I. FUNDAMENTAL THEORIES

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A DYNAMICAL SYSTEMS APPROACH TO NONLINEAR STELLAR PULSATIONS

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Abstract. Over the last decade we have seen the application of novel techniques to the old problem of nonlinear stellar pulsations. Together with numerical hydrodynamics this approach provides a more fundamental understanding of the systematics of the pulsational behavior. For weakly nonadiabatic pulsations, whether regular or multi-periodic, dimensional reduction techniques lead to amplitude equations and to a description in terms of modal interactions and resonances. In particular they shed new light on the bump progression in the classical Cepheids. In more dissipative stars numerical hydrodynamical modelling has uncovered the existence of irregular variability, both in radiative and in convective models. An application of modern dynamical systems techniques has shown that this behavior occurs according to well understood routes from regular to chaotic behavior. The mechanism is very robust and represents the first non *ad hoc* theoretical explanation of irregular stellar variability. Finally, we discuss how a comparison with observations of irregular variability shows the need for more suitable observations, on the one hand, and of better techniques of signal processing, on the other.

1. Introduction

There is hardly a need to stress the importance of the study of stellar pulsations. Almost all stars undergo some kind of pulsational phase during their lifetimes, and an understanding of the large variety of pulsational behavior poses a challenge to the astrophysicist. From an astronomical point of view, nonlinear pulsations yield information about the parameters of the stars which their static siblings do not reveal us. More generally, the study of stellar pulsations has greatly improved our understanding of stellar structure and evolution, of galactic evolution and especially of cosmology, where the variable stars have provided the pillar on which our knowledge of the Universe's distance scale rests. For the physicist the pulsating stars are intriguing giant natural heat engines. Because of the extreme conditions of density and temperature encountered in stellar interiors they provide an excellent testing ground for Physics. For example, it is a long-standing discrepancy between the predictions of stellar pulsation and stellar evolution that has stimulated (Simon 1982) a revisitation of the atomic physics calculations with a subsequent substantial change in the opacities. Finally, to the dynamicist, pulsating stars are of interest because they exhibit very characteristic low-dimensional behavior in spite of the quite complicated nonlinear hydrodynamical equations which govern their behavior.

The work on nonlinear stellar pulsations can be grouped into three categories, numerical hydrodynamical, simple modelling and nonlinear dynamics approaches. The first, the *numerical hydrodynamical approach* was pioneered

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by Christy in the mid 60s and has been the workhorse for nonlinear modelling. Although it clearly constitutes a brute force attack, it is also the approach that will yield the most detailed and accurate description of the pulsations. However, its obvious shortcomings are the difficulty of extracting the underlying systematics in the metamorphosis of the generated light and radial velocity curves when the stellar parameters are varied. In particular, it is easy to model and obtain the Hertzsprung progression in the bump Cepheids, but it is difficult to understand its origin and what governs its presence and shape.

The second approach of constructing *simple models* is very pedestrian, but is essential for developing physical intuition and guidance. In the case of the famous linear one-zone model of Baker (1966) it yielded a much clearer understanding the destabilization of vibrational modes through the effects of the equation of state and of the opacity, the so-called γ and κ mechanism. In the context of nonlinear pulsations, Baker, Moore and Spiegel (1966) suggested the use of such a model in the form of a simple set of three first order ODEs. Their suggestion came as a result of a model oscillator that they had constructed for studying overstability in a convectively unstable zone (Moor and Spiegel 1966). This simple oscillator model which is a little known contemporary of the now famous Lorenz oscillator (*e.g.*, Cvitanovich 1984), like the latter exhibits a myriad of different types of behavior, including chaotic oscillations. Buchler and Regev (1982) developed a simple one-zone model of interest for the oscillations of stars with extended convective partial ionization regions. It turned out this oscillator was very similar to the Moore-Spiegel oscillator as well. Buchler and Perdang (1979) introduced a two-zone model to understand the thermal relaxation oscillations found in the study of stars with thin burning shells. Barranco, Buchler and Livio (1981) and Livio and Regev (1984) used a similar model for X-ray bursters. Recently Tanaka and Takeuti (1988) introduced additional one-zone models for stellar pulsations, and Saitou, Takeuti and Tanaka (1989) showed that the famous Rössler attractor can be transformed into a model stellar oscillator (for a review cf. *e.g.*, Takeuti 1990). This is particularly interesting in view of the chaotic pulsations encountered in the numerical hydrodynamical modelling of W Vir stars which seem to have the topology of the Rössler attractor (Kovács and Buchler 1988b). Generally speaking, it is important to realize that such simple model equations (3 first order nonlinear ODEs) can have a variety solutions, from static, to regular periodic, to period-doubled and to chaotic, depending on the values of the model parameters. It would therefore be astonishing if the more complicated hydrodynamical modelling were not also to produce this type of behavior. The general drawback of this approach is that the predicted behavior is not robust to the addition of further zones and that it is therefore not easily generalizable and improvable in a systematic fashion.

Finally, the *dynamical systems approach* is complementary to numerical hydrodynamics in that it gives a natural framework within which to understand the sometimes overwhelming computer output. Its astrophysical origins go back to the Hamiltonian approach of Woltjer (1936, 1937, 1946). (Here we only note in parentheses, because we are concerned with dissipative systems, that this Hamiltonian approach has been continued and applied to Kolmogorov unstable systems by Perdang 1983). Papaloizou (1973), realizing the importance of dissipation, was the first to use asymptotic perturbation methods with which he studied the nonlinear pulsations of upper main sequence stars. He also drew attention to the essential role played by resonances. Vandakurov (1979) and Dziembowski (1980), with the help of averaging techniques used in plasma physics, studied resonant wave coupling, but did not consider nonlinear nonadiabatic effects. At the same time Takeuti and Aikawa (1980, 1981, 1985) employed another asymptotic perturbation technique, the method of harmonic balance, again with an adiabatic approximation, and they added *ad hoc* van der Pol nonlinear dissipation terms. Buchler, Yueh and Perdang (1977) and Barranco, Buchler and Regev (1982) applied a multi-time method, but used a quasi-adiabatic approach which turned out not to be very useful from a practical point of view. Although all these studies were not fully satisfactory in some way or other they gave rise to amplitude equations of very similar form. This is not astonishing in retrospect as there exists a very general, systematical formalism to derive such equations provided rather general physical assumptions are satisfied.

2. The Dynamical Systems Approach

The first basic assumption that underlies this approach is one of *weak nonlinearity* which allows us to describe the nonlinear behavior of the system in terms of *modes*. Let use denote the deviation from static equilibrium of the basic variables (radius, velocity and a thermal variable, *e.g.*, the temperature) by $z(t) = (\delta\boldsymbol{R}, \ldots, \delta\boldsymbol{v}, \ldots, \delta T, \ldots, \ldots)$, where $\delta\boldsymbol{R}$ and $\delta\boldsymbol{v}$ respectively are vectors or scalars in the nonradial or radial cases. When for practical purposes the model is discretized into N mass zones, each of these quantities then has N components. We can write the equations of hydrodynamics and radiative transport (in the Lagrangean description) in the very compact form

$$\frac{\partial z}{\partial t} = \mathcal{L}z + \mathcal{N}(z) \tag{1}$$

where $\mathcal{N}(z)$ is the strictly nonlinear part of the right-hand side. Linearization with a time-dependence $\exp(\sigma t)$ leads to the eignevalue problem

$$\mathcal{L}e_{\rm k} = \sigma_k e_{\rm k} \tag{2}$$

with

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$$\sigma_k = i\omega_k + \kappa_k \tag{3}$$

The complex eigenvectors $\boldsymbol{e}_k$, together with their dual adjoints, form a complete orthogonal set and the displacement can be expressed in terms of the amplitudes $\{a_k\}$ of all the modes, *viz.* $\boldsymbol{z}(t) = \sum_k a_k(t)\boldsymbol{e}_k$. Substitution into Eq. (1) then leads to an equivalent system of coupled equations

$$\frac{da_k}{dt} = \sigma_k a_k + \sum_{lm} n_{klm} a_l a_m + \sum_{lmn} n_{klmn} a_l a_m a_n + \ldots \tag{4}$$

the sum extending over all modes and over all combinations of the $\{a_k\}$ and complex conjugate $(c.c)$ amplitudes $\{a_k^*\}$. The nonlinear coefficients $n_{klmn..}$ are constructible from the operator $\mathcal{N}$ and from the complex eigenvectors $\boldsymbol{e}_k$ and their duals (*e.g.*, Buchler and Goupil 1984, Buchler 1985). At this stage such equations can represent Hamiltonian as well as dissipative systems.

2.1. The Galerkin Approximation

This approximation is nothing but a truncation both in the number of modes and in the number of nonlinear terms in system (4). For example, for a single mode one finds

$$\frac{da_1}{dt} = (i\omega_1 + \kappa_1)a_1 + (n_{111}a_1^2 + c.c. + 2n_{11-1}|a_1|^2) + (n_{111}a_1^3 + 3n_{111-1}|a_1|^2 a_1 + c.c) \tag{5}$$

In general, a Galerkin approximation is not a good approximation unless a large number of modes are included. The problem is that the predicted behavior is not robust in the sense that it can depend sensitively on the number of modes. The Lorenz equations, for example, have their origin in a truncation in terms of 3 Fourier components and have become famous not for the physics they represent, but for their interesting solutions.

2.2. The Amplitude Equation Formalism

The fundamental assumption of this approach is the existence of a *slow manifold*. What this means is that the modes can be split into two groups, the *principal modes* characterized by $|\kappa_k/\omega_k| \ll 1$ and the *slave modes* for which κ_k/ω_k is negative and of order unity. The physical idea which underlies the dimensional reduction method is very simple. When a system is disturbed away from its static equilibrium it is only during a short transient time-interval that it rings with all the eignefrequencies, an interval during which the amplitudes of the slave modes decay away very fast. This is followed by a slow evolution in which the system is completely specified by the behavior of the amplitudes of the principal modes, *i.e.*,

$$z(t) = z(\{a_k(t)\}) = \sum_{k=1}^{p} a_k(t)\boldsymbol{e}_k + \textit{quadratic terms} + \textit{cubic terms} + \ldots \tag{6}$$

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These amplitudes can be considered generalized coordinates which parametrize in the slow manifold, *i.e.*, the subspace of phase-space corresponding to the slow manifold in which the system evolves. The behavior of the principal amplitudes themselves is described by a system of ordinary differential equations, the *amplitude equations* whose general form is

$$\frac{da_k}{dt} = (i\omega_k + \kappa_k)a_k + g_k(\{a_k\}), \quad k = 1, \ldots, p \tag{7}$$

where p is the number of principal modes. The expressions for $g_k(\{a_k\})$, the *normal forms* as they are also called depend on the number of modes and on what resonances, if any, are present, and are generic (*e.g.*, Guckenheimer and Holmes 1983). *Which specific form one needs to use can therefore be decided on the basis of the linear spectrum.*

The amplitude equations (Eq. (7) and the nonlinear terms $g_k(\{a_k\})$) can be obtained in a number of ways. Coullet and Spiegel (1984) presented a very elegant and general method for deriving the amplitude equations, whereas, concomitantly, Buchler and Goupil (1984; *cf.* also Buchler 1985) used a more intuitive multi-time method and gave explicit expressions for the case of radial stellar pulsations.

Because of the basic assumption of weak nonlinearity for the principal modes we can factor out a rapidly varying term from the amplitudes, *viz.*

$$a_k(t) = \exp(i\omega_k t)\tilde{a}_k(t) \tag{8}$$

It is clear that expression (6) for $\boldsymbol{z}(t)$ represents a multi-periodic function with the eigen-frequencies of all the principal modes. Actually, the higher order terms gradually modify these frequencies because of nonlinear effects without however destroying the multi-periodicity.

To summarize, the dimensional reduction method reduces the complicated partial differential system (1) or its infinite dimensional counterpart (4) to as system of p ODEs for only the p principal modes. This is schematically illustrated in Fig. 1. In the amplitude equation formalism the slow manifold, while of the same dimension as the space of the principal modes, curves into the space of the slave modes. In contrast, in the Galerkin approximation the system is restricted to move in the space of the modes which are being included. In addition, the coefficients in the amplitude equations are different.

We now examine the predictions of the amplitude equation formalism for the simplest case, viz. that of a single principal mode. The amplitude equation, truncated at the lowest nonlinear terms, is given by

$$\frac{da_\alpha}{dt} = (i\omega_\alpha + \kappa_\alpha)a + Q|a_\alpha|^2 a_\alpha \tag{9}$$

Introducing into Eq. (9) the slowly varying amplitude-modulus and phase, defined by $\tilde{a}_\alpha(t) = A_\alpha(t)\exp(i\phi_\alpha(t))$ we obtain

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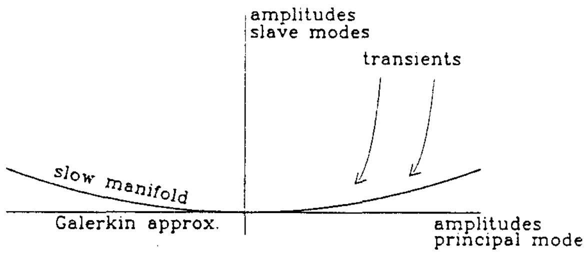

Fig. 1. The slow manifold.

$$\frac{dA_\alpha}{dt} = \kappa_\alpha A_\alpha + Re(Q_\alpha)A_\alpha^3 \qquad (10)$$

When $Re(Q_\alpha) < 0$, which is the usual situation in stellar pulsations, this equation has two *fixed points*, *i.e.*, values for which $dA_\alpha/dt = 0$. The first one, $A_\alpha = 0$, represents the stable equilibrium model, whereas the second one, $A_\alpha^2 = -\kappa_\alpha/Re(Q_\alpha)$, represents a *limit cycle*, or pulsation of the stellar model with constant amplitude. The point $\kappa_\alpha = 0$ is the *Hopf bifurcation* point or, in the parlance of stellar pulsations, the *blue edge* for the sequence.

One also obtains an equation for the phase $\phi_\alpha(t)$, given by $d\phi_\alpha(t)/dt = Im(Q_\alpha)A_\alpha^2$ which for the limit cycle has the trivial solution $\phi_\alpha(t) = Im(Q_\alpha)$ A_α^2. When inserted into Eq. (8) it represents the nonlinear correction to the frequency, $\omega_\alpha^{NL} = \omega_\alpha + Im(Q_\alpha)A_\alpha^2$.

It is easily shown (*e.g.*, Buchler 1985) that the cubic saturation coefficient Q_α is a (specific) combination of the cubic term and products of the quadratic terms from Eq. (4), *viz.*

$$Q_\alpha = 3n_{\alpha\alpha\alpha-\alpha} + \frac{2}{i\omega_\alpha}\sum_k(-2n_{\alpha\alpha k}n_{k\alpha-\alpha} + n_{\alpha-\alpha k}n_{k\alpha\alpha}) \qquad (11)$$

$$z(t) = (a_\alpha(t)e_\alpha + c.c.) + \frac{\alpha}{i\omega_\alpha}\sum_k\left(n_{k\alpha\alpha}a^2 - c.c. - n_{k\alpha-\alpha}|a_\alpha|^2\right)e_k \qquad (12)$$

where the sums extend over *all* modes, principal as well as slaves.

These results are very general and are derived from the full dynamical system (Eq. 1 or Eqs. 4). It is perhaps instructive to compare these results to the Galerkin approximation. If a corresponding dimensional reduction is performed on the Galerkin approximation (Eq. 5) we obtain the same amplitude equation (Eq. 9), but the sums both for the cubic term and for the correction to z now reduce to a single term, $k = \alpha$. The Galerkin solution vector $z(t)$ therefore lies in the space spanned by e_α and its complex conjugate. In contrast, in the amplitude equation formalism the second order and higher terms in $z(t)$ acorrectly extend into all of phase-space (as illustrated in Fig. 1). Of course when more modes are kept in the Galerkin

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approximation, these modes then also appear in the sums and improve the solution.

3. Applications of the Amplitude Equation Formalism

A number of formal studies of amplitude equations have been made for a variety of physical situations (*e.g.*, Dziembowski and Kovács 1984, Takeuti and Aikawa 1980, 1981, Buchler and Kovács 1986a, b, 1987a, Moskalik 1985, 1986, Buchler and Goupil 1988, and many others). Such studies are very valuable because they elucidate the various types of behavior that can be expected, *e.g.*, fixed points, limit cycles and chaos, corresponding to stellar pulsations with, respectively, constant amplitudes, periodically modulated and erratically modulated stellar pulsations. In addition to the nature of the solutions they also allow an assessment of their stability. In particular, they show the effects of various types of resonances and the bifurcations that they can produce. It is important to note that often these results do not depend on the exact values of the parameters which appear in the amplitude equations, but are valid for broad ranges of values, lending quite general validity to such studies.

On the quantitative side, two types of studies have been made. Truly *ab initio* calculations in realistic stellar models have only been preformed for quadratic coefficients. Dziembowski (1982) and Dziembowski *et al.* (1985, 1988) computed the coefficients appropriate for a resonant condition between nonradial modes (a p mode coupled to two g modes) in δ Scuti stars. Takeuti and Aikawa (1980, 1981) computed the quadratic coupling coefficients in the case of a 2:1 resonance in classical Cepheid models. These last two studies approximated the exact (complex) linear eigenvectors by (real) adiabatic ones. The general expressions for the coupling coefficients in classical Cepheids were computed by Klapp, Goupil and Buchler (1985). The expressions are very complicated an extension to the cubic terms will almost certainly require the use of a symbol manipulation program.

The second, different approach consists of using numerical hydrodynamical computations of the pulsations to extract the values of the nonlinear coupling coefficients. This approach is perhaps less accurate, but it yields some useful results which we shall describe in the next sections. In particular, it allows a quantitative comparison of hydro-model sequences, it sheds new light on the important role played by resonances and it allows a search for specific pulsational behavior in stellar models. For example, the amplitude equations predict quite generally that in the presence of a 2:1 resonance a limit cycle sees its stability decreased, sometimes to the point of instability. Such general guidance from amplitude equations allowed Kovács and Buchler (1988a) to conduct a successful numerical hydrodynamical search for persistent beat pulsations.

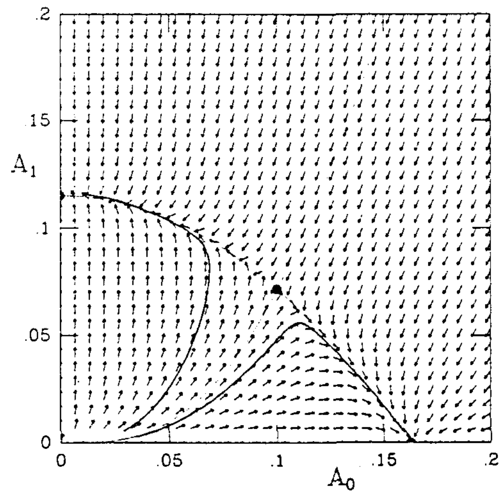

Fig. 2. Flow field for nonresonant two-mode case; Separatrices: *thin lines*; Evolution of hydro-model: *thick lines*.

3.1. Quantitative Studies

The question arises how good the amplitude equation formalism is from a *quantitative* point of view. We shall now illustrate this for the case of a nonresonant RR Lyrae model with stellar parameters, $0.45 M_\odot$, $50 L_\odot$, 5900K (taken from Buchler and Kovács 1987a). The model is linearly unstable in the fundamental (0) and first overtone (1) and has stable limit cycles in these two modes (talking about a "limit cycle in a mode" is appropriate because the *nonlinear* limit cycles have a predominant projection onto the space spanned by their respective *linear* eigenvectors, and additionally their frequencies are also very close to the linear ones).

After disturbing the static equilibrium model we can follow its evolution with the numerical hydro-code. A frequency analysis shows that after a very short transient only two principal frequencies survive and that they are very close to the linear fundamental and first overdone frequencies. A time-dependent Fourier decomposition of, say the stellar radius $R_*(t)$, then yields the time-dependent amplitudes $A_0(t)$ and $A_1(t)$ and the phases $\phi_0(t)$ and $\phi_1(t)$. The amplitudes are plotted as thick lines in an amplitude-amplitude plot in Fig. 2 for two separate initializations of the hydro-code. One notes that in each case both amplitudes first increase before evolving toward a fundamental and first overtone limit cycle, respectively. This evolution will be considered the experimental "data". We will show that this somewhat unexpected behavior of the amplitudes is explained not only qualitatively, but also very well quantitatively by the apposite amplitude equations.

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The amplitude equations for the *nonresonant two-mode* case to lowest order are given by

$$\frac{dA_0}{dt} = \kappa_0 A_0 + Re(Q_{00})A_0^3 + Re(Q_{01})A_1^2 A_0$$
$$\frac{dA_1}{dt} = \kappa_1 A_1 + Re(Q_{11})A_1^3 + Re(Q_{10})A_0^2 A_1 \quad (13)$$

The integral curves of these equations depend on the 2 growth-rates and on the 4 nonlinear cubic saturation coefficients. These are to be determined by requiring a fit of the integral curves of the amplitude equations to the "data". Details of the procedure can be found in Buchler and Kovács (1987a). The structure of the 2-D phase-space is perhaps best seen from the flow field of Eqs. (13) (with the fitted coefficients) which is shown in Fig. 2. The thin solid lines represent the separatrices of the flow field. The system has four fixed points, the origin (static model) which is an unstable node point, two stable node points, corresponding to the two limit cycles and a saddle point (double-mode beat solution). The nonmonotonic behavior of the amplitudes is now easily understood: The double-mode fixed point exerts a strong attraction at first, but then repels the integral curves toward the limit cycles. The phases decouple again from the amplitudes as for the single mode case (Eq. 9) and they can be computed when the temporal behavior of the latter is known. We just mention here that their behavior is also very well described by the amplitude equations.

The location and stability of the limit cycles are easily obtained from the amplitude equations. Table 1 shows that they compare very closely with the values obtained from a numerical computation of the limit cycles with a relaxation hydro-code. The quantities $\lambda_1(\lambda_0)$ represent the Floquet exponents for the growth of the first overtone (fundamental) in the fundamental (first overtone) limit cycles, respectively.

TABLE I
Limit Cycle Characteristics

Relaxation code:	$A_0 = 0.166$	$\lambda_1 = -0.039\Pi_1$;	$A_1 = 0.117$	$\lambda_0 = -0.0093\Pi_0$
AE fit:	$A_0 = 0.163$	$\lambda_1 = 0.033\Pi_1$;	$A_1 = 0.115$	$\lambda_0 = -0.0098\Pi_0$

We conclude that despite the very complicated input physics (equation of state with H and He ionization, realistic expression for opacity) the hydrodynamical evolution of the stellar model takes place in a 2-D space (4-D when the decoupled phases are included). This evolution is quite accurately described by the system of two nonresonant amplitude equations for the fundamental mode and the first overtone, truncated at the lowest order (cubic) nonlinearities. In addition, because of the simplicity of these equations all its

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fixed points and their stability can readily be studied analytically (Buchler and Kovács 1986b).

3.2. Effects of Resonances

Resonances have a strong effect on the appearance and stability of the pulsations. A model is said to be resonant when there exists a relation of the form $\sum l_k\omega_k \approx 0$, ($l_k$ positive or negative integers) between the frequencies of the eigenmodes. In general, only the resonances of low order, *i.e.*, for which $\sum_k |l_k|$ is small, can have an appreciable effect.

Since the two-mode resonances play a particularly important role we shall examine them in some detail. Let the resonance be characterized by the condition $n\omega_\alpha \approx m\omega_\beta$. The apposite amplitude equations are given by

$$\frac{da_\alpha}{dt} = \sigma_\alpha a_\alpha + Q_{\alpha\alpha}|a_\alpha|^2 a_\alpha + Q_{\alpha\beta}|a_\beta|^2 a_\alpha + P_\alpha a_\alpha^{*n-1} a_\beta^m$$
$$\frac{da_\beta}{dt} = \sigma_\beta a_\beta + Q_{\beta\beta}|a_\beta|^2 a_\beta + Q_{\beta\alpha}|a_\alpha^2|a_\beta + P_\beta a_\alpha^n a_\beta^{*m-1} \tag{14}$$

The nonlinear coupling coefficients P_γ and $Q_{\gamma\delta}$ are complex in general.

Let modes α and β be linearly unstable and stable, respectively. Under these conditions, in the absence of a resonance these equations would reduce to Eqs. (13) and the system would have a single stable limit cycle corresponding to mode α.

We can distinguish three types of solutions depending on the value of m.

3.2.1. m=1, integer resonances: $n\omega_\alpha = \omega_\beta$

The characteristics of this type of resonance are that $A_\beta \neq 0$, always, and that the fixed points represent phase-locked (or synchronized) solutions, which are therefore singly periodic. These resonances thus cause a distortion of the light and velocity curves (*e.g.*, bumps and shoulders). They have been found to play a major role in stellar pulsations.

The 2:1 resonance between the fundamental mode and the second overtone ($2\omega_0 \approx \omega_2$) has been found to be responsible for the Hertzsprung progression in the classical Cepheids. This was first conjectured by Simon and Schmidt (1976), but a full understanding had to await the development of the amplitude equation formalism. The observational low order Fourier phases both for the magnitude and for the radial velocity curves show a great deal of structure (*e.g.*, Simon and Moffett, 1985 for the light curves and Kovács *et al.*, 1990 for the radial velocity curves). It has been shown that cubic amplitude equations with terms describing the effects of the 2:1 resonance reproduce rather well this behavior of the Fourier phases (Kovács and Buchler 1989). Physically, the resonant, linearly stable second overtone gets entrained through its resonance with the fundamental mode. Because of phase-lock the solution remains periodic and the excitation of the second

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overtone manifests itself through the appearance of a secondary maximum on the light and radial velocity curves whose position varies with the proximity of the resonance. The amplitude equation formalism is the only formalism that is capable of explaining the Hertzsprung progression, which must be considered one of its major successes to date. We note that the same resonance is responsible for essentially identical progressions in BL Her model sequences (Buchler and Moskalik 1992).

The higher order, 3:1 resonance ($3\omega_0 \approx \omega_4$) occurs in low period classical Cepheids and in BL Her stars and gives rise to a deformation of the Fourier phases and amplitudes (Kovács and Buchler 1989, Buchler and Moskalik 1992). Moskalik and Buchler (1989) have analyzed the formal solutions of the amplitude equations, but a quantitative study has not been undertaken yet because this resonance overlaps with the wing of the 2:1 resonance and a fit involves too many terms to be practical.

3.2.2. m=2, half-integer resonances: $n\omega_\alpha = 2\omega_\beta$

The characteristics of this resonance are that they either have no effect on the pulsation (*i.e.*, $A_\beta = 0$) or they lead to a parametric excitation of the resonant overtone, depending on the parameters of the amplitude equations. In the latter case they lead to phase-locked period 2 pulsations because their period Π is given by $\Pi = n\Pi_\beta = 2\Pi_\alpha$). In other words these resonances affect the stability of the limit cycles, but not their appearance.

The 1:2 resonance has been found to be important in δ Scuti stars (Dziembowski 1982). Actually, he studied the 3-mode resonance between a p mode and tow g modes, but because the two g modes are required to have very similar frequencies, the 3-modes resonance has the same properties as the 2-mode resonance. Because of the large number of such resonances Dziembowski finds that amplitude saturation in these stars occurs through this resonance, rather that through the cubic terms; this has as a consequence a lower saturation amplitude. His numerical calculations of realistic stellar models yield amplitudes in agreement with observed ones. In an extension of this work Dziembowski *et al.* (1985, 1988) propose that the shortage of observed rapidly rotating variables is due to increased chances of resonances due to rotational splitting, and thus to unobservably low saturation amplitudes.

All numerical hydrodynamical studies of classical Cepheid model sequences found windows in which the pulsations displayed steady strictly periodically alternating cycles. This behavior has been trace to the 3:2 resonance ($3\omega_0 \approx 2\omega_1$) and the appropriate amplitude equations again give an analytical explanation of this behavior. Subsequently the same resonance was also found to be responsible for similar windows in BL Her models (Buchler and Moskalik 1992). In W Vir models a 5:2 resonance ($5\omega_0 \approx 2\omega_2$) is associated with period-doubling and the subsequent cascade of subsequent period-doublings to chaos which have been found in the radiative hydrodynamical models. This will be further discussed in §4.

3.2.3. m>2, integer resonances:
As in the case $m = 1$ the fixed points of the amplitude equations represent phase-locked solutions with $A_\beta \neq 0$. These solutions are not as likely to be important because, generally speaking, the effects of a resonance decrease with the order. Such a criterion must be used with caution, however; the previous section showed that a relatively high order resonance (5:2) was found responsible for a period 2 bifurcation. So far no hydrodynamical calculations have shown any good evidence for the importance $m > 2$ resonances.

Generally speaking we still have a relatively poor *a priori* understanding of when a particular resonance is going to have an important effect on the pulsation.

3.2.4. Other Resonances
The 3-mode resonance $\omega_\alpha + \omega_\beta \approx \omega_\gamma$, for a while was thought to be the cause of beat behavior. Although this resonance can in principle give rise to beat behavior (*e.g.*, Takeuti and Aikawa 1980, Kovács and Kolláth 1988), in fact, no evidence thereof has been found in hydrodynamical models. Other resonances which may turn out to play an important role are the higher order 3-mode resonances $\omega_\alpha + \omega_\beta \approx 2\omega_\gamma$ and $\omega_\alpha + \omega_\beta \approx 3\omega_\gamma$ (*cf.* Kovács in this Volume).

3.2.5. Overlapping Resonances
In many real situations several resonances are sufficiently close so that their combined effect may give rise to very complicated behavior. For example, a particularly dramatic accumulation of resonances occurs in very low temperature BL Her models (Buchler and Moskalik 1992). The corresponding amplitude equations become very cumbersome, contain many coefficients and are very difficult to study. As our modelling of stellar pulsations progresses the next stage of refinement will require the consideration of resonance overlaps.

4. The Nature of Chaos in W Vir Stars

4.1. Hydrodynamical Models

The standard astronomical techniques are not adequate for understanding the behavior of W Vir model sequences. However techniques from nonlinear dynamics can be used to decipher the nature of the observed bifurcations, and amplitude equations help clarify the physical situation.

The hydrodynamical study of W Vir model sequences by Buchler and Kovács (1987b), Kovács and Buchler (1988b) found that several sequences

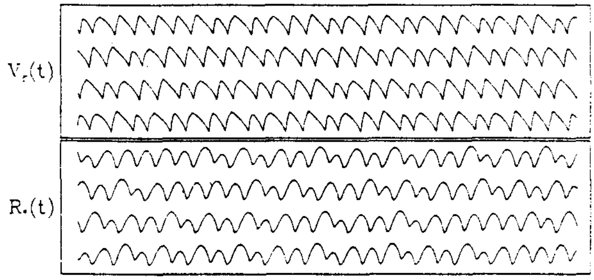

Fig. 3. Temporal behavior of the radial velocity and of the stellar radius for the W Vir model.

of models undergo text-book successions of *period-doublings* (*cf.* Cvitanovich 1986, Bergé *et al.* 1986) as a control parameter, here T_{eff}, is gradually lowered. For higher luminosity sequences, on the other hand, the transition to chaos occurs through a *tangent bifurcation* (Aikawa 1987, 1988, Buchler, Goupil and Kovács 1987, Kovács and Buchler, 1988). Since this transition to chaos can happen after just a few a few period-doublings (Aikawa 1990) this seems to imply that the bifurcation diagram folds backward. Instead of displaying the metamorphosis of the radius or radial velocity curves here, we refer the reader to the quoted papers (*cf.* also Aikawa's review in this Volume, or Buchler 1990).

Here we merely examine one such W Vir model, with stellar parameters $M = 0.6M_{\odot}$, $L = 500L_{\odot}$, $T_{\text{eff}} = 4200$K and a Pop. II composition of $X = 0.745$, $Z = 0.005$. The model comes from a sequence which has undergone a typical period-doubling cascade to chaos and its inverse chaotic undoublings. Figure 3 displays four short consecutive stretches of the temporal behavior of the stellar radius $R_*(t)$ (radius of the outermost zone) at the bottom and of the radial velocity $V_r(t)$ at the top. The fluctuations are much more apparent in the radius than in the velocity. Referring to the former we notice occasional intervals over which the oscillation is almost singly periodic and other intervals over which it exhibits strong RV Tau-like behavior.

The Fourier power spectrum, taken over 400 pulsations, is exhibited in Fig. 4. It displays a remarkably sharp peak at the fundamental frequency, but has a strong sub-harmonic structure, indicative of something interesting. Had the spectrum been obtained with gapped data beset with observational noise one might have concluded that the oscillation is periodic. However an O-C diagram, shown in Fig. 5, displays fairly large phase variations, up to 10%.

Perhaps the most powerful technique of the dynamicists is the phase-space reconstruction of an attractor. It allows one to construct a topologically equivalent structure from a *single* temporal signal; barring pathological cases, this can be any variable, observationally, for example the magnitude

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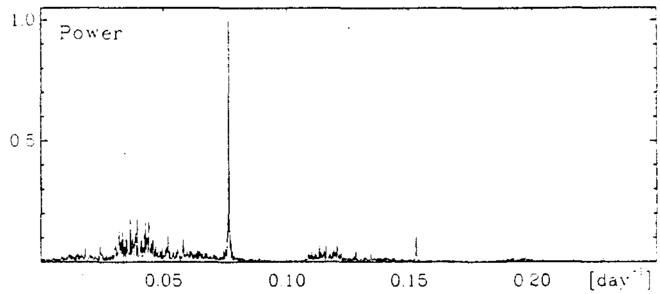

Fig. 4. Power spectrum for W Vir model of Fig. 3.

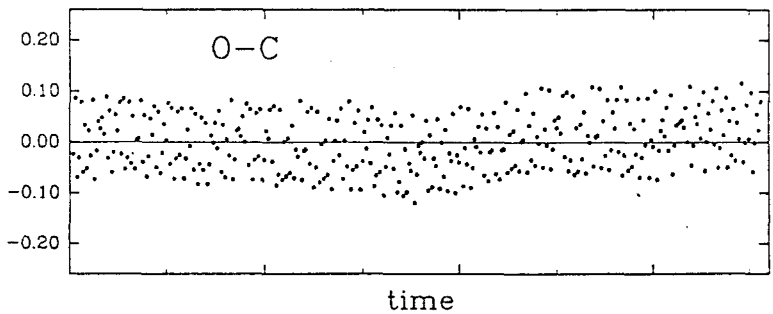

Fig. 5. O-C diagram for W Vir model of Fig. 3.

or the radial velocity. For the numerical hydrodynamical output we find it convenient to use the stellar radius. While a mathematical theorem says that any n-D dynamic can be embedded in a (2n+1)-D space, in practice it has been found that many physical systems are not pathologicl and an n-D embedding space is sufficient. Indeed, while a limit cycle (single loop) is embeddable in 2 dimensions (it can be represented by 2 first order ODEs), a period 2 cycle (double loop) clearly cannot occur in a 2-D space because this would require an intersection point for the trajectory (which is not allowed by the uniquencess theorem of ODEs). A period-doubling therefore implies that the dimension of phase-space must be greater that 2 and that the reconstruction must at least be made in a 3-D embedding space.

In Fig. 6 we display such a 3-D phase-space reconstruction of the dynamic obtained by plotting the triplets of values, $\{R(t), R(t+\tau), R(t+2\tau)\}$ with τ equal to 40% of the pulsation "period". The signal shows a remarkably tight structure, strongly reminiscent of the *Rössler attractor*. (We note that a multi-periodic signal would have given rise to a space-filling structure, *cf. e.g.*, Fig. 9 in Kovács and Buchler 1988b). The Rössler attractor of course arises in a 3-D embedding space since it is generated by a set of 3 ODEs. It is therefore of interest to attempt to compute the dimension of our attractor. The Grassberger-Procaccia correlation method yields a low dimension, but

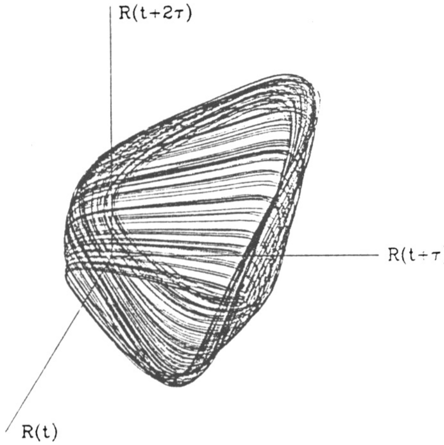

Fig. 6. 3-D phase-space reconstruction for W Vir model of Fig. 3.

is rather inaccurate and beset by uncertainties. An analysis based on a dynamical systems prediction method (Serre and Buchler 1992) also gives a low dimension, probably between 3 and 5. While the exact value of the embedding dimension is not certain, there seems to be little doubt that it is small.

We note in passing that there are many other useful techniques for exhibiting low dimensional structure, such as Poincaré sections, 1-D return maps, Lyapunov exponents, entropy methods, *etc.*

Two important questions arise. First, since the dimension of phase-space must be increased, in our modal description this necessitates the *excitation of another mode.* In the Rössler attractor this additional mode is necessarily a real, nonoscillatory mode. We will show that in our case, in contrast, it is oscillatory rather than secular. The second question concerns the robustness of period-doubling. Indeed, it occurs in many sequences of models and with a very different code (Aikawa 1987). It even survives when the heat flux includes time-dependent convection (Glasner and Buchler 1990).

A clue as to the origin of period-doubling comes from the classical Cepheid models. First of all, every sequence of classical Cepheid models (Buchler, Moskalik and Kovács 1990) has windows of strictly period 2 behavior in some range of T_{eff}, and a linear stability analysis of the models reveals a correlation of this behavior with the location of the half-integer resonance $3\omega_0 \approx 2\omega_1$. Second, a period-doubling bifurcation indicates that one of the Floquet coefficients $F_k = \exp(i\Phi_k + \lambda_k)$ must cross the negative real axis at -1, or that the Floquet phase, Φ_k passes through π. Since in the lowest approximation, $\Phi_k = \omega_k \Pi_0 = 2\pi \Pi_0/\Pi_k$, we may anticipate that the bifurcation is associated with a half-integer resonance condition $\Pi_0/\Pi_k = n/2$,

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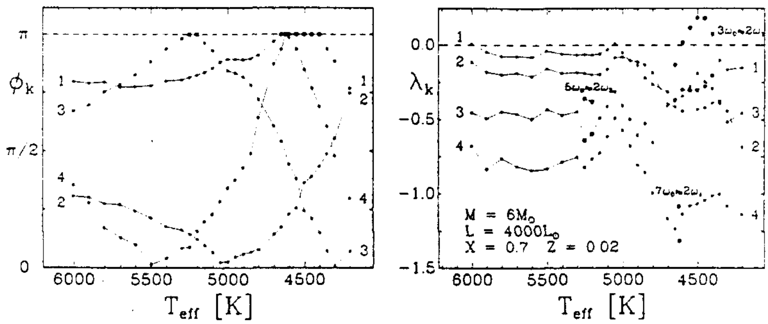

Fig. 7. Low order Floquet phases and coefficients for classical Cepheid sequence.

where n is integer. The bifurcation itself, however, is strictly nonlinear, and to uncover its true nature we need to turn to numerical hydrodynamics and to the amplitude equation formalism.

It is found that the results of the Floquet stability analysis of the limit cycles for a sequence of Cepheid models (sequence I of Moskalik and Buchler 1990, 1991) display very characteristic behavior, namely bubbles in the Floquet exponents and concomitant plateaux in the phases, which are connected with the first overtone. Figure 7 presents the phases Φ_k and exponents λ_k of the Floquet coefficients for the lowest few modes as a function of the control parameter $T_{\rm eff}$. (How this mode identification is carried out is described in Moskalik and Buchler 1991). Again it is seen that the bubbles and plateaux are associated with the linear modes which are in a half-integer resonance condition, as indicated in Fig. 7. One notes, however, that only for the first overtone does the Floquet exponent λ_1 pierce the stability boundary ($\lambda_1 = 0$) and give rise to period 2 behavior. As we have seen the classical Cepheids are weakly dissipative and the conditions for the validity of the amplitude equation formalism are satisfied. From Eqs. (14) it follows that the proper resonance terms to be added to Eqs. (13) are $P_0 a_0^{*n-1} a_k^2$ and $P_k a_k^{*n-1} a_0$, respectively. Moskalik and Buchler (1990) have shown that with the simplifying assumption of $Q_{01} = 0$ and of constant nonlinear coupling coefficients along the sequence as the resonance is traversed, the fixed points of these equations can be obtained analytically.

The amplitude equations predict that the Floquet coefficient for the resonant overtone, and the corresponding phase and exponent, behave as shown in Fig. 8. This is exactly the behavior seen in the Floquet coefficients of the numerical hydro-models.

For the W Vir models, on the other hand, we are beyond the range of validity of the amplitude equation formalism with $e.g.$, $\kappa_0 \Pi_0 \approx 0.20$. In order to obtain the cascade of period-doublings and chaos it would be necessary to introduce non-"normal" terms in the amplitude equations which

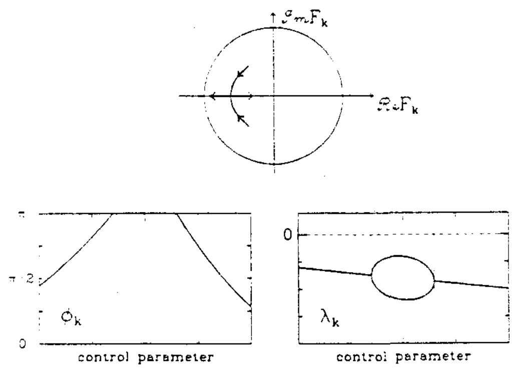

Fig. 8. Schematic behavior of the Floquet coefficient near the instability.

couple the oscillation to the modulation which now varies on the same time-scale. At the present time there is no known procedure for selecting such non-normal terms among all the nonlinear terms. Still, the Floquet analysis of the model sequences displays the same general type of behavior as for the classical Cepheids (Fig. 2 of Moskalik and Buchler 1991) and the same association of the bubbles and plateaux with the half-integer resonances. In these models it is the resonance $5\omega_0 \approx 2\omega_2$ that is responsible for the parametric excitation of the overtone. Now, however, inside the most important unstable bubble further period-doubling bifurcations and a transition to chaos occur. Still, it is the first period-doubling bifurcation which increases the dimension of phase-space and allows the subsequent period-doubling cascade to chaos to occur. We also note that a set of two coupled oscillators, designed to undergo a half-integer resonance, display the same type of period-doublings and chaos, including tangent bifurcations, as are observed in the hydrodynamical sequences.

We conclude that the bifurcation to period 2 behavior occurs because of the parametric excitation of a vibrational overtone which is brought about by a half-integer resonance. It is this association with a resonance which makes the occurrence of period-doubling so robust. Indeed, the locations of the resonances generally vary slowly and smoothly as the stellar parameters are varied, and they are not very much affected either by the numerical methods.

4.2. Observations

Traditional astronomical methods are well suited for analyzing periodic and multi-periodic signals, but they are inadequate when the purpose is to detect low-dimensional nonlinear behavior, and to determine its nature. It is true that almost 40 years ago small alternations were observed for the long period W Vir stars (15–35^d) in globular clusters by Arp (1955) in the light curves,

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and subsequently confirmed by Wallerstein (1958) in the radial velocities; similarly Lloyd-Evans (1971) reports such behavior in SZ Mon. Unfortunately, these investigations were not pursued, perhaps for want of suitable techniques of analysis and theoretical motivation. The observed alternations are very gentle in contrast to those found in the more luminous, longer period RV Tauri stars, and they appear very similar to the alternations in our hydrodynamical models. A systematic observation of long period W Vir stars has exciting prospect of confirming the period-doubling scenario.

Recently some attempts have been made to search for chaotic behavior. In the stellar context we note the work of Goupil *et al.* (1988) on white dwarfs, of Canizzo *et al.* (1990) and of Yanagida *et al.* (in this Volume) on long-period variables and of Saitou, Takeuti and Tanaka (1989) on semiregular stars. The only really thorough study is that of Kolláth (1990) who, by applying a myriad of techniques to the study of 150 years of data on R Sct gives credible evidence for low dimensional chaos in this star.

Generally, the application of modern analyses are hampered by large observational noise and by erratically timed and gapped observations (*cf.* Baglin in this Volume). Phase-space reconstructions require observations at equal time intervals, or at least data which can be interpolated as such with good accuracy. In order to make theoretical progress we have to convince the observers for the need of long-term observational programs of specific stars with good phase coverage, possibly multi-site to avoid too many gaps. Because of their relatively short periods and of the irregularities W Vir stars seem ideal candidates for that purpose.

5. Stochastic Effects

So far we have assumed that the model around which we expand Eq. (1) is truly static. In reality, the stellar interior is not quiescent, but undergoes complicated convective or turbulent motions. The latter can be considered to be the result of the nonlinear interaction of a large number of convective modes (of an originally truly static model), in static on average and treat the fluctuations as stochastic noise. We then substitute

$$\frac{\partial z}{\partial t} = \mathcal{L}(t)z + \mathcal{N}(z,t) + \Xi(t) \tag{15}$$

for Eq. (1), where the function $\Xi(t)$ represents additive noise and the time-dependence in $\mathcal{L}$ and $\mathcal{N}$ is parametric noise.

The amplitude equation formalism can be generalized to handle this situation (Stratonovich 1965, Buchler, Kovács and Goupil 1992) provided that, in addition to weak nonlinearity and weak dissipation, we assume (a) that the correlation time of the noise, τ_c is much smaller than the time-scale of modulation of the principal amplitudes, *i.e.*, $\tau_c \ll \tau_m \sim 1/\kappa_0$ and (b) that

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the stochastic processes are stationary, a reasonable assumption. Generally, τ_c is expected to be smaller than the period, therefore *a fortiori* smaller than τ_m. The result is a *Fokker-Planck equation* for the *probability distribution of the amplitudes* $w(\{a_k\})$.

If we look at the asymptotic case when the distribution has become stationary, and if, for illustration, we consider the 2 mode nonresonant case with additive noise only, the distribution $w(A_0, A_1)$ satisfies

$$\frac{\partial}{\partial A_0}\left[\left(\kappa_0 A_0 + Re(Q_{00})A_0^3 + Re(Q_{01})A_1^2 A_0 + \frac{C_0}{A_0}\right) w - C_0 \frac{\partial w}{\partial A_0}\right] + \frac{\partial}{\partial A_1}\left[\left(\kappa_1 A_1 + Re(Q_{11})A_1^3 + Re(Q_{10})A_0^2 A_1 + \frac{C_1}{A_1}\right) w - C_1 \frac{\partial w}{\partial A_1}\right] = 0 \quad (16)$$

where C_k is related to the spectral density of the noise.

It is well known that noise can have a critical effect on the behavior of a system near bifurcation points (*e.g.*, Moss and McClintock 1989). It is therefore of interest to explore how the modal selection problem is affected by noise, in particular whether it can generate beat behavior (Buchler and Kovács 1992; *cf.* also Kovács in this Volume).

Here we just give an example of how noise affects the behavior of an RR Lyrae model similar to that treated in section 3.1 (in particular Fig. 3). For a very low noise level the distribution has sharply spiked peaks at the fixed points of the noiseless amplitude equations. As the noise strength increases these peaks broaden and concomitantly move away from the axes. This leads to what is called precursor-noise in which the fluctuations are of the same order as the average amplitude of the mode. As pointed out by Kovács (in this Volume) the small, slowly varying overtone amplitudes detected by Walraven *et al.* in AI Vel could well be an example of such a stochastic excitation.

Noise can not only affect the appearance of the pulsation, it can also provoke qualitative changes in the nature of the pulsation. In Fig. 9 we show the distribution function for four increasing noise-levels (*a* to *d*, from left to right, top to bottom). Clearly, the peaks correspond to the limit cycles of the noiseless case. Although the peaks occur at a finite distance from the axes, *i.e.*, although both amplitudes are nonzero on average, the amplitude of the secondary mode remains of precursor type throughout. This situation therefore does not correspond to a true beat behavior. One notes, however, that sufficiently large noise can cause the disappearance of one of the two original states.

A more interesting situation comes about when the two linearly unstable modes are coupled to a large number of linearly stable modes which are stochastically driven. It then becomes possible for noise to convert the two peaks (single-mode limit cycles) into saddle points (unstable double-mode cycles) and the original saddle point into a peak. It is remarkable that the

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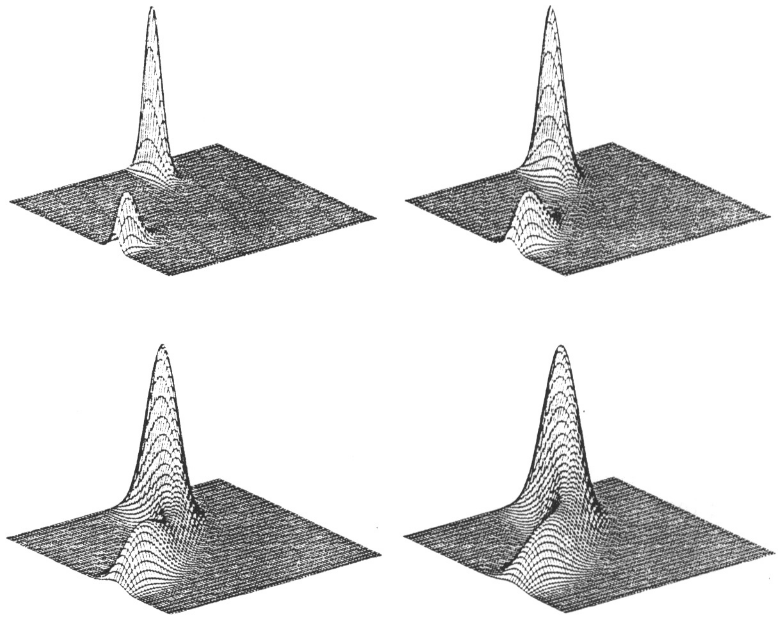

Fig. 9. Probability densities for nonresonant two-mode case; origin at the leftmost corner; A_1 and A_2 axes upward and downward, resp.; increasing noise level from left to right and top to bottom.

primary amplitudes of this double-mode solution can be made to have arbitrarily small fluctuations when the number of secondary modes is made very large (Buchler and Kovács 1992). Numerical hydrodynamical tests with additive noise confirm the predictions of the Fokker-Planck treatment.

The astrophysical applications of this formalism are still in their infancy and have interesting potential applications both in radial and nonradial stellar pulsations. In particular, it is well known that *parametric noise* which we have not considered here can have very dramatic effects on the bifurcations of the models (Moss and McClintock 1989).

6. Prospects

The application of techniques from dynamical systems has provided a new outlook on the problem of nonlinear stellar pulsations. The amplitude equation formalism not only gives a more basic understanding of the nature of the pulsations in terms of bifurcations, it also clarifies the role played by specific resonances. However, much more remains to be learned about the relative importance of resonances and especially about the consequences of their overlaps. It has also become clear that new tools of data analysis are necessary to understand not only the behavior of numerical hydrodynamical

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models, but also of observational data. It is hoped that the new theoretical developments will stimulate an observational effort specifically geared toward detecting an analyzing nonlinear effects.

We have not even touched on some areas in which dynamical systems techniques have also been applied, or where they are expected to make an impact. For example, the problems of the interaction between the pulsation and rotation, magnetic fields, and convection are clearly going to receive increase attention. Astrophysical disks are expected to be another active area of application. Perhaps one of the most exciting topics is the study of spatio-temporal structures, their formation, propagation, interaction and destruction with applications to convection, turbulence, magneto-hydrodynamics and dynamo theory. Finally, another topic in nonlinear science is that of patterns and fractal structures for which astrophysics is destined to be a fertile ground.

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