Can. J. Math., Vol. XXX, No. 5, 1978, pp. 961–982

SULLIVAN'S MINIMAL MODELS AND HIGHER ORDER WHITEHEAD PRODUCTS

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1. Introduction. The theory of minimal models, as developed by Sullivan [**6**; **8**; **16**] gives a method of computing the rational homotopy groups of a space X (that is, the homotopy groups of X tensored with the additive group of rationals Q). One associates to X a free, differential, graded-commutative algebra $\mathscr{M}$ over Q, called the *minimal model* of X, from which one can read off the rational homotopy groups of X. More importantly, the rational homotopy type of X is determined by $\mathscr{M}$. Thus all rational homotopy invariants of X can theoretically be derived from $\mathscr{M}$. It is indicated in the above works how to obtain two important homotopy invariants from $\mathscr{M}$, namely, the rational Hurewicz homomorphism and rational Whitehead products. It is stated (without proof) in [**16**] that the quadratic term in the formula for the differential of the minimal algebra $\mathscr{M}$ determines rational Whitehead products in X. The main goal of this paper is to prove and generalize this latter result. We show that the rth order homotopy operation, the rth order Whitehead product, can be obtained from r-fold products in the decomposition of the differential of the minimal algebra. (Higher order Whitehead products are discussed in [**8**, pp. 183–184] in connection with minimal models. However, it is clear from the context that iterated ordinary Whitehead products and not higher order Whitehead products are being considered.) In point of fact, Sullivan's theory does not give the rational homotopy groups, the rational Hurewicz homomorphism, or rational Whitehead products, but rather the dual (in the vector space sense) of these objects. Thus in our main result we determine the dual of the rth order Whitehead product set from the minimal model.

The paper is organized as follows. In Sections 2 and 3 we present preliminary material on higher order Whitehead products, localization, Postnikov systems, linear algebra, and minimal models. In Section 3 we make explicit the pairing between elements of the minimal algebra $\mathscr{M}$ and elements of the homotopy of X. We consider in Section 4 the universal rth order Whitehead product element in the homotopy of the fat wedge of localized spheres. We give a complete calculation of the pairing of this element with all the appropriate generators of the minimal model of the fat wedge. This result enables us, in Section 5, to determine the pairing of rth order Whitehead product elements in a rational space with those elements of the minimal algebra whose differential decomposes into a sum of products with at least r factors. The paper concludes with

Received April 22, 1977 and in revised form, April 5, 1978.

several applications. We compute some higher order Whitehead products in two stage Postnikov systems and we show that the vanishing of all Whitehead products in a rational space implies the existence of an H-structure on that space.

2. Preliminaries on higher order Whitehead products, localization and linear algebra. All spaces in this paper will be 1-connected, pointed spaces having the homotopy type of CW-complexes. Maps and homotopies are to preserve base points. We shall not distinguish notationally between a map and its homotopy class and between two spaces of the same homotopy type. If f is a map, then $f_\#$ denotes the induced homomorphism on homotopy groups and $f_*(f^*)$ the induced homomorphism on homology (cohomology) groups. Notationally we suppress coefficients in homology and cohomology but *all homology will be with integer coefficients Z and all cohomology with rational coefficients Q.*

Let $A_1, A_2, \ldots, A_r$ be any r spaces, $r > 1$. We define the following two subspaces of the cartesian product $A_1 \times A_2 \times \ldots \times A_r$:

(1) the *wedge* $A_1 \vee A_2 \vee \ldots \vee A_r$ consisting of all r-tuples with at most one coordinate not at the base point;

(2) the *fat wedge* $T(A_1, A_2, \ldots, A_r)$ consisting of all r-tuples with at least one coordinate at the base point.

For homology elements $w_1 \in H_{n_1}(A_1)$, $w_2 \in H_{n_2}(A_2)$, $\ldots$, $w_r \in H_{n_r}(A_r)$ (Z coefficients), we denote the *homology cross product* by

$$w_1 \times w_2 \times \ldots \times w_r \in H_{n_1+\ldots+n_r}(A_1 \times A_2 \times \ldots \times A_r) \quad [\mathbf{7}, \text{ p. } 190].$$

For cohomology elements $u_1 \in H^{n_1}(A_1)$, $u_2 \in H^{n_2}(A_2)$, $\ldots$, $u_r \in H^{n_r}(A_r)$ (Q coefficients), the *cohomology cross product* is

$$u_1 \times u_2 \times \ldots \times u_r \in H^{n_1+\ldots+n_r}(A_1 \times A_2 \times \ldots \times A_r) \quad [\mathbf{7}, \text{ p. } 215].$$

Next let n_i, $i = 1, 2, \ldots, r$ be integers > 1 ($r > 1$), $N = n_1 + n_2 + \ldots + n_r$ and S^{n_i} the n_i-sphere. Denote the product $S^{n_1} \times \ldots \times S^{n_r}$ by P', the wedge $S^{n_1} \vee \ldots \vee S^{n_r}$ by W' and the fat wedge $T(S^{n_1}, \ldots, S^{n_r})$ by T'. If $v_i' \in H_{n_i}(S^{n_i}) \approx Z$ are generators, then $v_1' \times \ldots \times v_r' \in H_N(P') \approx Z$ is a generator. Let $j: P' \to (P', T')$ be the inclusion and $\partial: \pi_N(P', T') \to \pi_{N-1}(T')$ the boundary homomorphism in the homotopy sequence of the pair (P', T'). Since the pair (P', T') is $(N-1)$-connected, the Hurewicz homomorphism $h: \pi_N(P', T') \to H_N(P', T')$ is an isomorphism. Define *the universal* rth *order Whitehead product element* (of type $n_1, n_2, \ldots, n_r$) $w' \in \pi_{N-1}(T')$ by $w' = \partial h^{-1} j_*(v_1' \times \ldots \times v_r')$:

$$H_N(P') \xrightarrow{j_*} H_N(P', T') \underset{\approx}{\xleftarrow{h}} \pi_N(P', T') \xrightarrow{\partial} \pi_{N-1}(T').$$

Now suppose X is a space and $x_i \in \pi_{n_i}(X)$, $i = 1, 2, \ldots, r$, $n_i > 1$ and $r \geqq 2$. The elements x_i define a map $g': W' \to X$. Following Porter [**13**], define the (possibly empty) rth *order Whitehead product set* $[x_1, x_2, \ldots, x_r] \subseteq \pi_{N-1}(X)$ to be

$$\{f'_{\#}(w')|f'\colon T' \to X \text{ an extension of } g'\}.$$

We next summarize some facts about localization [**2**; **9**]. For a space X, let $X_\emptyset$ denote the *localization* of X at the empty set $\emptyset$. Then $X_\emptyset$ is also called the *rationalization* of X. In this paper localization shall always mean localization at $\emptyset$. If $W = S_\emptyset^{n_1} \vee \ldots \vee S_\emptyset^{n_r}$, $T = T(S_\emptyset^{n_1}, \ldots, S_\emptyset^{n_r})$ and $P = S_\emptyset^{n_1} \times \ldots \times S_\emptyset^{n_r}$ and W', T' and P' are as above, then the localization maps $e_i\colon S^{n_i} \to S_\emptyset^{n_i}$ induce maps

$$\tilde{e}\colon W' \to W, \quad e\colon T' \to T \quad \text{and} \quad \bar{e}\colon P' \to P$$

each of which is an extension of the previous one. Since $\tilde{e}$, e and $\bar{e}$ localize homology, it follows [**2**, pp. 45–48] that each is a localization map. Thus $W = W_\emptyset'$, $T = T_\emptyset'$ and $P = P_\emptyset'$. If $e_{i*}\colon H_{n_i}(S^{n_i}) \to H_{n_i}(S_\emptyset^{n_i})$ and $e_\#\colon \pi_{N-1}(T') \to \pi_{N-1}(T)$, then define $v_i \in H_{n_i}(S_\emptyset^{n_i})$ and $w \in \pi_{N-1}(T)$ by $v_i = e_{i*}(v_i')$ and $w = e_\#(w')$. We call w the *rational universal* rth *order Whitehead product element* (of type $n_1, n_2, \ldots, n_r$).

By a *rational space* is meant the rationalization of some space. If X is a rational space and $x_i \in \pi_{n_i}(X)$, $i = 1, 2, \ldots, r$, then, since $W = W_\emptyset'$, the map $g'\colon W' \to X$ determined by the x_i induces a unique map $g\colon W \to X$ such that $g\tilde{e} = g'$. We have the following characterization of higher order Whitehead products in X.

LEMMA 2.1. *If X is a rational space and $x_i \in \pi_{n_i}(X)$, then the rth order Whitehead product set $[x_1, x_2, \ldots, x_r] \subseteq \pi_{N-1}(X)$ is*

$$\{f_\#(w)|\, f\colon T \to X \text{ an extension of } g\}$$

where w is the rational universal rth Whitehead product element.

The proof is an immediate consequence of elementary properties of localization and hence omitted. Since only rational space will be considered in the sequel, this characterization of Whitehead products will be used.

Next we turn to a few simple facts about Postnikov systems [**15**, Chapter 8]. For any space X let

$$\ldots \to X_{n+1} \to X_n \to X_{n-1} \to \ldots$$

denote the *Postnikov tower* of X, where X_n is the nth *Postnikov section* of X. In particular, we consider Postnikov towers for $P = S_\emptyset^{n_1} \times \ldots \times S_\emptyset^{n_r}$ and $T = T(S_\emptyset^{n_1}, \ldots, S_\emptyset^{n_r})$. Since the inclusion map $T \to P$ is an $(N-2)$-equivalence, $N = \sum n_i$, we may assume $T_{N-2} = P_{N-2}$. If we denote $(S_\emptyset^{n_i})_{N-2}$ by L_i, then

$$P_{N-2} = (S_\emptyset^{n_1})_{N-2} \times \ldots \times (S_\emptyset^{n_r})_{N-2} = L_1 \times \ldots \times L_r.$$

Note that

$$L_i = \begin{cases} K(Q, n_i) = S_\emptyset^{n_i} & \text{if } n_i \text{ odd} \\ K(Q, n_i) & \text{if } n_i \text{ even and } N - 2 < 2n_i - 1 \\ S_\emptyset^{n_i} & \text{if } n_i \text{ even and } N - 2 \geqq 2n_i - 1. \end{cases}$$

We now define basic homology classes in $H_{n_i}(L_i)$ and $H_{n_i}(K(Q, n_i))$. Let q_i: $S_\emptyset^{n_i} \rightarrow (S_\emptyset^{n_i})_{N-2} = L_i$ be the $(N - 2)$-equivalence of the Postnikov system of $S_\emptyset^{n_i}$ and let

$$\nu: L_i = (S_\emptyset^{n_i})_{N-2} \rightarrow (S_\emptyset^{n_i})_{n_i} = K(Q, n_i)$$

be the composition of maps in the Postnikov tower. (If n_i is odd, q_i and ν are identity maps.)

Definition 2.3. The *basic homology classes* $\gamma_i \in H_{n_i}(L_i)$ and $b_i \in H_{n_i}(K(Q, n_i))$ are

$$\gamma_i = q_{i*}(v_i) \quad \text{and} \quad b_i = \nu_*(\gamma_i),$$

where $v_i \in H_{n_i}(S_\emptyset^{n_i})$ has been defined above.

If q: $P = S_\emptyset^{n_i} \times \ldots \times S_\emptyset^{n_r} \rightarrow P_{N-2} = L_1 \times \ldots \times L_r$ is the $(N - 2)$-equivalence of the Postnikov system of P, then clearly

$$(2.4) \qquad q_*(v_1 \times \ldots \times v_r) = \gamma_1 \times \ldots \times \gamma_r \quad \text{in } H_N(L_1 \times \ldots \times L_r).$$

We obtain from the Postnikov towers of T and P a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\text{inclusion}} & P \\ \downarrow{\scriptstyle l_{N-1}} & & \\ T_{N-1} & & \downarrow{\scriptstyle q} \\ \downarrow{\scriptstyle \nu_{N-1}} & & \\ T_{N-2} & = & P_{N-2} \end{array}$$

where l_{N-1} is the $(N - 1)$-equivalence of the Postnikov system of T and ν_{N-1} the fibre map of the Postnikov tower of T. Thus (q, l_{N-1}) is a map of pairs (or rather a map of maps)

$$(q, l_{N-1}): (P, T) \rightarrow (T_{N-2}, T_{N-1}).$$

If F_{N-1} denotes the fibre of ν_{N-1} with inclusion map i_{N-1}: $F_{N-1} \rightarrow T_{N-1}$ and τ: $H_N(T_{N-2}) \rightarrow H_{N-1}(F_{N-1})$ is the homology transgression [**10**, p. 284], then

https://doi.org/10.4153/CJM-1978-083-6 Published online by Cambridge University Press

the following diagram commutes (cf. [**1**, Remark 3.2])

$$\begin{array}{ccccccc}
H_N(P) & \xrightarrow{j_*} & H_N(P, T) & \xrightarrow{h^{-1}} & \pi_N(P, T) & \xrightarrow{\partial} & \pi_{N-1}(T) \\
\downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle (q,\, l_{N-1})_*} & & \downarrow{\scriptstyle (q,\, l_{N-1})_\#} & & \downarrow{\scriptstyle l_{N-1\#}} \\
H_N(T_{N-2}) & \xrightarrow{j_*} & H_N(T_{N-2}, T_{N-1}) & \xrightarrow{h^{-1}} & \pi_N(T_{N-2}, T_{N-1}) & \xrightarrow{\partial} & \pi_{N-1}(T_{N-1}) \\
& \searrow{\scriptstyle \tau} & & & & \nearrow{\scriptstyle i_{N-1\#}} & \\
& & H_{N-1}(F_{N-1}) & \xrightarrow{h^{-1}} & \pi_{N-1}(F_{N-1}). & &
\end{array}$$

Here h^{-1} denotes the inverse of the Hurewicz isomorphism, ∂ the homotopy boundary homomorphism and j the inclusion map into a pair. This diagram and (2.4) yield the following useful result.

LEMMA 2.5. *With the above notation,*

$$l_{N-1\#}(w) = i_{N-1\#}h^{-1}\tau(\gamma_1 \times \ldots \times \gamma_r)$$

in the group $\pi_{N-1}(T_{N-1})$, *where* w *is the rational universal* rth *order Whitehead product element and the* γ_i *are basic homology classes.*

We conclude this section with some linear algebra. Let $M(r, Q)$ denote the set of $r \times r$ matrices with entries from Q.

Definition 2.6. For fixed positive integers $n_1, n_2, \ldots, n_r$, define a function $K: M(r, Q) \rightarrow Q$ by

$$K(A) = \sum_{\sigma \in S_r} (-1)^{\epsilon(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{r\sigma(r)}$$

where $A = (a_{ij}) \in M(r, Q)$, S_r is the permutation group on $\{1, 2, \ldots, r\}$ and (Cf. [**5**, p. 473])

$$\epsilon(\sigma) = \sum_{i=1}^{r} \sum_{\substack{1 \leq j < \sigma^{-1}(i) \\ \sigma(j) > i}} n_i n_{\sigma(j)}.$$

If there are no summands in the latter sum, then let $\epsilon(\sigma) = 0$.

These formulas appear complicated, but they describe a fairly simple idea. Suppose $w_1, w_2, \ldots, w_r$ are elements of a graded (anti)commutative algebra with degree $w_i = n_i$. Use the matrix $A = (a_{ij})$ to construct formal expressions

$$p_1 = a_{11}w_1 + \ldots + a_{1r}w_r, \quad p_2 = a_{21}w_1 + \ldots + a_{2r}w_r, \quad \text{etc.}$$

Then $K(A)$ gives the coefficient of the term $w_1 w_2 \ldots w_r$ in the product $p_1 p_2 \ldots p_r$. The $(-1)^{\epsilon(\sigma)}$ introduces a $(-1)^{mn}$ whenever two adjacent elements of degree m and n are interchanged. Thus in the graded algebra,

$$w_1 w_2 \ldots w_r = (-1)^{\epsilon(\sigma)} w_{\sigma(1)} w_{\sigma(2)} \ldots w_{\sigma(r)}.$$

When all the n_i's are odd, then it is easily seen that $K(A)$ is the determinant of A. When all the n_i's are even, then $K(A)$ is the permanent of A [**11**].

3. Background on minimal models. Unless otherwise stated all spaces will now be the rationalization of spaces of finite type. By a *space of finite type* we mean a 1-connected space of the homotopy type of a CW-complex with finitely generated homotopy groups in each dimension. We further assume that each space X comes with a fixed Postnikov system, that is, a Postnikov tower

$$\ldots \longrightarrow X_{n+1} \xrightarrow{\nu_{n+1}} X_n \xrightarrow{\nu_n} X_{n-1} \longrightarrow \ldots$$

and compatible n-equivalences $l_n \colon X \to X_n$. Each ν_n is a fibre map with fibre F_n an Eilenberg-MacLane space $K(\pi_n(X), n)$,

$$(3.1) \quad F_n \xrightarrow{i_n} X_n \xrightarrow{\nu_n} X_{n-1}.$$

We recall some facts about minimal models [**6**; **8**; **16**]. The *minimal model* $\mathcal{M}_X$ of X is a free, commutative, differential, graded algebra (DGA) over Q with differential d a degree 1, decomposable homomorphism. The cohomology algebra $H^*(\mathcal{M}_X)$ is isomorphic to $H^*(X)$. The construction of $\mathcal{M}_X$ can proceed inductively from the Postnikov system of X. One inductively defines free commutative DGAs $\mathcal{M}_X(n)$ for all $n \geqq 1$ and then sets $\mathcal{M}_X = \bigcup_n \mathcal{M}_X(n)$. As an algebra

$$(3.2) \quad \mathcal{M}_X(n) = \mathcal{M}_X(n-1) \otimes H^*(F_n).$$

The differential d of $\mathcal{M}_X(n)$ is defined on $\mathcal{M}_X(n-1)$ to be the (inductively) given one. On $H^*(F_n)$, d is determined by the cohomology transgression

$$\tau \colon H^n(F_n) \to H^{n+1}(X_{n-1}) \approx H^{n+1}(\mathcal{M}_X(n-1))$$

of the fibration (3.1). There are two important points to note here:

(1) $\mathcal{M}_X(n)$ is the subalgebra of $\mathcal{M}_X$ generated by all elements of degree $\leqq n$;

(2) $\mathcal{M}_X(n)$ is the minimal algebra of X_n.

Thus there is a sequence of algebra isomorphisms

$$\chi_n \colon H^*(\mathcal{M}_X(n)) \to H^*(X_n)$$

and they are related by the following commutative diagram

$$(3.3) \quad \begin{array}{ccc} H^*(\mathcal{M}_X(n)) & \xrightarrow{\chi_n} & H^*(X_n) \\ \uparrow & & \uparrow \nu_n{}^* \\ H^*(\mathcal{M}_X(n-1)) & \xrightarrow{\chi_{n-1}} & H^*(X_{n-1}). \end{array}$$

It is part of the general theory that there is a one-one correspondence between Postnikov towers of X

$$\ldots \to X_{n+1} \to X_n \to X_{n-1} \to \ldots$$

and sequences of subalgebras

$$\ldots \subset \mathcal{M}_X(n-1) \subset \mathcal{M}_X(n) \subset \mathcal{M}_X(n+1) \subset \ldots$$

of the minimal algebra $\mathcal{M}_X$. Indeed, to specify a Postnikov system of X is it sufficient to give the minimal algebra $\mathcal{M}_X$ and its sequence of minimal subalgebras $\mathcal{M}_X(n)$ generated by all elements of degree $\leqq n$. Then the subalgebras $\mathcal{M}_X(n)$ and the resulting Postnikov tower of X satisfy all the relations mentioned above. Furthermore, these considerations apply to mappings. In particular, a map $f: X \to Y$ induces a *DGA* homomorphism $\varphi: \mathcal{M}_Y \to \mathcal{M}_X$. For f induces a map of Postnikov towers $f_n: X_n \to Y_n$ which inductively gives rise to homomorphisms $\varphi(n): \mathcal{M}_Y(n) \to \mathcal{M}_X(n)$ by (3.2). The isomorphisms χ_n are compatible with $\varphi(n)$ and f_n.

For any *DGA* $\mathcal{A}$ over Q with $\mathcal{A}^0 = Q$, define the graded vector space of *indecomposables* $I(\mathcal{A})$ to be the quotient $\mathcal{A}/\mathcal{A}^+ \cdot \mathcal{A}^+$, where $\mathcal{A}^+$ denotes the elements of positive degree. Let $\alpha \to \bar{\alpha}$ denote the quotient map $\mathcal{A} \to I(\mathcal{A})$. We call $I^n(\mathcal{A})$, the image of $\mathcal{A}^n$ under this map, the *indecomposables of degree* n. From the definition of $\mathcal{M}_X$ and (3.2), there is a natural isomorphism

(3.4) $\quad \omega: I^n(\mathcal{M}_X) = I^n(\mathcal{M}_X(n)) \xrightarrow[\approx]{} H^n(F_n).$

Moreover, a careful look at the construction yields the following commutative diagram [**8**, p. 163]

(3.5)
$$\begin{array}{ccc}
 & H^n(\mathcal{M}_X(n),\mathcal{M}_X(n-1)) & \xrightarrow{\ \delta\ } & H^{n+1}(\mathcal{M}_X(n-1)) \\
 & \Vert & & \\
I^n(\mathcal{M}_X) = & I^n(\mathcal{M}_X(n)) & & \downarrow \chi_{n-1} \\
 & \downarrow \omega & & \\
 & H^n(F_n) & \xrightarrow{\ \hat{\tau}\ } & H^{n+1}(X_{n-1})
\end{array}$$

Here $(\mathcal{M}_X(n), \mathcal{M}_X(n-1))$ is the relative cochain complex and δ is the boundary homomorphism in the exact cohomology sequence. It can also be shown that for $\beta \in \mathcal{M}_X(n)$ of degree n

(3.6) $\quad \delta\bar{\beta} = \{d\beta\}_{n-1},$

where $\{\quad\}_p$ denotes the cohomology class in $\mathcal{M}_X(p)$ of a cocycle in $\mathcal{M}_X(p)$. Diagram (3.5) now gives the relation

(3.7) $\quad \hat{\tau}\omega(\bar{\beta}) = \chi_{n-1}(\{d\beta\}_{n-1}).$

We next define a basic pairing in the theory.

Definition 3.8. Define the *Sullivan pairing*

$$\langle\langle\, ,\, \rangle\rangle: I^n(\mathcal{M}_X) \otimes \pi_m(X) \to Q$$

as follows. Let $\gamma \in I^n(\mathcal{M}_X) = I^n(\mathcal{M}_X(n))$ and $x \in \pi_m(X)$ and set

$$\langle\langle \gamma, x\rangle\rangle = \begin{cases} 0 & \text{if } n \neq m \\ \langle \omega(\gamma), hi_{n\#}^{-1}l_{n\#}(x)\rangle & \text{if } n = m. \end{cases}$$

where $\langle\ ,\ \rangle: H^n(F_n) \otimes H_n(F_n) \to Q$ is the Kronecker pairing of cohomology and homology [**7**, p. 187].

A map of spaces induces a homomorphism of homotopy groups and of minimal models, and it can be shown that the Sullivan pairing is natural with respect to these homomorphisms.

The existence of the Sullivan pairing implies that $I^n(\mathcal{M}_X)$ is isomorphic to Hom $(\pi_n(X), Q)$. Thus the theory of minimal models encompasses the theory of dual homotopy groups of rational spaces. The rest of this paper will show how to compute the operations dual to the rth order Whitehead product, $r \geqq 2$.

In the remainder of this section we examine the minimal models of localized spheres $S_\emptyset^{n_i}$. We first introduce basic cohomology classes.

Definition 3.9. A *basic cohomology class* $\hat{b}_i \in H^{n_i}(K(Q, n_i))$ is defined by the condition that the Kronecker pairing $\langle \hat{b}_i, b_i\rangle = 1$, where b_i is the basic homology class (2.3). Now define a *basic cohomology class* $\hat{\gamma}_i \in H^{n_i}(L_i)$ by $\hat{\gamma}_i = \nu^*(\hat{b}_i)$, where ν: $L_i = (S_\emptyset^{n_i})_{N-2} \to (S_\emptyset^{n_i})_{n_i} = K(Q, n_i)$ is the composition of Postnikov fibrations.

It follows from the naturality of the Kronecker pairing that $\langle \hat{\gamma}_i, \gamma_i\rangle = 1$, where γ_i is the basic homology class (2.3).

We now determine the minimal model of $S_\emptyset^{n_i}$ which we denote by $\mathcal{S}_i$. We first note that $\mathcal{S}_i(n_i)$ is a free algebra on one generator σ_i of dimension n_i and $d\sigma_i = 0$. Indeed, the fibration (3.1) reduces to

$$K(Q, n_i) \xrightarrow[=]{} (S_\emptyset^{n_i})_{n_i} \longrightarrow (S_\emptyset^{n_i})_{n_i-1} = *$$

and we see by (3.2) that we may identify $\mathcal{S}_i(n_i)$ with $H^*((S_\emptyset^{n_i})_{n_i}) = H^*(K(Q, n_i))$, the free algebra generated by an element σ_i in dimension n_i. Thus there are identifications $H^{n_i}(K(Q, n_i)) = H^{n_i}(\mathcal{S}_i(n_i)) = I^{n_i}(\mathcal{S}_i(n_i))$. The isomorphisms

$$\chi_{n_i}: H^{n_i}(\mathcal{S}_i(n_i)) \to H^{n_i}(K(Q, n_i)) \quad \text{and}$$
$$\omega_i: I^{n_i}(\mathcal{S}_i(n_i)) \to H^{n_i}(K(Q, n_i))$$

can be assumed to have the property

(3.10) $\chi_{n_i}\{\sigma_i\}_{n_i} = \hat{b}_i = \omega_i(\bar{\sigma}_i)$.

If n_i is odd, then $\mathcal{S}_i(n_i) = \mathcal{S}_i$. If n_i is even then $S_\emptyset^{n_i}$ can be represented as a two stage Postnikov system (3.1)

$$K(Q, 2n_i - 1) \to S_\emptyset^{n_i} = (S_\emptyset^{n_i})_{2n_i-1} \to (S_\emptyset^{n_i})_{2n_i-2} = (S_\emptyset^{n_i})_{n_i}$$
$$= K(Q, n_i).$$

Thus $\mathcal{S}_i$ is a free algebra generated by $\sigma_i \in \mathcal{S}_i^{n_i}$ and $\theta_i \in \mathcal{S}_i^{2n_i-1}$ such that

$d\sigma_i = 0$ and $d\theta_i = \sigma_i^2$. This describes the minimal model $\mathscr{S}_i$ of $S_\emptyset^{n_i}$. We conclude by examining the Sullivan pairing in $\mathscr{S}_i$.

LEMMA 3.11. *If $\sigma_i \in \mathscr{S}_i^{n_i}$ is the generator described above and $e_i \in \pi_{n_i}(S_\emptyset^{n_i})$ is the localization map, then $\langle\langle \bar{\sigma}_i, e_i \rangle\rangle = 1$.*

Proof.
$$\begin{aligned}\langle\langle \bar{\sigma}_i, e_i\rangle\rangle &= \langle \omega_i(\bar{\sigma}_i), h(\nu q_i)_\#(e_i)\rangle \\ &= \langle \hat{b}_i, \nu_* q_{i*} h(e_i)\rangle \\ &= \langle \hat{b}_i, \nu_* q_{i*} e_{i*}(v_i')\rangle \\ &= \langle \hat{b}_i, \nu_* q_{i*}(v_i)\rangle \\ &= \langle \hat{b}_i, \nu_*(\gamma_i)\rangle \\ &= \langle \hat{b}_i, b_i\rangle \\ &= 1.\end{aligned}$$

4. The minimal model of the fat wedge. To compute Whitehead products from the minimal model, it will be necessary to know the minimal model of T_{N-1}, the $(N-1)$st Postnikov stage of the fat wedge $T = T(S_\emptyset^{n_1}, \dots, S_\emptyset^{n_r})$. As was noted in § 2 we can choose Postnikov towers for the product $P = S_\emptyset^{n_1} \times \dots \times S_\emptyset^{n_r}$ and T such that $P_n = (S_\emptyset^{n_1})_n \times \dots \times (S_\emptyset^{n_r})_n$ for all n and $T_n = P_n$ for $n \leqq N-2$. In particular, $P_{N-2} = T_{N-2} = L_1 \times \dots \times L_r$ (2.2).

Now let $\mathscr{M}$ denote $\mathscr{M}_T$ and let $\mathscr{S}_i$ denote $\mathscr{M}_{S_\emptyset^{n_i}}$. Then, since $\mathscr{M}(N-2)$ is the minimal model of T_{N-2}, $\mathscr{M}(N-2) \approx \mathscr{S}_1(N-2) \otimes \dots \otimes \mathscr{S}_r(N-2)$. Thus the algebra $\mathscr{M}(N-2)$ has generators α_i in degree n_i for all $i = 1, \dots, r$ and generators β_i in degree $2n_i - 1$ whenever n_i is even and $2n_i - 1 \leqq N-2$. Furthermore, $d\alpha_i = 0$ and $d\beta_i = \alpha_i^2$. We can make the relationship between $\mathscr{M}$ and the $\mathscr{S}_i$ more precise in the following way. Let $p_i: T \to S_\emptyset^{n_i}$ be the projection onto the ith factor. Then p_i induces maps of Postnikov sections

$$p_i': T_{N-2} = L_1 \times \dots \times L_r \to (S_\emptyset^{n_i})_{N-2} = L_i \quad \text{and} \quad p_i'': T_{n_i} \to (S_\emptyset^{n_i})_{n_i} = K(Q, n_i)$$

which are also projections onto factors. The p_i induce homomorphisms of minimal models

$$\varphi_i: \mathscr{S}_i \to \mathscr{M}, \quad \varphi_i': \mathscr{S}_i(N-2) \to \mathscr{M}(N-2) \quad \text{and} \quad \varphi_i'': \mathscr{S}_i(n_i) \to \mathscr{M}(n_i).$$

We thus have commutative diagrams

$$(4.1) \qquad \begin{array}{ccc} T & \xrightarrow{p_i} & S_\emptyset^{n_i} \\ \downarrow{\scriptstyle l_{N-2}} & & \downarrow{\scriptstyle q_i} \\ T_{N-2} & \xrightarrow{p_i'} & (S_\emptyset^{n_i})_{N-2} = L_i \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \nu} \\ T_{n_i} & \xrightarrow{p_i''} & (S_\emptyset^{n_i})_{n_i} = K(Q, n_i) \end{array} \qquad \begin{array}{ccc} \mathscr{M} & \xleftarrow{\varphi_i} & \mathscr{S}_i \\ \cup & & \cup \\ \mathscr{M}(N-2) & \xleftarrow{\varphi_i'} & \mathscr{S}_i(N-2) \\ \cup & & \cup \\ \mathscr{M}(n_i) & \xleftarrow{\varphi_i''} & \mathscr{S}_i(n_i) \end{array}$$

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where l_{N-2} and q_i are $(N-2)$-equivalences and π and ν are compositions of Postnikov fibrations. It easily follows that

(4.2) $\varphi_i''(\sigma_i) = \alpha_i$.

In calculating $H^N(\mathscr{M}(N-2))$, we consider three cases:

(4.3) *Case* 1: $r = 2$ and $n_1 = n_2$ is even.

In this case $H^N(\mathscr{M}(N-2)) = Q \oplus Q \oplus Q$ and $\{\alpha_1\alpha_2\}_{N-2}$, $\{\alpha_1^2\}_{N-2}$ and $\{\alpha_2^2\}_{N-2}$ form a basis.

(4.4) *Case* 2: $r > 2$, $2 \max\{n_1, \ldots, n_r\} = N$ and $\max\{n_1, \ldots, n_r\}$ is even.

Let $n_i = \max\{n_1, \ldots, n_r\}$. Then $H^N(\mathscr{M}(N-2)) = Q \oplus Q$ and $\{\alpha_1 \ldots \alpha_r\}_{N-2}$ and $\{\alpha_i^2\}_{N-2}$ form a basis.

(4.5) *Case* 3: all other possibilities.

In this case $H^N(\mathscr{M}(N-2)) = Q$ and $\{\alpha_1 \ldots \alpha_r\}_{N-2}$ is a basis.

We are now able to determine $\mathscr{M}(N-1)$ from $\mathscr{M}(N-2)$ using the inductive construction by means of cohomology instead of Postnikov towers (see [**6**, p. 251] and [**8**, pp. 153–155]). Since $H^{N-1}(T) = 0$ and $H^N(T) = 0$, to obtain $\mathscr{M}(N-1)$ from $\mathscr{M}(N-2)$ it is only necessary to adjoint generators in dimension $N-1$ to kill the cohomology group $H^N(\mathscr{M}(N-2))$. In Case 1, $\mathscr{M}(N-1)^{N-1}$ will have three new generators: δ, ϵ_1 and ϵ_2 with $d\delta = \alpha_1 \ldots \alpha_r$, $d\epsilon_1 = \alpha_1^2$ and $d\epsilon_2 = \alpha_2^2$. In Case 2, $\mathscr{M}(N-1)^{N-1}$ will have two new generators δ and ϵ_i with $d\delta = \alpha_1 \ldots \alpha_r$ and $d\epsilon_i = \alpha_i^2$. In Case 3, $\mathscr{M}(N-1)^{N-1}$ will have only one new generator δ with $d\delta = \alpha_1 \ldots \alpha_r$. This defines $\mathscr{M}(N-1)$ in all cases. As we observed in § 3, this determines the Postnikov section T_{N-1} of T. The results of § 2 hold for this Postnikov tower of T.

Before proving the main result of this section we easily establish a lemma. Recall that χ_{N-2}: $H^{n_i}(\mathscr{M}(N-2)) \to H^{n_i}(T_{N-2})$ is the isomorphism defined in § 3 and $p_i'^*$: $H^{n_i}(L_i) \to H^{n_i}(T_{N-2})$ is induced by the projection p_i': $T_{N-2} = L_1 \times \ldots \times L_r \to L_i$.

LEMMA 4.6. *If* $\alpha_i \in \mathscr{M}(N-2)$ *is the generator of degree* n_i *and* $\hat{\gamma}_i \in H^{n_i}(L_i)$ *is the basic class, then*

$$\chi_{N-2}\{\alpha_i\}_{N-2} = p_i'^*(\hat{\gamma}_i).$$

Proof. $\chi_{N-2}\{\alpha_i\}_{N-2} = \pi^*\chi_{n_i}\{\alpha_i\}_{n_i}$ (by 3.3)

$= \pi^*\chi_{n_i}\tilde{\varphi}_i''\{\sigma_i\}_{n_i}$ (by 4.2)

where $\tilde{\varphi}_i''$ is the cohomogy homomorphism induced by φ_i'': $\mathscr{S}_i(n_i) \to \mathscr{M}(n_i)$. But

$$\begin{aligned}\pi^*\chi_{n_i}\tilde{\varphi}_i''\{\sigma_i\}_{n_i} &= \pi^*p_i''^*\chi_{n_i}\{\sigma_i\}_{n_i}\\ &= p_i'^*\nu^*\chi_{n_i}\{\sigma_i\}_{n_i} \quad \text{(by 4.1)}\\ &= p_i'^*\nu^*(\hat{b}_i) \quad \text{(by 3.10)}\\ &= p_i'^*(\hat{\gamma}_i) \quad \text{(by 3.9).}\end{aligned}$$

We now give the main results of this section.

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PROPOSITION 4.7. *If* $\lambda \in \mathscr{M}^{N-1}$ *is any element such that* $d\lambda = \alpha_1 \ldots \alpha_r$ *and* $w \in \pi_{N-1}(T)$ *is the rational universal Whitehead product element, then the Sullivan pairing*

$$\langle\langle \bar{\lambda}, w \rangle\rangle = (-1)^\wedge$$

where $\wedge = \sum_{i<j} n_i n_j$.

The sign $(-1)^\wedge$ here and in subsequent propositions is a result of our convention regarding the cohomology cross product. We follow Dold [**7**, Chapter 7] who uses the standard sign-changing convention for interchanging graded objects.

Proof.

$$\begin{aligned}\langle\langle \bar{\lambda}, w \rangle\rangle &= \langle \omega(\bar{\lambda}), h i_{N-1\#}^{-1} l_{N-1\,\#}(w) \rangle && \text{(by (3.8))}\\ &= \langle \omega(\bar{\lambda}), \tau(\gamma_1 \times \ldots \times \gamma_r) \rangle && \text{(by (2.5))}\\ &= \langle \hat{\tau}\omega(\bar{\lambda}), \gamma_1 \times \ldots \times \gamma_r \rangle \\ &= \langle \chi_{N-2}\{d\lambda\}_{N-2}, \gamma_1 \times \ldots \times \gamma_r \rangle && \text{(by (3.7))}\\ &= \langle \chi_{N-2}\{\alpha_1 \ldots \alpha_r\}_{N-2}, \gamma_1 \times \ldots \times \gamma_r \rangle \\ &= \langle \chi_{N-2}\{\alpha_1\}_{N-2} \ldots \chi_{N-2}\{\alpha_r\}_{N-2}, \gamma_1 \times \ldots \times \gamma_r \rangle \\ &= \langle p_1'^*(\hat{\gamma}_1) \ldots p_r'^*(\hat{\gamma}_r), \gamma_1 \times \ldots \times \gamma_r \rangle && \text{(by (4.6))}\\ &= \langle \hat{\gamma}_1 \times \ldots \times \hat{\gamma}_r, \gamma_1 \times \ldots \times \gamma_r \rangle \\ &= (-1)^\wedge \langle \hat{\gamma}_1, \gamma_1 \rangle \ldots \langle \hat{\gamma}_r, \gamma_r \rangle && \text{(by [7, 7.14])}\\ &= (-1)^\wedge.\end{aligned}$$

In cases (4.3) and (4.4) it is possible to have a non-zero cohomology class $\{\alpha_i^2\}_{N-2} \in H^N(\mathscr{M}(N-2))$.

PROPOSITION 4.8. *If* $\lambda \in \mathscr{M}^{N-1}$ *is such that* $d\lambda = \alpha_i^2$, *then* $\langle\langle \bar{\lambda}, w \rangle\rangle = 0$.

Proof. As in the proof of Proposition 4.7,

$$\begin{aligned}\langle\langle \bar{\lambda}, w \rangle\rangle &= \langle \chi_{N-2}\{d\lambda\}_{N-2}, \gamma_1 \times \ldots \times \gamma_r \rangle \\ &= \langle \chi_{N-2}\{\alpha_i^2\}_{N-2}, \gamma_1 \times \ldots \times \gamma_r \rangle \\ &= \langle p_i'^*(\hat{\gamma}_i^2), \gamma_1 \times \ldots \times \gamma_r \rangle \\ &= \langle \hat{\gamma}_i^2, p_{i*}'(\gamma_1 \times \ldots \times \gamma_r) \rangle \\ &= 0.\end{aligned}$$

Note from the proof of Proposition 4.7 that we could have required the weaker hypothesis $\{d\lambda\}_{N-2} = \{\alpha_1 \ldots \alpha_r\}_{N-2}$. A similar remark holds for Proposition 4.8.

5. The main theorem. Let $\mathscr{A}$ be a free, commutative *DGA* over Q and let us fix an ordered set of generators $\{\eta_1, \eta_2, \ldots, \eta_t, \ldots\}$ of $\mathscr{A}$ with $\dim \eta_1 \leqq \dim \eta_2 \leqq \ldots \leqq \dim \eta_t \leqq \ldots$. (We allow the set of generators to be finite or infinite.) We now define $\mathscr{F}^p(\mathscr{A})$ to be the graded vector space generated by all elements $\{\eta_{i_1}\eta_{i_2}\ldots\eta_{i_s}|s \geqq p$ and $1 \leqq i_1 \leqq i_2 \leqq \ldots \leqq i_s\}$. This gives a decreasing sequence of graded vector spaces

$$\mathscr{A} = \mathscr{F}^1(\mathscr{A}) \supseteq \mathscr{F}^2(\mathscr{A}) \supseteq \ldots \supseteq \mathscr{F}^p(\mathscr{A}) \supseteq \ldots .$$

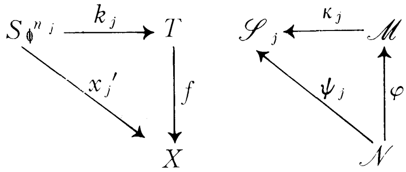
We now consider the minimal algebra $\mathscr{N}$ of a rational space X. We assume that $\mathscr{N}$ has an ordered set of generators $\{\eta_1, \eta_2, \ldots, \eta_t, \ldots\}$ as above. Let $x_j \in \pi_{n_j}(X)$, $j = 1, 2, \ldots, r$, be any r homotopy elements, $r \geqq 2$. Our main result will deal with the following situation. The elements x_j determine x_j': $S_\emptyset^{n_j} \to X$ with $x_j'e_j = x_j$. The x_j' give rise to a map g: $W = S_\emptyset^{n_1} \vee \ldots \vee S_\emptyset^{n_r} \to X$ and we now assume that there exists an extension f: $T \to X$ of g. Then f determines a homomorphism $\varphi: \mathscr{N} \to \mathscr{M}$ of minimal models. If η_i is a generator of dimension $\leqq N - 2$ then $\varphi(\eta_i) \in \mathscr{M}(N - 2)$ and so we can write

$$(5.1) \quad \varphi(\eta_i) = \sum_{j=1}^{r} d_{ij}\alpha_j + a_i$$

where $d_{ij} \in Q$ and $a_i \in \mathscr{M}(N - 2)$ is a linear combination of terms each of which is in $\mathscr{F}^2(\mathscr{M}(N - 2))$ or is a multiple of some generator β_m of $\mathscr{M}(N - 2)$.

LEMMA 5.2. $d_{ij} = \langle\langle\bar{\eta}_i, x_j\rangle\rangle$.

Proof. If dim $\eta_i \neq n_j$ then from (5.1) $d_{ij} = 0$. But in this case $\langle\langle\bar{\eta}_i, x_j\rangle\rangle = 0$. Therefore suppose that dim $\eta_i = n_j$. Let k_j: $S_\emptyset^{n_i} \to T$ be the inclusion. Thus we have a commutative diagram of maps of spaces and of resulting homomorphisms of minimal models



where κ_j and ψ_j are induced by k_j and x_j' respectively. It is clear that if α_i and β_i are the generators of $\mathscr{M}(N - 2)$, then

$$\kappa_j(\alpha_i) = \begin{cases} 0 & i \neq j \\ \sigma_j & i = j \end{cases} \quad \text{and} \quad \kappa_j(\beta_i) = \begin{cases} 0 & i \neq j \\ \theta_j & i = j \end{cases}$$

where σ_j and θ_j are the generators of $\mathscr{S}_j$. We apply κ_j to (5.1) and obtain

$$\psi_j(\eta_i) = \kappa_j\varphi(\eta_i) = d_{ij}\sigma_j + \kappa_j(a_i).$$

But a_i consists of terms which are multiples of β_m or are products of generators. The only possible β_m which is not annihilated by κ_j is β_j. However, this term cannot occur in a_i since dim $\beta_j = 2n_j - 1 \neq n_j =$ dim η_i. Thus $\kappa_j(a_i)$ is a linear combination of products of generators of $\mathscr{S}_j$ and so

$$\overline{\psi_j(\eta_i)} = d_{ij}\bar{\sigma}_j.$$

Therefore

$$\begin{aligned}\langle\langle\bar{\eta}_i, x_j\rangle\rangle &= \langle\langle\bar{\eta}_i, x_{j\#}'(e_j)\rangle\rangle \\ &= \langle\langle\overline{\psi_j(\eta_i)}, e_j\rangle\rangle \\ &= d_{ij}\langle\langle\bar{\sigma}_j, e_j\rangle\rangle \\ &= d_{ij} \qquad \text{(by Lemma 3.11).}\end{aligned}$$

Before stating the main result we need one more definition. Recall that we have fixed generators $\{\eta_1, \eta_2, \ldots, \eta_t, \ldots\}$ of the minimal model $\mathscr{N}$ of X and homotopy elements $x_j \in \pi_{n_j}(X)$, $j = 1, \ldots, r$. We now define a function $\tilde{K}: \mathscr{F}^r(\mathscr{N}) \to Q$ as follows. Let $\alpha \in \mathscr{F}^r(\mathscr{N})$ and write

$$\alpha = \sum_{1 \leqq i_1 \leqq \ldots \leqq i_r} q_{i_1 \ldots i_r}\eta_{i_1} \ldots \eta_{i_r} + \beta$$

where $q_{i_1 \ldots i_r} \in Q$ and $\beta \in \mathscr{F}^{r+1}(\mathscr{N})$. Setting $I = (i_1, \ldots, i_r)$ with $1 \leqq i_1 \leqq \ldots \leqq i_r$ and $\eta_I = \eta_{i_1} \ldots \eta_{i_r}$, we rewrite this as

$$\alpha = \sum_I q_I\eta_I + \beta.$$

We then define

$$(5.3) \quad \tilde{K}(\alpha) = \sum_I q_I K(A_I)$$

where A_I is the $r \times r$ matrix in $M(r, Q)$ whose (p, q)-entry is $\langle\langle\bar{\eta}_{i_p}, x_q\rangle\rangle$ and $K: M(r, Q) \to Q$ is the function given by Definition 2.6.

We now state the main theorem.

THEOREM 5.4. *Let X be a rational space with homotopy elements $x_j \in \pi_{n_j}(X)$, $j = 1, \ldots, r$, whose minimal algebra $\mathscr{N}$ has a fixed set of generators. Suppose that the higher order Whitehead product set $[x_1, x_2, \ldots, x_r] \subseteq \pi_{N-1}(X)$ is non-empty and that $\mu \in \mathscr{N}$ is an element of degree $N - 1$ with $d\mu \in \mathscr{F}^r(\mathscr{N})$. Then for each $x \in [x_1, x_2, \ldots, x_r]$, the Sullivan pairing*

$$\langle\langle\bar{\mu}, x\rangle\rangle = (-1)^{\wedge}\tilde{K}(d\mu)$$

where $\wedge = \sum_{i<j} n_i n_j$ and $\tilde{K}$ is defined by (5.3).

Proof. The x_i determine a map $g: W = S_\emptyset^{n_1} \vee \ldots \vee S_\emptyset^{n_r} \to X$. By Lemma 2.1, there is an extension $f: T = T(S_\emptyset^{n_1}, \ldots, S_\emptyset^{n_r}) \to X$ such that $f_\#(w) = x$. Then f induces a homomorphism $\varphi: \mathscr{N} \to \mathscr{M}$ and

$$\langle\langle\bar{\mu}, x\rangle\rangle = \langle\langle\bar{\mu}, f_\#(w)\rangle\rangle = \langle\langle\overline{\varphi(\mu)}, w\rangle\rangle.$$

This brings the computation back into $\mathscr{M}$ and from Propositions 4.7 and 4.8 it will suffice to find $d\varphi(\mu)$. We recall from § 4 that in constructing $\mathscr{M}(N - 1)$ from $\mathscr{M}(N - 2)$ new generators δ (in all cases (4.3)–(4.5)), ϵ_i (in Cases 1 and 2 ((4.3) and (4.4)) with $i = 1$ in Case 1) and ϵ_2 (in Case 1 (4.3)) of dimension

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$N - 1$ were adjoined. Since $\varphi(\mu) \in \mathscr{M}(N-1)$,

$$\varphi(\mu) = a\delta + b\epsilon_i + c\epsilon_2 + \rho$$

for $a, b, c \in Q$ and $\rho \in \mathscr{M}(N-2)$. In Case 3, $b = c = 0$, in Case 2, $c = 0$, and in Case 1, $i = 1$. Thus

$$\langle\langle\overline{\varphi(\mu)}, w\rangle\rangle = a\langle\langle\bar{\delta}, w\rangle\rangle + b\langle\langle\bar{\epsilon}_i, w\rangle\rangle + c\langle\langle\bar{\epsilon}_2, w\rangle\rangle$$

since $\bar{\rho} = 0$. But $d\delta = \alpha_1\alpha_2 \ldots \alpha_r$ and $d\epsilon_i = \alpha_i^2$. It now follows from Propositions 4.7 and 4.8 that

$$\langle\langle\overline{\varphi(\mu)}, w\rangle\rangle = (-1)^{\hat{}}a.$$

All that needs to be calculated then is a, the coefficient of $\alpha_1\alpha_2 \ldots \alpha_r$ in the expansion of $d\varphi\mu = \varphi d\mu$. By hypothesis, $d\mu \in \mathscr{F}^r(\mathscr{N})$ and so we can write

$$(5.5) \quad d\mu = \sum_I q_I\eta_I + \beta$$

where $I = (i_1, \ldots, i_r)$ with $1 \leqq i_1 \leqq \ldots \leqq i_r$, $\eta_I = \eta_{i_1} \ldots \eta_{i_r}$ with $\dim \eta_{i_1} + \ldots + \dim \eta_{i_r} = N$, and $\beta \in \mathscr{F}^{r+1}(\mathscr{N})$. Thus

$$(5.6) \quad \varphi(d\mu) = \sum_I q_I\varphi(\eta_I) + \varphi(\beta)$$

where $\varphi(\eta_I) = \varphi(\eta_{i_1}) \ldots \varphi(\eta_{i_r})$ and $\varphi(\beta) \in \mathscr{F}^{r+1}(\mathscr{M})$. Because $\varphi(\beta) \in \mathscr{F}^{r+1}(\mathscr{M})$ it can give no contribution to the $\alpha_1\alpha_2 \ldots \alpha_r$ term. We therefore examine more closely each term $\varphi(\eta_I) = \varphi(\eta_{i_1}) \ldots \varphi(\eta_{i_r})$ in (5.6). Since for each $i_j \in I$, $\dim \eta_{i_j} \leqq N-2$, we can by (5.1) write

$$\varphi(\eta_{i_j}) = \sum_{m_j=1}^{r} d_{i_jm_j}\alpha_{m_j} + a_{i_j}$$

with each a_{i_j} a linear combination of terms in $\mathscr{F}^2(\mathscr{M})$ or of multiples of some β_m. Thus we must determine the coefficient of $\alpha_1\alpha_2 \ldots \alpha_r$ in

$$\varphi(\eta_I) = \varphi(\eta_{i_1}) \ldots \varphi(\eta_{i_r}) = \prod_{j=1}^{r} \left(\sum_{m_j=1}^{r} d_{i_jm_j}\alpha_{m_j} + a_{i_j}\right).$$

When this product is expanded out, any summand which contains an a_{i_j} does not contribute to the coefficient of $\alpha_1\alpha_2 \ldots \alpha_r$. Hence we must determine the coefficient of $\alpha_1\alpha_2 \ldots \alpha_r$ in the expansion of

$$\prod_{j=1}^{r} \left(\sum_{m_j=1}^{r} d_{i_jm_j}\alpha_{m_j}\right).$$

But this is just $K(A_I)$, where A_I is the $r \times r$ matrix whose (p, q)-entry is d_{i_pq} (see Definition 2.6 and the ensuing discussion). Thus the coefficient of $\alpha_1\alpha_2 \ldots \alpha_r$ in (5.6) is $\sum_I q_IK(A_I)$ and so

$$a = \sum_I q_IK(A_I).$$

But $d_{i_p q} = \langle\langle \bar{\eta}_{i_p}, x_q \rangle\rangle$ by Lemma 5.2. Therefore it follows from (5.5) and Definition 5.3 that

$$a = \tilde{K}(d\mu).$$

Hence

$$\langle\langle \bar{\mu}, x \rangle\rangle = (-1)^{\wedge} a = (-1)^{\wedge} \tilde{K}(d\mu).$$

This completes the proof.

We conclude this section with some remarks on the theorem. First of all, the function $\tilde{K}: \mathscr{F}^r(\mathscr{N}) \to Q$ depends on the choice of elements $x_j \in \pi_{n_j}(X)$ and on the choice of generators of the minimal algebra $\mathscr{N}$. When a different Whitehead product set is being considered, a different function $\tilde{K}$ will be needed. Secondly, the theorem gives a calculation of a higher Whitehead product x by showing how x is paired with indecomposables $\bar{\mu}$ of degree $N - 1$. The right hand side of the equality in Theorem 5.4 is computable once one knows $d\mu$ as a sum of products of generators and how these generators pair with the homotopy elements x_i to form the matrices A_I. The rational number $\tilde{K}(d\mu)$ is then a linear combination of the $K(A_I)$, and the determination of the latter is a straightforward operation in linear algebra. We will illustrate this method in the next section by computing higher order Whitehead products of rational spaces from a knowledge of the minimal algebra. Finally, we comment on the hypothesis that $d\mu \in \mathscr{F}^r(\mathscr{N})$. It is sometimes possible to compute $\langle\langle \bar{\mu}, x \rangle\rangle$ when $d\mu \notin \mathscr{F}^r(\mathscr{N})$. However, if one could compute $\langle\langle \bar{\mu}, x \rangle\rangle$ for all possible $\bar{\mu}$, then x would be uniquely determined and hence $[x_1, \ldots, x_r] = \{x\}$. We can give an example of a space one of whose Whitehead product sets has a non-trivial indeterminacy. Thus there is no formula for $\langle\langle \bar{\mu}, x \rangle\rangle$ in terms of $d\mu$ and $x_1, \ldots, x_r$ in the case $d\mu \notin \mathscr{F}^r(\mathscr{N})$.

In conclusion we observe that the hypotheses of Theorem 5.4 are always satisfied when $r = 2$. This is because for any $\mu \in \mathscr{N}^{N-1}$, $d\mu \in \mathscr{F}^2(\mathscr{N})$, since the differential d is decomposable. Furthermore, ordinary Whitehead products $[x_1, x_2]$ always exist (and are unique). Therefore the equality in Theorem 5.4 holds without any restriction in the case $r = 2$. Thus we have proved Sullivan's result in [**16**, Theorem B] which asserts that (dual) Whitehead products are described by the quadratic terms of the d-images of generators of the minimal model. The formation of this result as hinted in [**6**, p. 250] can also be obtained. We do this in the next section.

6. Applications. In this section we give several applications of the main theorem. We first explicitly state and prove Sullivan's result for ordinary Whitehead products which is indicated in [**6**, p. 250]. After that we establish some general results on the existence, uniqueness and vanishing of higher order Whitehead products. Then we make some computations of higher order Whitehead products in two stage Postnikov systems. We conclude the section with some results on higher order Whitehead products and H-spaces.

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Let V be a graded vector space over Q which is finite dimensional in each degree. Denote the symmetric product of V with itself by $V \wedge V$, and the full symmetric algebra on V by $S(V)$ [**5**, Chapter 3, § 6]. We define an isomorphism

$$\Phi: \mathrm{Hom}\,(V, Q) \wedge \mathrm{Hom}\,(V, Q) \rightarrow \mathrm{Hom}\,(V \wedge V, Q)$$

by

$$\Phi(f \wedge g)(v \wedge w) = \begin{cases} (-1)^{pq} f(v) g(w) + g(v) f(w) & \text{if } \{p, q\} = \{m, n\} \\ 0 \quad \text{otherwise,} & \end{cases}$$

where $f \in \mathrm{Hom}\,(V^m, Q)$, $g \in \mathrm{Hom}\,(V^n, Q)$, $v \in V^p$, and $w \in V^q$. It is understood that $f(v) = 0$ when $m \neq p$. If $\mathcal{N}$ denotes the minimal algebra of a space X, then the Sullivan pairing induces an isomorphism

$$\Psi: I(\mathcal{N}) \rightarrow \mathrm{Hom}\,(\pi_*(X), Q).$$

Since $\mathcal{N}$ is free, it is isomorphic, as an algebra, with $S(I(\mathcal{N}))$ and $\mathcal{F}^2(\mathcal{N})/\mathcal{F}^3(\mathcal{N})$ is isomorphic, as a graded vector space, with $I(\mathcal{N}) \wedge I(\mathcal{N})$. Let $\tilde{d}: I(\mathcal{N}) \rightarrow I(\mathcal{N}) \wedge I(\mathcal{N})$ be the degree 1 homomorphism defined by the composition

$$I(\mathcal{N}) \xrightarrow{d'} \mathcal{F}^2(\mathcal{N})/\mathcal{F}^3(\mathcal{N}) \approx I(\mathcal{N}) \wedge I(\mathcal{N}),$$

where $d'(\bar{\lambda}) = \pi(d\lambda)$ with $\pi: \mathcal{F}^2(\mathcal{N}) \rightarrow \mathcal{F}^2(\mathcal{N})/\mathcal{F}^3(\mathcal{N})$ the quotient map.

THEOREM 6.1 [**6**]. *The following diagram commutes:*

$$\begin{array}{ccc}
I(\mathcal{N}) & \xrightarrow{\quad \tilde{d} \quad} & I(\mathcal{N}) \wedge I(\mathcal{N}) \\
 & & \downarrow \approx \; \Psi \wedge \Psi \\
\approx \big\downarrow \Psi & & \mathrm{Hom}\,(\pi_*(X), Q) \wedge \mathrm{Hom}\,(\pi_*(X), Q) \\
 & & \downarrow \approx \; \Phi \\
\mathrm{Hom}\,(\pi_*(X), Q) & \xrightarrow{\; WP^* \;} & \mathrm{Hom}\,(\pi_*(X) \wedge \pi_*(X), Q),
\end{array}$$

where $WP: \pi_*(X) \wedge \pi_*(X) \rightarrow \pi_*(X)$ *is the degree* -1 *homomorphism defined by* $WP(x \wedge y) = [x, y]$ *and* WP^* *is its vector space dual.*

Proof. From the definition of Φ it is easy to check that

$$\Phi(\Psi \wedge \Psi)\tilde{d}(\bar{\lambda})(x \wedge y) = (-1)^{pq}\tilde{K}(d\lambda),$$

where $\lambda \in \mathcal{N}^{p+q-1}$, $x \in \pi_p(X)$, and $y \in \pi_q(X)$. By Theorem 5.4 the latter term is precisely $\langle\langle \bar{\lambda}, [x, y] \rangle\rangle = (WP^* \circ \Psi)(\bar{\lambda})(x \wedge y)$.

Since Ψ and Φ are isomorphisms, Theorem 6.1 implies that the dual Whitehead product homomorphism WP^* can be identified with $\tilde{d}$. Note that $\tilde{d}$ is completely determined by the quadratic terms in the d formula in $\mathcal{N}$.

We next give some general results on higher order Whitehead products which are both useful in the sequel and interesting in themselves. We begin by observing that in Theorem 5.4 the r homotopy elements $x_j \in \pi_{n_j}(X)$ were arbitrarily chosen. Thus it was necessary to assume the higher order Whitehead product set non-empty. However, if one chooses the homotopy elements dual to the generators of the minimal model, then one can prove certain Whitehead product sets are non-empty. We do this next.

Let X be a rational space with minimal model $\mathcal{N}$. Assume that $\eta_1, \eta_2, \ldots, \eta_t, \ldots$ is an ordered set of algebra generators for $\mathcal{N}$ with $\deg \eta_i = n_i$ such that $n_1 \leqq n_2 \leqq \ldots \leqq n_t \leqq \ldots$. Let $z_j \in \pi_{n_j}(X), j = 1, 2, \ldots, t, \ldots$ be dual to the generators. That is, $\langle\langle \bar{\eta}_i, z_j \rangle\rangle = \delta_{ij}$, the Kronecker delta.

LEMMA 6.2. *Suppose for all $i \leqq k$, $d\eta_i \in \mathcal{F}^r(\mathcal{N})$. Let $q = q_{i_1 \ldots i_r} \in Q$ be the coefficient of $\eta_{i_1} \ldots \eta_{i_r}$ in the expansion of $d\eta_k$, where $1 \leqq i_1 \leqq \ldots \leqq i_r$ and $N = n_{i_1} + \ldots + n_{i_r} = n_k + 1$. Then the Whitehead product set $[z_{i_1}, \ldots, z_{i_r}] \subseteq \pi_{N-1}(X)$ is non-empty. Furthermore, if $y \in [z_{i_1}, \ldots, z_{i_r}]$ and we rewrite $\eta_{i_1} \ldots \eta_{i_r}$ as $\eta_{j_1}{}^{t_1} \ldots \eta_{j_s}{}^{t_s}$ with $t_\iota > 0$ and $1 \leqq j_1 < \ldots < j_s$, then*

$$\langle\langle \bar{\eta}_k, y \rangle\rangle = (-1)^{\wedge} q t_1! \ldots t_s!,$$

where $\wedge = \sum_{a<b} n_{i_a} n_{i_b}$.

Proof. By Theorem 6.1 we need only consider the case when $r > 2$. To show the set non-empty, it suffices by [**13**, p. 127] to prove the following: If $z_{k_1}, \ldots, z_{k_s}$ is a proper subsequence of $z_{i_1}, \ldots, z_{i_r}$, then $[z_{k_1}, \ldots, z_{k_s}] = \{0\}$. We do this by induction on s. Let $s = 2$ and let η_i be any generator of dimension $n_{k_1} + n_{k_2} - 1$. Then $\deg \eta_i < \deg \eta_k$. Thus $i < k$ and so $d\eta_i \in \mathcal{F}^r(\mathcal{N})$. Since $2 = s < r$, it follows from Theorem 5.4 that $\langle\langle \bar{\eta}_i, [z_{k_1}, z_{k_2}] \rangle\rangle = 0$. Thus $[z_{k_1}, z_{k_2}] = 0$. Now let $s < r$ and assume the result for $s - 1$. By inductive assumption, $[z_{k_1}, \ldots, z_{k_s}] \neq \emptyset$. We let $x \in [z_{k_1}, \ldots, z_{k_s}]$ and show $x = 0$. If η_i is a generator of $\mathcal{N}$ of dimension $n_{k_1} + \ldots + n_{k_s} - 1$, then as before $d\eta_i \in \mathcal{F}^r(\mathcal{N})$. Since $r > s$, $\langle\langle \bar{\eta}_i, x \rangle\rangle = 0$ by Theorem 5.4. This shows $x = 0$ and completes the induction. Therefore $[z_{i_1}, \ldots, z_{i_r}] \neq \emptyset$.

Now with $d\eta_k \in \mathcal{F}^r(\mathcal{N})$ and $y \in [z_{i_1}, \ldots, z_{i_r}]$, we have

$$\langle\langle \bar{\eta}_k, y \rangle\rangle = (-1)^{\wedge} \tilde{K}(d\eta_k).$$

We write $d\eta_k$ as a linear combination of products of r or more generators. It is not hard to show that each term with r factors which occurs in $d\eta_k$ other than $q\eta_{j_1}{}^{t_1} \ldots \eta_{j_s}{}^{t_s}$ gives rise to a matrix (as in (5.3)), such that K of it is zero. The $\eta_{j_1}{}^{t_1} \ldots \eta_{j_s}{}^{t_s}$ term in $d\eta_k$ yields a matrix A of the form

$$A = \begin{bmatrix} A_1 & & & & 0 \\ & A_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & A_s \end{bmatrix}$$

https://doi.org/10.4153/CJM-1978-083-6 Published online by Cambridge University Press

where A_i is a $t_i \times t_i$ matrix with 1 in each entry. We consider two cases: (i) Some n_{j_i} is odd and $t_i > 1$. Then $\eta_{j_k}{}^{t_i} = 0$ and so $q = 0$. (ii) All other cases. In (ii) it easily follows from Definition 2.6 that $K(A) = t_1! \ldots t_s!$. Thus in either case $\tilde{K}(d\eta_k) = qt_1! \ldots t_s!$. This completes the proof of Lemma 6.2.

Before giving a consequence of this lemma we need a simple definition.

Definition 6.3. We say *all Whitehead products of order r vanish in X* if for any r elements $x_j \in \pi_{n_j}(X)$, $j = 1, 2, \ldots, r$, $[x_1, \ldots, x_r] = \{0\}$. We say *all Whitehead products vanish in X* if all Whitehead products of order r vanish in X for all $r \geqq 2$.

PROPOSITION 6.4. *Let X be a rational space whose minimal model $\mathscr{N}$ has a fixed set of generators. Then all Whitehead products of order less that s vanish in X if and only if $d\mu \in \mathscr{F}^s(\mathscr{N})$ for every element μ of $\mathscr{N}$.*

Proof. Suppose all Whitehead products of order less than s vanish. If suffices to show $d\eta_i \in \mathscr{F}^s(\mathscr{N})$ for every generator η_i of $\mathscr{N}$. Suppose this is not the case, and let η_k be the first generator such that $d\eta_k \notin \mathscr{F}^s(\mathscr{N})$. Thus $d\eta_i \in \mathscr{F}^s(\mathscr{N})$ if $i < k$. Let r be the largest integer such that $d\eta_k \in \mathscr{F}^r(\mathscr{N})$. Therefore $2 \leqq r < s$. Hence we have $d\eta_i \in \mathscr{F}^r(\mathscr{N})$ for all $i \leqq k$. Choose a term $q\eta_{i_1} \ldots \eta_{i_r}$ in the expansion of $d\eta_k$ with $q \neq 0$ and $1 \leqq i_1 \leqq \ldots \leqq i_r$. By hypothesis $[z_{i_1}, \ldots, z_{i_r}] = \{0\}$, and so Lemma 6.2 implies

$$0 = \langle\langle \bar{\eta}_k, 0\rangle\rangle = (-1)^{\wedge} ql$$

for some non-zero integer l. Thus $q = 0$, which is a contradiction. Therefore $d\eta_k \in \mathscr{F}^s(\mathscr{N})$.

We now prove the opposite implication. Let $x_j \in \pi_{n_j}(X)$, $j = 1, \ldots, r$, be r elements with $r < s$. By induction we may assume all Whitehead products of order $< r$ vanish. Therefore by [**13**, p. 127] there is an element $x \in [x_1, \ldots, x_r] \subseteq \pi_{N-1}(X)$. By Theorem 5.4, for any $\mu \in \mathscr{N}^{N-1}$,

$$\langle\langle \bar{\mu}, x\rangle\rangle = (-1)^{\wedge}\tilde{K}(d\mu) = 0$$

since $d\mu \in \mathscr{F}^s(\mathscr{N})$ and $s > r$. Thus $x = 0$. This completes the proof.

As a consequence we obtain the following corollary which, though it may be known, we have not found in the literature. It answers a question in Porter's thesis [**12**, p. 51] for rational spaces.

COROLLARY 6.5. *Let X be a rational space in which all Whitehead products of order $< s$ vanish. Then any* sth *order Whitehead product set in X is non-empty and consists of a single element.*

Proof. Let $x_j \in \pi_{n_j}(X)$ be any s homotopy elements, $j = 1, \ldots, s$. As before the set $[x_1, \ldots, x_s]$ is non-empty since lower order Whitehead products vanish. Now let $x, y \in [x_1, \ldots, x_s]$ and let μ be any element of $\mathscr{N}^{N-1}$, $N = \sum n_j$. Because $d\mu \in \mathscr{F}^s(\mathscr{N})$ by Proposition 6.4, we have by Theorem 5.4 that

$$\langle\langle \bar{\mu}, x\rangle\rangle = (-1)^{\wedge}\tilde{K}(d\mu) = \langle\langle \bar{\mu}, y\rangle\rangle.$$

Since this is true for all $\mu \in \mathscr{N}^{N-1}$, $x = y$. Thus $[x_1, \ldots, x_s]$ consists of a single element.

We now turn to calculating some higher order Whitehead products in two stage Postnikov systems. These propositions are included more to illustrate the computational possibilities of Theorem 5.4 than to present the most general results.

Suppose X is rational space with $\pi_i(X) = 0$ when $i < n$ and $n < i < kn - 1$. Let $\mathscr{N}$ be the minimal algebra of X. Denote the generators of $\mathscr{N}(kn - 1)$ in dimension n by $\alpha_1, \alpha_2, \ldots, \alpha_r$ and the generators in dimension $kn - 1$ by $\beta_1, \beta_2, \ldots, \beta_t$. Let $x_1, x_2, \ldots, x_r \in \pi_n(X)$ and $y_1, y_2, \ldots, y_t \in \pi_{kn-1}(X)$ be dual homotopy elements. Then $d\alpha_1 = 0$, $i = 1, \ldots, r$ and $d\beta_j = p_j(\alpha_1, \ldots, \alpha_r)$, $j = 1, \ldots, t$, where each p_j is a homogeneous polynomial of degree k in r variables with rational coefficients. It follows from Corollary 6.5 that all kth order Whitehead product sets in $\pi_{kn-1}(X)$ are non-empty and consist of a single element.

Our first computation concerns a kth order Whitehead product where all the homotopy elements are the same (Cf. [**1**, § 4]).

PROPOSITION 6.6. *Let $x \in \pi_n(X)$ and let y be the* kth *order Whitehead product element* $[x, x, \ldots, x]$.

(i) *If n is odd, then $y = 0$.*

(ii) *If n is even, then*

$$y = k! \sum_{j=1}^{t} p_j(\langle\langle \bar{\alpha}_1, x\rangle\rangle, \langle\langle \bar{\alpha}_2, x\rangle\rangle, \ldots, \langle\langle \bar{\alpha}_r, x\rangle\rangle) y_j.$$

Proof. Since $\bar{\beta}_1, \ldots, \bar{\beta}_t$ form a basis for $I^{kn-1}(\mathscr{N})$ dual to $y_1, \ldots, y_t$, $y = \sum_{j=1}^{t} \langle\langle \bar{\beta}_j, y\rangle\rangle y_j$. Hence it suffices to compute the rational numbers $\langle\langle \bar{\beta}_j, y\rangle\rangle$. Let $p_j(\alpha_1, \ldots, \alpha_r) = \sum_I q_I \alpha_I$, where $I = (i_1, \ldots, i_k)$ is a multi-index with $1 \leqq i_1 \leqq \ldots \leqq i_k \leqq r$, $\alpha_I = \alpha_{i_1} \ldots \alpha_{i_k}$, $q_I \in Q$. The matrix A_I corresponding to the term α_I in $p_j(\alpha_1, \ldots, \alpha_r)$ will have pqth entry $\langle\langle \bar{\alpha}_{i_p}, x\rangle\rangle$, $p = 1, \ldots, k$. Thus all the columns of each A_I will be equal. When n is odd, $K(A_I) = \det(A_I) = 0$, and so

$$\langle\langle \beta_j, y\rangle\rangle = \pm\tilde{K}(p_j(\alpha_1, \ldots, \alpha_r)) = \sum_I q_I K(A_I) = 0$$

for all j. When n is even,

$$\begin{aligned} K(A_I) &= \sum_{\sigma \in S_k} \langle\langle \bar{\alpha}_{i_1}, x\rangle\rangle \ldots \langle\langle \bar{\alpha}_{i_k}, x\rangle\rangle \\ &= k! \langle\langle \bar{\alpha}_{i_1}, x\rangle\rangle \ldots \langle\langle \bar{\alpha}_{i_k}, x\rangle\rangle. \end{aligned}$$

Thus, in this case,

$$\begin{aligned} \langle\langle \bar{\beta}_j, y\rangle\rangle &= \tilde{K}(p_j(\alpha_1, \ldots, \alpha_r)) \\ &= \sum_I q_I K(A_I) \\ &= k! p_j(\langle\langle \bar{\alpha}_1, x\rangle\rangle, \ldots, \langle\langle \bar{\alpha}_r, x\rangle\rangle). \end{aligned}$$

This completes the proof of the proposition.

Let X be the same space as described above, but now consider the kth order Whitehead Product element

$$y = [\underbrace{x_{j_1}, \ldots, x_{j_1}}_{t_1}, \underbrace{x_{j_2}, \ldots, x_{j_2}}_{t_2}, \ldots, \underbrace{x_{j_s}, \ldots, x_{j_s}}_{t_s}],$$

where x_{j_1} is repeated t_1 times, x_{j_2} is repeated t_2 times, etc. We assume $1 \leqq j_1 < \ldots < j_s$, each $t_i > 0$ and $t_1 + t_2 + \ldots + t_s = k$. Let a_j be the coefficient of $\alpha_{j_1}{}^{t_1} \ldots \alpha_{j_s}{}^{t_s}$ in $p_j(\alpha_1, \ldots, \alpha_r)$, where $d\beta_j = p_j(\alpha_1, \ldots, \alpha_r)$.

PROPOSITION 6.7. *With the above hypotheses,*
(i) *if n is odd and $t_i > 1$ for some i, then the Whitehead product element $y = 0$;*
(ii) *otherwise the Whitehead product element*

$$y = \pm t_1! t_2! \ldots t_s! \sum_{j=1}^{t} a_j y_j.$$

(The sign is $-$ if n and $k(k-1)/2$ are odd, and $+$ in all other cases.)

Proof. As in the proof of Proposition 6.6 we need only compute the rational numbers $\langle\langle \bar{\beta}_j, y\rangle\rangle$. The result now follows from Lemma 6.2.

It is an easy consequence of the proof that a similar result holds for any kth order Whitehead product of the dual generators in $\pi_n(X)$. It should also follow from "anti-commutativity" of higher order Whitehead products.

We can apply Proposition 6.6 to the localization of complex and quaternionic projective spaces, $CP_\emptyset(m-1)$ and $HP_\emptyset(m-1)$. The minimal model of $CP_\emptyset(m-1)$ is generated by elements α in dimension 2 and β in dimension $2m-1$, with $d\alpha = 0$ and $d\beta = \alpha^m$. Then if $x \in \pi_2(CP_\emptyset(m-1))$ is dual to $\bar{\alpha}$ and $y \in \pi_{2m-1}(CP_\emptyset(m-1))$ is dual to $\bar{\beta}$, Proposition 6.6 implies that the mth order Whitehead product $[x, \ldots, x] = m!y$. This agrees with a similar result of Porter [**14**] for $CP(m-1)$.

Likewise, it follows that in $HP_\emptyset(m-1)$ there are homotopy elements $x \in \pi_4(HP_\emptyset(m-1))$ and $y \in \pi_{4m-1}(HP_\emptyset(m-1))$ with the mth order Whitehead product $[x, \ldots, x] = m!y$. For $HP(m-1)$ however, Barry has shown [**4**, p. 24] that if $x' \in \pi_4(HP(m-1))$ is a generator, then the mth order Whitehead product $[x', \ldots, x']$ is the empty set. He does show [**4**, p. 17] that there is an integer s for which $[sx', \ldots, sx']$ is non-empty and equals $s^m m!$ times the Hopf map plus an element of finite order. This agrees with the calculation of $[sx, \ldots, sx]$ in $\pi_{4m-1}(HP_\emptyset(m-1))$ using Proposition 6.6. This gives an example of a space Y and homotopy elements $x_i \in \pi_{n_i}(Y)$, $i = 1, \ldots, r$, for which $[x_1, \ldots, x_r] \subseteq \pi_{N-1}(Y)$ is empty, but $[e_\#(x_1), \ldots, e_\#(x_r)] \subseteq \pi_{N-1}(Y_\emptyset)$ is non-empty.

For the remainder of the paper we shall have occasion to consider both topological spaces of finite type (see the beginning of § 3) and their rationalizations. We shall consistently denote a space of finite type by Y and the localization of such a space by X.

The computations of Whitehead products, like those in Propositions 6.6 and 6.7, for the rationalization of a space can yield information about Whitehead products in the space itself. As an illustrative example, we mention the following proposition.

PROPOSITION 6.8. *Let Y be a space with minimal model $\mathcal{N}$ (with a fixed set of generators). Suppose $x_1, \ldots, x_r$ are homotopy elements in $\pi_*(Y)$ such that $[x_1, \ldots, x_r] \neq \emptyset$. If there is an element λ in $\mathcal{N}$ with $d\lambda \in \mathcal{F}^r(\mathcal{N})$ and $\tilde{K}(d\lambda) \neq 0$, then all elements of $[x_1, \ldots, x_r]$ have infinite order.*

We conclude this section with some results on H-spaces and higher order Whitehead products.

PROPOSITION 6.9. *If X is a rational space, then X is an H-space if and only if all Whitehead products vanish in X.*

Proof. It is proved in [**13**, p. 126] that all Whitehead products vanish in an H-space. Thus we prove the reverse implication. Suppose all Whitehead products vanish in X. We show the differential d of the minimal model $\mathcal{N}$ of X is zero. For every element μ of $\mathcal{N}$, Proposition 6.4 implies that $d\mu \in \mathcal{F}^s(\mathcal{N})$ for all $s \geqq 1$. Thus $d\mu = 0$. We finish the proof by showing that $d = 0$ implies that X is an H-space. We have that $H^*(X) = H^*(\mathcal{N}) = \mathcal{N}$, a free algebra. Hence $H^*(X)$ has the cohomology of an appropriate product of Eilenberg-MacLane spaces $K(Q, n)$. From this it easily follows that there is a map of X into this product which induces an isomorphism of cohomology groups. Consequently this map is a homotopy equivalence. Therefore X is an H-space. This completes the proof.

We conclude the paper by proving an analogue of Proposition 6.9 for topological spaces. The following is a generalization of [**3**, Satz] from spaces of the homotopy type of finite CW-complexes to spaces of finite type.

PROPOSITION 6.10. *Let Y be a space of finite type. Then $Y_\emptyset$ is an H-space if and only if every element in every higher order Whitehead product set in Y has finite order.*

Proof. It suffices by Proposition 6.9 to establish the equivalence of the following two assertions:

(i) All Whitehead products in $Y_\emptyset$ vanish.

(ii) Every element in every higher order Whitehead product set in Y has finite order.

(i) $\Rightarrow$ (ii): Let $e: Y \rightarrow Y_\emptyset$ be the localization map and let $x \in [x_1, \ldots, x_r] \subseteq \pi_{N-1}(Y)$. Then

$$e_\#(x) \in [e_\#(x_1), \ldots, e_\#(x_r)] \subseteq \pi_{N-1}(Y_\emptyset),$$

and so $e_\#(x) = 0$. Thus x is in the kernel of $e_\#$, and hence x has finite order.

(ii) $\Rightarrow$ (i): We assume (ii) and prove by induction on r that every Whitehead product of order r in $Y_\emptyset$ vanishes. For $r = 2$, consider $[x_1, x_2] \in \pi_{N-1}(Y_\emptyset)$. Then $M_i x_i = e_\#(z_i)$ for integers M_i and elements z_i in the homotopy groups of Y. Thus

$$M_1M_2[x_1, x_2] = [M_1x_1, M_2x_2] = [e_\#(z_1), e_\#(z_2)] = e_\#[z_1, z_2].$$

But $[z_1, z_2]$ has finite order and hence so does $[x_1, x_2]$. Therefore $[x_1, x_2] = 0$. Now assume that (i) holds for all Whitehead products of order $< r$. Let $x \in [x_1, \ldots, x_r] \subseteq \pi_{N-1}(Y_\emptyset)$ be an rth order Whitehead product element. Then $M_ix_i = e_\#(z_i)$ for integers M_i and homotopy elements z_i. Unfortunately the set $[z_1, \ldots, z_r] \subseteq \pi_{N-1}(Y)$ may be empty. However, since all Whitehead products in Y have finite order, a result of [**3**, § 3b] asserts that there exists an integer M such that $0 \in [Mz_1, \ldots, Mz_r]$. Hence

$$0 = e_\#(0) \in [e_\#(MZ_1), \ldots, e_\#(Mz_r)] = [MM_1x_1, \ldots, MM_rx_r].$$

But $x \in [x_1, \ldots, x_r]$, and so it easily follows that $M^rM_1 \ldots M_rx$ is also in $[MM_1x_1, \ldots, MM_rx_r]$. By Corollary 6.5, $M^rM_1 \ldots M_rx = 0$. Therefore $x = 0$, and hence $[x_1, \ldots, x_r] = \{0\}$. This ends the inductive argument, establishes (i), and completes the proof.

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