \otimes \vartheta_0 + \cdots + \tau_i^3 \Delta \otimes \Delta \otimes \Delta$ so that $\det_{i,j,k,l}\left(\vartheta, \tau^2, \tau, 1\right)|_{\bigoplus^3 \mathcal{H}_1} = \det_{i,j,k,l}\left(\Delta \otimes \Delta \otimes \Delta \tau^3, \tau^2, \tau, 1\right) = \det_{i,j,k,l}\left(\tau^3, \tau^2, \tau, 1\right) \Delta \otimes \Delta \otimes \Delta$.

Since $\dim S_3^{12} = 1$ [**14**, p.832] and since we have four distinct Coxeter numbers $\tau_0, \tau_1, \tau_2, \tau_3$, we may employ the same linear algebra techniques as above to deduce that $\det_{i,j,k,l,m}\left(\vartheta, \tau^3, \tau^2, \tau, 1\right)$ is identically zero on $\mathcal{H}_3$ and is a cusp form on $\mathcal{H}_4$. Restriction to $\bigoplus^4 \mathcal{H}_1$ gives $\det_{i,j,k,l,m}\left(\vartheta, \tau^3, \tau^2, \tau, 1\right)|_{\bigoplus^4 \mathcal{H}_1} = \det_{i,j,k,l,m}\left(\tau^4, \tau^3, \tau^2, \tau, 1\right) \Delta \otimes \Delta \otimes \Delta \otimes \Delta$ in the same manner as above. ∎

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The simple pattern of this Lemma does not continue because $\dim S_4^{12} = 2$, as we shall deduce from a Theorem of Erokhin [**3**, **4**].

THEOREM 2.3. (Erokhin) *We have* $\dim_{\mathbb{C}} \mathrm{Span}\{\vartheta_i \text{ on } \mathcal{H}_4 : i \in \{0,1,\ldots,23\}\} = 6$.

PROOF: In the notation of [4] the assertion of the Theorem is $\dim \mathrm{Im}\, \phi_4 = 6$, where $\phi_g : \mathbb{C}^{24} \to M_g^{12}$ is defined by $c \mapsto \sum_i c_i \vartheta_i$. Let $V_g = \{v \in \mathbb{C}^{24} : \forall c \in \ker \phi_g, {}^t vc = 0\}$, then we have $\dim V_g = \dim \mathrm{Im}\, \phi_g$. Corollary 1 of [**4**, p.1017] and Theorem 2 of [**4**, p.1018] assert that V_4 has a basis of six elements. (These are $\{1, v_{A_1}, v_{A_1}^2, v_{A_1}^3, v_{A_1}^4, v_{D_4}\}$ in the notation of [4].) □

COROLLARY 2.4. *We have* $\dim M_4^{12} = 6$, $\dim S_4^{12} = 2$.

PROOF: From Theorem 1.2, and $12 > 2 \cdot 4$, we see that the theta series of the Niemeier lattices span M_4^{12} and so $\dim M_4^{12} = 6$ using the previous Theorem of Erokhin. From the surjectivity of $\Phi_g : M_g^k \to M_{g-1}^k$ for even $k > 2g$ [**12**, p.68] the following sequence of complex vector spaces is exact: $0 \to S_4^{12} \to M_4^{12} \to M_3^{12} \to 0$. Since $\dim M_3^{12} = 4$ [**14**, p.835] we have $\dim S_4^{12} = 2$. □

PROPOSITION 2.5. *For any five indices* $i,j,k,l,m \in \{0,1,\ldots,23\}$ *such that the Coxeter numbers* τ_i, τ_j, τ_k, τ_l, τ_m *are distinct, we have that* S_4^{12} *is spanned by* $f_4 j_8$ *and* $\det_{i,j,k,l,m}(\vartheta, \tau^3, \tau^2, \tau, 1)$.

PROOF: Since $\dim S_4^{12} = 2$ by Corollary 2.4 it suffices to show that the cusp forms $f_4 j_8$ and $\det_{i,j,k,l,m}(\vartheta, \tau^3, \tau^2, \tau, 1)$ are linearly independent. By Lemma 2.1 the cusp form $\det_{i,j,k,l,m}(\vartheta, \tau^3, \tau^2, \tau, 1)$ does not vanish on the diagonal locus whereas $f_4 j_8$ does because j_8 vanishes on the Jacobian locus and hence on the diagonal locus. □

THEOREM 2.6. *We have* $\dim S_4^8 = 1$ *and* j_8 *spans* S_4^8.

PROOF: Since there exist five Niemeier lattices with distinct Coxeter numbers [**2**, p.407], let $\psi_{12} = \det_{i,j,k,l,m}(\vartheta, \tau^3, \tau^2, \tau, 1)$ be such that $f_4 j_8$ and ψ_{12} span S_4^{12} as in Proposition 2.5. Take any $f \in S_4^8$. By Theorem 1.3 f must vanish on the hyperelliptic locus and hence on the diagonal locus. We also have $f_4 f \in S_4^{12}$ and so $f_4 f = a f_4 j_8 + b\psi_{12}$ for some $a, b \in \mathbb{C}$. Upon restriction to $\bigoplus^4 \mathcal{H}_1$ we obtain $0 = b \det(\tau^4, \tau^3, \tau^2, \tau, 1)\, \Delta \otimes \Delta \otimes \Delta \otimes \Delta$ by Lemma 2.1. Therefore $b = 0$ and $f_4 f = a f_4 j_8$. We conclude that $f = a j_8$ because the ring of Siegel modular forms has no zero divisors. □

COROLLARY 2.7. *We have* $\dim M_4^8 = 2$.

PROOF: Since M_3^8 is spanned by f_4^2 [**9**, p.854], the map $\Phi_4 : M_4^8 \to M_3^8$ is onto and the sequence $0 \to S_4^8 \to M_4^8 \to M_3^8 \to 0$ is exact. Hence we have $\dim M_4^8 = 2$. □

REMARK. We see that the theta series also span M_4^8 in this particular case where $k = 2g$.

THEOREM 2.8. *We have* $S_4^6 = 0$.

PROOF: Let the notation be as in the proof of Theorem 2.6. Any $f \in S_4^6$ vanishes on the hyperelliptic locus by Theorem 1.3 and hence on the diagonal locus. Since $f^2 \in S_4^{12}$ we have $f^2 = af_4j_8 + b\psi_{12}$, for some constants a and b. Considering the restriction to $\bigoplus^4 \mathcal{H}_1$, since f and j_8 both vanish there and ψ_{12} does not, we must have $b = 0$. Thus we have $f^2 = af_4j_8$. There are two ways to show that $a = 0$ and $f = 0$, completing the proof. First, the ring of Siegel modular forms is a unique factorisation ring for $g \geqslant 3$ [**7**, **8**] and j_8 is irreducible by Theorem 1.4. Second, since $f^2 = af_4j_8$ vanishes on the Jacobian locus, so must f itself. But since j_8 generates the ideal of forms vanishing on the Jacobian locus by Theorem 1.4, we must have that $f = f_{-2}j_8$ for some modular form f_{-2} of weight $6 - 8 = -2$, which is necessarily 0. ∎

REMARK. Since $\dim M_3^6 = 1$ [**9**, p.852] this Theorem shows that either $\dim M_4^6 = 0$ or $\dim M_4^6 = 1$.

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