A NOTE ON A PRIME RING WITH A MAXIMAL ANNIHILATOR RIGHT IDEAL

Kwangil Koh

A ring R is called a prime ring [1] if and only if $a \cdot R \cdot b = 0$ implies that $a = 0$ or $b = 0$ for all a, $b \in R$. Hence if R is a prime ring and a is a non-zero element of R, $a \cdot R \neq 0$ and $R \cdot a \neq 0$. In the present note we prove that a prime ring with a maximal annihilator right ideal has no non-zero nil right or left ideal.

LEMMA. If R is a prime ring with a maximal annihilator right ideal then every nil right ideal I in R is zero.

Proof. Let J be a maximal annihilator right ideal of R. Then there is an element $0 \neq a \in R$ such that $J = (a)^r = \{ r \in R : ar = 0\}$. Suppose $I \neq 0$. If i is a non-zero element of I then $i \cdot R \cdot a \neq 0$ since R is a prime ring. Hence there is $b \in R$ such that $iba \neq 0$. Note that $R \cdot (iba)$ is a non-zero nil left ideal of R since R is prime and I is a nil right ideal. Let x, y be arbitrary elements of $R \cdot (iba)$. If x and y are non-zero elements then $(x)^r = (a)^r = (y)^r$ since $(a)^r$ is a maximal annihilator right ideal of R. Since y is nilpotent there is a positive integer m such that $y^m = 0$ and $y^{m-1} \neq 0$. $(y^{m-1})^r = (a)^r$ since $(y^{m-1})^r \geq (y)^r$. Now if r is an arbitrary element of R then $0 = y^m \cdot r = y^{m-1} \cdot (y \cdot r)$ and $x(yr) = 0$ since $(y^{m-1})^r = (a)^r = (x)^r$. This proves that $[R \cdot (iba)]^2 = (0)$ and thus $iba = 0$. This is a contradiction.

THEOREM. If R is a prime ring with a maximal annihilator right ideal then every nil right or left ideal of R is zero.

Proof. Let L be a nil left ideal of R. Then for each $x \in L$, $x \cdot R$ is a nil right ideal of R. Hence by the lemma,

https://doi.org/10.4153/CMB-1965-014-x Published online by Cambridge University Press

$0 = x \cdot R \leq L$. Thus L is also a right ideal of R. Hence by the lemma, $L = 0$.

REFERENCE

1. N.H. McCoy, Prime ideals in general rings, Amer. J. Math., vol. 71(1949), pp. 823-833.

North Carolina State of the
University of North Carolina at Raleigh

https://doi.org/10.4153/CMB-1965-014-x Published online by Cambridge University Press