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PROJECTIVELY AND AFFINELY INVARIANT PDES ON HYPERSURFACES

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Abstract In *Communications in Contemporary Mathematics* **24** 3, (2022), the authors have developed a method for constructing G-invariant partial differential equations (PDEs) imposed on hypersurfaces of an $(n+1)$-dimensional homogeneous space G/H, under mild assumptions on the Lie group G. In the present paper, the method is applied to the case when $G = \mathsf{PGL}(n+1)$ (respectively, $G = \mathsf{Aff}(n+1)$) and the homogeneous space G/H is the $(n+1)$-dimensional projective $\mathbb{P}^{n+1}$ (respectively, affine $\mathbb{A}^{n+1}$) space, respectively. The main result of the paper is that projectively or affinely invariant PDEs with n independent and one unknown variables are in one-to-one correspondence with invariant hypersurfaces of the space of *trace-free cubic forms* in n variables with respect to the group $\mathsf{CO}(d, n-d)$ of conformal transformations of $\mathbb{R}^{d,n-d}$.

Keywords: Homogeneous manifolds; Lie symmetries of PDEs; G-invariant PDEs; Jet spaces

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1. Introduction

In this paper, we go on constructing G-invariant partial differential equations (PDEs) in one unknown variable defined on $(n+1)$-dimensional G-homogeneous manifolds, following the general theoretical scheme developed by the authors in [1]: there, the cases when G is either the Euclidean $\mathsf{SE}(n+1)$ or the conformal group $\mathsf{CO}(n+1)$ were treated: here, we will deal with the cases when G is either the projective $\mathsf{PGL}(n+1)$ or the affine group $\mathsf{Aff}(n+1)$.

The reason why we treat these last two cases together in a separate paper is that, unlike the two before, they give rise to *third-order* invariant PDEs; in particular, this casts an important bridge with the differential geometry of affine hypersurfaces and, in particular, with the Fubini–Pick invariant. On this concern, see, e.g., [2, § 2.2], [4, § 1], [5, § 3.5], [9], as well as the original work of Blaschke [3].

The vanishing of this invariant defines a G-invariant third-order PDE that can be constructed, according to the general scheme developed in [1], by a suitable choice of a *fiducial hypersurface* of order 3. In view of the tight relationship between the affine and the projective case (see also [8]), we will state a result concerning both in § 5; technical computations concerning the projective case, that is when $\mathbb{P}^{n+1}$ is regarded as homogeneous space of $\mathsf{SL}(n+2)$, will be carried out in § 3. Analogous computations for the affine case, that is when $\mathbb{A}^{n+1}$ is regarded as homogeneous space of the affine group $\mathsf{Aff}(n+1)$, which can be thought of as a 'restriction' of the projective case, will be carried out in § 3.

The main result is Theorem 5.1; each projectively or affinely invariant PDE imposed on hypersurfaces of the $(n+1)$-dimensional projective or affine space is uniquely given by a $\mathsf{CO}(d, n-d)$-invariant hypersurface of the space of *trace-free cubic forms* in n variables, were symbol $\mathsf{CO}(d, n-d)$ denotes the Lie group of conformal transformations of the space $\mathbb{R}^n$, equipped with a metric of signature $(d, n-d)$.

A local coordinate description for the $\mathsf{Aff}(n+1)$-case, which obviously works for the $\mathsf{PGL}(n+1)$-case as well, will be given in § 5.2, whereas in the last § 5.4, we focus on the case $n=2$.

1.1. Notations and conventions

The symmetric product will be denoted by $\odot$, and the symmetric ℓ-power of a vector space V will be denoted by $S^\ell(V)$. If $f : M \to N$ is a differentiable map, then pull-back via f of a bundle $\pi : E \to N$ is denoted by $f^*(E)$. Symbol $\mathbb{R}^\times$ denotes the multiplicative group of real numbers.

2. A general construction of G-invariant PDEs on a homogeneous manifold $M = G/H$

We will review here, without proofs, the main definitions and results, as well as all the necessary preliminary material, contained in [1, § 2 and 3]. Throughout this section, $M = G/H$ will be an $(n+1)$-dimensional homogeneous manifold and $S \subset M$ an embedded hypersurface of M; in § 3 and 4, M will be either the projective space $\mathbb{P}^{n+1}$ or the affine space $\mathbb{A}^{n+1}$, respectively.

2.1. Preliminary definitions

Locally, in an appropriate local chart

$$(u, \boldsymbol{x}) = (u, x^1, \dots, x^n) \tag{1}$$

of M, the hypersurface S can be described by an equation $u = f(\boldsymbol{x}) = f(x^1, \dots, x^n)$, where f is a smooth function of the variables $x^1, \dots, x^n$, that we refer to as the *independent* variables, to distinguish them from the remaining coordinate u, that is the

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dependent one.[1] We say that such a chart is *admissible* for S or, equivalently, that the hypersurface S is (locally) admissible for the chart $(u, \boldsymbol{x})$. We denote by $S_f = S$ the graph of f:

$$S_f := \{(f(\boldsymbol{x}), \boldsymbol{x})\} = \{u = f(\boldsymbol{x})\} .$$

Given two hypersurfaces S_1 and S_2 through a common point $\boldsymbol{p}$, one can always choose a chart $(u, \boldsymbol{x})$ about $\boldsymbol{p}$ that is admissible for both: $S_1 = S_{f_1}$, $S_2 = S_{f_2}$.

Definition 2.1. *Two hypersurfaces S_{f_1}, S_{f_2} passing through a common point $\boldsymbol{p} = (u, \boldsymbol{x})$ are called ℓ-equivalent at $\boldsymbol{p}$ if the Taylor expansions of f_1 and f_2, in a chart admissible for both, coincide at $\boldsymbol{x}$ up to order ℓ. The class of ℓ-equivalent hypersurfaces to a given hypersurface S at the point $\boldsymbol{p}$ is denoted by $[S]^\ell_{\boldsymbol{p}}$, and the union*

$$J^\ell(n, M) := \bigcup_{\boldsymbol{p}\in M} \{[S]^\ell_{\boldsymbol{p}} \mid S \text{ is a hypersurface of } M \text{ passing through } \boldsymbol{p}\}$$

of all these equivalence classes is the space of ℓ-jets of hypersurfaces of M.

Note that $J^1(n, M) = \mathbb{P}T^*M$, that is the Grassmanian bundle $Gr_n(TM)$ of tangent n-planes to the $(n + 1)$-dimensional manifold M. From now on, when there is no risk of confusion, we let

$$J^\ell := J^\ell(n, M) .$$

The natural projections

$$\pi_{\ell,m} : J^\ell \longrightarrow J^m , \quad [S]^\ell_{\boldsymbol{p}} \longmapsto [S]^m_{\boldsymbol{p}} , \quad \ell > m ,$$

define a tower of bundles

$$\cdots \longrightarrow J^\ell \longrightarrow J^{\ell-1} \longrightarrow \cdots \longrightarrow J^1 = \mathbb{P}T^*M \longrightarrow J^0 = M .$$

It is well known that $\pi_{\ell,\ell-1}$ are affine bundle for $\ell \geq 2$. For any $a^m \in J^m$, the fibre of $\pi_{\ell,m}$ over a^m will be denoted by the symbol

$$J^\ell_{a^m} := \pi^{-1}_{\ell,m}(a^m) .$$

Definition 2.2. *A system of m PDEs of order k is an m-codimensional submanifold $\mathcal{E} \subset J^k$. A solution of the system $\mathcal{E}$ is a hypersurface $S \subset M$ such that $S^{(k)} \subset \mathcal{E}$.*

[1] A reader who is familiar with the standard literature about jet spaces may have noticed that we reversed the order of $\boldsymbol{x}$ and u; this choice will be more convenient for us as the coordinate u will play the role of the '0^{th} coordinate'.

2.2. Assumptions on the Lie group G

Before introducing the conditions, the Lie group G will have to fulfill (see § 2.2.2) in order to make Theorem 1.1 work, we recall some basic facts about the affine group that will help understand the meaning of these conditions.

2.2.1. *Affine groups and their subgroups of affine type*

Let V be a vector space, treated as an affine space, then the group $\mathsf{Aff}(V)$ of affine transformations of V fits into the short exact sequence of groups:

$$0 \longrightarrow V \xrightarrow{T} \mathsf{Aff}(V) \xrightarrow{L} \mathsf{GL}(V) \longrightarrow 0. \tag{2}$$

The monomorphism T maps a vector $v \in V$ into the corresponding parallel translation T_v: one has then a canonical normal subgroup T_V, made of parallel translations, which acts on V in a simply transitive way.

The action of T_V defines even an absolute parallelism on V, i.e., it allows to canonically identify the tangent space T_vV at an arbitrary point $v \in V$ of the *affine* space V, with the *vector* space V; in particular, if an origin $o \in V$ is chosen, then the differential

$$L(g) := d_o g : T_oV \longrightarrow T_{g\cdot o}V$$

of $g \in \mathsf{Aff}(V)$ at o can be regarded as an isomorphism of V, that is, as an element of $\mathsf{GL}(V)$. This explains the rightmost arrow of (2) and allows to regard $\mathsf{GL}(V)$ as the *linear group of the affine group*, that is, as the subgroup $\mathsf{Aff}(V)_o = \mathsf{GL}(T_oV)$ of the group $\mathsf{Aff}(V)$ that stabilizes the origin $o \in V$; this leads to the semidirect decomposition

$$\mathsf{Aff}(V) = T_V \rtimes \mathsf{GL}(V) \tag{3}$$

of the affine group $\mathsf{Aff}(V)$, associated with the origin $o \in V$.

If now a subgroup $H \subset \mathsf{Aff}(V)$ is given, decomposition (3) does not need to descend to H, in the sense that the sequence

$$0 \longrightarrow T_W := T^{-1}(H) \xrightarrow{T} H \xrightarrow{L} L_H := L(H) \longrightarrow 0 \tag{4}$$

may be still exact but not split. This remark motivates the following definition.

Definition 2.3. *We say that a subgroup $H \subset \mathsf{Aff}(V)$ is of affine type if H admits a semidirect decomposition*

$$H = T_W \rtimes L_H \tag{5}$$

for some $o \in V$. The subgroup $L_H = H_o$ is called the linear subgroup of H, whereas T_W is its subgroup of translations.

In condition (A2), we shall require that $\tau(H^{(k-1)})$ be a subgroup of affine type; indeed, as a direct consequence of Definition 2.3, if H is a subgroup of affine type of $\mathsf{Aff}(V)$, then the orbit $H \cdot o$ of o coincides with the affine subspace W of V.

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Let $H = T_W \rtimes L_H$ be a subgroup of affine type, where $L_H = H_o$ denotes its linear subgroup, and let us fix a complementary subspace U to W in V, then any $h \in H$ can be decomposed into a product

$$h = T_h \cdot L_h \,. \tag{6}$$

Therefore, in terms of the decomposition $V = U + W$, the action of the linear part L_h takes the form

$$L_h = \begin{pmatrix} * & * \\ 0 & \overline{L}_h \end{pmatrix} . \tag{7}$$

We let $\overline{L}_H := \{\overline{L}_h \mid h \in H\}$.

Lemma 2.1. *Let $H \subset \mathsf{Aff}(V)$ be a subgroup of affine type. Then, there exists a one-to-one correspondence between $\overline{L}_H$-invariant hypersurfaces $\overline{\Sigma} \subset \overline{V} = V/W$ and (cylindrical) H-invariant hypersurfaces $\Sigma = W + \overline{\Sigma}$ in V.*

Proof. Let $\pi : V \to \overline{V} = V/W$ be the projection. Then, if $\overline{\Sigma}$ is an L_H-invariant hypersurface in $\overline{V}$, then $\Sigma := \pi^{-1}(\bar{\Sigma})$ is an H-invariant hypersurface in V, see also [1, Lemma 3.1]. □

2.2.2. k-*admissible homogeneous manifolds*

In what follows, unless otherwise specified, o is a fixed point of $M = G/H$ (an 'origin'), so that $M = G \cdot o$, and o^ℓ is a point of J^ℓ projecting onto o. This allows us to consider, $\forall\, \ell \geq 2$, the fibre $J^\ell_{o^{\ell-1}}$ as a vector space with the origin o^ℓ playing the role of zero vector. The group G acts naturally on each ℓ-jet space J^ℓ:

$$\begin{aligned} g : J^\ell &\longrightarrow J^\ell \,, \\ o^\ell = [S]^\ell_o &\longrightarrow g \cdot o^\ell := [g(S)]^\ell_{g(o)} \,, \end{aligned}$$

with $o \in S$, for all $g \in G$.

Definition 2.4. *The system $\mathcal{E}$ is called G-invariant if $G \cdot \mathcal{E} = \mathcal{E}$.*

We denote by $H^{(\ell)}$ the stability subgroup G_{o^ℓ} in G of the point o^ℓ:

$$H^{(\ell)} := G_{o^\ell} \,.$$

We are going to assume that there exists a point $o^k \in J^k$, with $k \geq 2$, such that:
(A1) the orbit

$$\check{J}^{k-1} := G \cdot o^{k-1} = G/H^{(k-1)} \subset J^{k-1}$$

through the projection $o^{k-1} \in J^{k-1}$ of o^k is open;

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(A2) the orbit

$$W^k := \tau(H^{(k-1)}) \cdot o^k \subset J^k_{o^{k-1}} \tag{8}$$

of the natural affine action

$$\tau : H^{(k-1)} \to \mathsf{Aff}(J^k_{o^{k-1}}) \tag{9}$$

in the fibre $J^k_{o^{k-1}}$ is an affine subspace and the group $\tau(H^{(k-1)})$ is a subgroup of affine type, i.e.,

$$\tau(H^{(k-1)}) = T_{W^k} \rtimes L_{H^{(k-1)}}\,, \tag{10}$$

where $L_{H^{(k-1)}}$ is the stabilizer of o^k, see Definition 2.3.

Assumption (A2) implies that there is a point $o^k \in J^k_{o^{k-1}}$ such that the restriction of the affine bundle $\pi_{k,k-1} : J^k \to J^{k-1}$ to the orbit $G \cdot o^k$ is an affine subbundle of $\pi_{k,k-1}$ (over the base $\check{J}^{k-1}$).

Definition 2.5. *A homogeneous manifold $M = G/H$ is called k-admissible for $k \geq 2$ if assumptions (A1) and (A2) are satisfied.*

The problem of classifying all G-invariant PDEs $\mathcal{E} \subset J^k$ on a given $(n+1)$-dimensional manifold M acted upon by a Lie group G will be made more workable by assuming M to be a G-homogeneous manifold of a particular kind, namely a k-admissible one.

2.3. Natural bundles on jet spaces

2.3.2. The lift of hypersurfaces of M *to* J^ℓ

The space J^ℓ has a natural structure of smooth manifold: one way to see this is to extend the local coordinate system (1) on M to a coordinate system

$$(u, \boldsymbol{x}, \dots, u_i, \dots, u_{ij}, \dots, u_{i_1\cdots i_l}, \dots) = (u, x^1, \dots, x^n, \dots, u_i, \dots, u_{ij}, \dots, u_{i_1\cdots i_l}, \dots) \tag{11}$$

on J^ℓ, where each coordinate function[2] $u_{i_1\cdots i_k}$, with $k \leq \ell$, is unambiguously defined by

$$u_{i_1\cdots i_k}\left([S_f]^\ell_{\boldsymbol{p}}\right) = \partial^k_{i_1\cdots i_k} f(\boldsymbol{x})\,, \quad \boldsymbol{p} = (u, \boldsymbol{x})\,, \quad k \leq \ell\,. \tag{12}$$

In formula (12), the symbol ∂_i denotes the partial derivative ∂_{x^i}, for $i = 1, \dots, n$; we recall that the hypersurface $S = S_f$ is the graph of the function $u = f(\boldsymbol{x})$ and, as such, it is admissible for the chart $(u, \boldsymbol{x})$.

[2] The $u_{i_1\cdots i_k}$'s are symmetric in the lower indices.

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The ℓ-lift of S is defined by

$$S^{(\ell)} := \{[S]^\ell_{\boldsymbol{p}} \mid \boldsymbol{p} \in S\}\,.$$

It is an n-dimensional submanifold of J^ℓ. If $S = S_f$ is the graph of $u = f(\boldsymbol{x})$, then $S^{(\ell)}_f$ can be naturally parametrized as follows:[3]

$$\left(u = f(\boldsymbol{x}), \boldsymbol{x}, \dots u_i = \frac{\partial f}{\partial x^i}(\boldsymbol{x}), \dots u_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}(\boldsymbol{x}), \dots \right)\,.$$

1.3.2. The tautological bundle and the higher order contact distribution on J^ℓ

Lemma 2.2. *Any point $a^\ell = [S]^\ell_{\boldsymbol{p}} \in J^\ell$ canonically defines the n-dimensional subspace*

$$T_{a^{\ell-1}} S^{(\ell-1)} \subset T_{a^{\ell-1}} J^{\ell-1}\,, \quad a^{\ell-1} := \pi_{\ell,\ell-1}(a^\ell)\,. \tag{13}$$

Definition 2.6. *The tautological rank-n vector bundle $\mathcal{T}^\ell \subset \pi^*_{\ell,\ell-1}(TJ^{\ell-1})$ is the bundle over J^ℓ whose fibre over the point a^ℓ is given by (13), i.e.,*

$$\mathcal{T}^\ell = \left\{ (a^\ell, v) \in J^\ell \times TJ^{\ell-1} \mid v \in T_{a^{\ell-1}} S^{(\ell-1)} \right\}\,.$$

The (truncated) total derivatives

$$D^{(\ell)}_i := \partial_{x^i} + \sum_{k=1}^{\ell} \sum_{j_1 \leq \cdots \leq j_{k-1}} u_{j_1 \cdots j_{k-1}\, i}\, \partial_{u_{j_1 \cdots j_{k-1}}}\,, \quad i = 1 \dots n\,, \tag{14}$$

constitute a local basis of the bundle $\mathcal{T}^\ell$.

The pre-image $\mathcal{C}^\ell := (d\pi_{\ell,\ell-1})^{-1}\mathcal{T}^\ell$ of the tautological bundle on J^ℓ, via the differential $d\pi_{\ell,\ell-1}$ of the canonical projection $\pi_{\ell,\ell-1}$, is a distribution on J^ℓ.

Definition 2.7. *$\mathcal{C}^\ell$ is called the ℓ^{th} order contact structure or the Cartan distribution (on J^ℓ).*

We will also need the *vertical subbundle* $T^v J^\ell := \ker(d\pi_{\ell,\ell-1})$ of TJ^ℓ. The distribution $\mathcal{C}^\ell$ has been called the 'higher order contact structure' [6, 7, 14] because, for $\ell = 1$, if (u, x^i, u_i) is a chart on J^1, then $\mathcal{C} := \mathcal{C}^1 = \ker(\theta)$, where $\mathrm{d}\theta = \mathrm{d}u - u_i \mathrm{d}x^i$, is the contact distribution.

[3] We stress once again that a switch has occurred between the first and the second entry, with respect to a more standard literature.

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2.3.3. The affine structure of the bundles $J^\ell \to J^{\ell-1}$ for $\ell \geq 2$

According to Definition 2.6, the tautological bundle $\mathcal{T} := \mathcal{T}^1$ is the vector bundle over J^1 defined by

$$\mathcal{T}_{[S]^1_{\boldsymbol{p}}} := \mathcal{T}^1_{[S]^1_{\boldsymbol{p}}} = T_{\boldsymbol{p}} S\,.$$

Definition 2.8. *The normal bundle $\mathcal{N}$ is the line bundle*

$$\mathcal{N}_{[S]^1_{\boldsymbol{p}}} := N_{\boldsymbol{p}} S = T_{\boldsymbol{p}} M / T_{\boldsymbol{p}} S$$

over J^1.

Remark 2.1. To simplify notations, we denote by ∂_u the equivalence class $\partial_u \bmod \mathcal{T}$.

Lemma 2.3 and Proposition 2.1 are both well known (see, for instance, [6, 12]).

Lemma 2.3. *For $\ell \geq 1$, the following vector bundle isomorphism holds:*

$$T^v J^\ell \simeq \pi^*_{\ell,1}(S^\ell \mathcal{T}^* \otimes \mathcal{N})\,.$$

Proposition 2.1. *For $\ell \geq 2$, the bundles $J^\ell \to J^{\ell-1}$ are affine bundles modelled by the vector bundles $\pi^*_{\ell-1,1}(S^\ell \mathcal{T}^* \otimes \mathcal{N})$. In particular, once a chart $(u, \boldsymbol{x})$ has been fixed, a choice of a point $[S]^\ell_{\boldsymbol{p}}$ (the origin) defines an identification of $J^\ell_{[S]^{\ell-1}_{\boldsymbol{p}}}$ with $S^\ell T^*_{\boldsymbol{p}} S$.*

2.4. Constructing G-invariant PDEs $\mathcal{E}$

Let $M = G/H = G \cdot o$, $o \in M$, be an $(n+1)$-dimensional homogeneous manifold and recall (see § 2.2.2) that G acts on each jet space $J^\ell = J^\ell(n, M)$. To further simplify the setting, we will assume that $M = G/H$ possess a *fiducial hypersurface* of order k, defined below.

2.4.1. The fiducial hypersurface

Definition 2.9. *Let $S \subset M$ be a hypersurface, such that $S \ni o$. The hypersurface S is called a a fiducial hypersurface (of order k), if S is homogeneous with respect to a subgroup of G, such that (A1) and (A2) of § 2.2.2 are satisfied with $o^k := [S]^k_o$.*

Plainly, if $M = G/H$ admits a fiducial hypersurface of order k, then it is k-admissible as well (see Definition 2.5). Let S be a fiducial hypersurface of order k in the sense of

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Definition 2.9; therefore, for any $\ell \leq k$, we will regard the point

$$o^\ell := [S]^\ell_o \in J^\ell$$

as the origin of J^ℓ. Furthermore, the identification

$$J^\ell_{o^{\ell-1}} = S^\ell(T^*_o S) \otimes N_o S\,,$$

in the case when the fiducial hypersurface S is the graph S_f of a f, reads (see Proposition 2.1):

$$J^\ell_{o^{\ell-1}} = S^\ell(T^*_o S_f)\,. \tag{15}$$

We will use this identification in the sequel.

2.4.2. A general method for constructing G*-invariant PDEs*

We apply now Lemma 2.1 to the subgroup $\tau(H^{(k-1)}) \subset \mathsf{Aff}(J^k_{o^{k-1}})$ of affine type, which eventually leads to [1, Theorem 3.1].

Corollary 2.1. *Let $M = G/H$ be a k-admissible homogeneous manifold. Then, there exists a one-to-one correspondence between $L_{H^{(k-1)}}$-invariant hypersurfaces $\overline{\Sigma} \subset J^k_{o^{k-1}}/W^k$ and (cylindrical) $\tau(H^{(k-1)})$-invariant hypersurfaces $\Sigma = p^{-1}(\overline{\Sigma}) \subset J^k_{o^{k-1}}$, where*

$$p : J^k_{o^{k-1}} \to J^k_{o^{k-1}}/W^k \tag{16}$$

is the natural projection.

The aforementioned main result of [1], that is Theorem 3.1, is a direct consequence of Corollary 2.1 and Lemma 2.4, applied to the bundle $\pi_{k,k-1}$.

Lemma 2.4. *Let $\pi : P \longrightarrow B$ be a bundle. Assume that a Lie group G of automorphisms of π, such that $B = G/H$, acts transitively on B, where H is the stabilizer of a point $o \in B$. Then:*

i) *any H-invariant function F on $P_o := \pi^{-1}(o)$ extends to a G-invariant function $\widehat{F}$ on P (where $\widehat{F}(gy) = F(y)$ for $y \in P_o$ and $g \in G$), and $F \longmapsto \widehat{F}$ is a bijection;*
ii) *any H-invariant hypersurface Σ of the fibre P_o extends to a G-invariant hypersurface $\mathcal{E}_\Sigma := G \cdot \Sigma$ of P, and this gives a bijection between H-invariant hypersurfaces of P_o and G-invariant hypersurfaces of P.*

Proof. See [1, Lemma 3.2]. □

Theorem 2.1. *Let $M = G/H$ be a k-admissible homogeneous manifold (see Definition 2.5). Then, there is a natural one-to-one correspondence between $L_{H^{(k-1)}}$-invariant hypersurfaces $\overline{\Sigma}$ (see also (10)) of $J^k_{o^{k-1}}/W^k$ and G-invariant hypersurfaces $\mathcal{E}_{\overline{\Sigma}} := \mathcal{E}_{p^{-1}(\overline{\Sigma})} = G \cdot p^{-1}(\overline{\Sigma})$ of $J^k = J^k(n, M)$, where p is the natural projection (16).*

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Theorem 2.1 closes the summary of the theory developed by the authors in [1] that is a strategy for constructing G-invariant PDEs imposed on the hypersurfaces of a k-admissible homogeneous manifold $M = G/H$:

(1) calculate the orbit $W^k = \tau(H^{(k-1)}) \cdot o^k$ and decompose $\tau(H^{(k-1)})$ accordingly to (10);
(2) describe $L_{H^{(k-1)}}$-invariant hypersurfaces $\overline{\Sigma} \subset V^k = J^k_{o^{k-1}}/W^k$;
(3) write down the G-invariant equations $\mathcal{E}_{\overline{\Sigma}} = G \cdot p^{-1}(\overline{\Sigma})$ in the coordinates (11).

In § 3, we begin implementing this strategy for the projective space $\mathbb{P}^{n+1}$, whereas in § 4, we will be dealing with the affine space $\mathbb{A}^{n+1}$; the G-invariant PDE itself is obtained, in an unified manner, in § 5.

3. Stabilizers of the $\mathsf{SL}(n+2)$-action on $J^\ell(n, \mathbb{P}^{n+1})$

We consider the linear space $\mathcal{W} := \mathbb{R}^{n+2}$ with the basis

$$\{p, e_1, \dots, e_n, q\},$$

and we let $G = \mathsf{SL}(n+2)$ act naturally on it; therefore, G acts on the projectivization $M := \mathbb{P}\mathcal{W}$ of $\mathcal{W}$. The projective coordinates

$$[u : x^1 : \cdots : x^n : t]$$

on $\mathbb{P}\mathcal{W} = \mathbb{P}^{n+1}$ will be given by the dual coordinates to the basis above. We shall also need a scalar product

$$g = \langle \cdot, \cdot \rangle \tag{17}$$

on $E := \langle e_1, \dots, e_n \rangle$, of signature $d, n-d$. Let $\mathcal{S}_g$ denote the *projective quadric*

$$\mathcal{S}_g := \mathbb{P}\mathcal{W}_0 \,, \tag{18}$$

where $\mathcal{W}_0$ is the null cone of the pseudo-Euclidean metric

$$g_{\mathcal{W}} := g - \mathrm{d}u \odot \mathrm{d}t \,, \tag{19}$$

that is, $\mathcal{W}_0 := \{w \in \mathcal{W} \,|\, g_{\mathcal{W}}(w, w) = 0\}$. In § 5, we shall prove that $\mathcal{S}_g$ is a fiducial hypersurface, see Proposition 5.1.

The point

$$o := [p] = [1 : 0 : \cdots : 0 : 0]$$

clearly belongs to the hypersurface $\mathcal{S}_g$, so that it makes sense to consider

$$o^{(k)} := [\mathcal{S}_g]^k_o$$

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for $k \geq 0$. In particular, the point $o^{(1)} = [\mathcal{S}_g]^1_o$, that is the tangent space $T_o\mathcal{S}_g = T_o(\mathcal{S}_g \cap \mathcal{U}) \in J^1$, in the affine coordinate neighbourhood

$$\mathcal{U} := \{[1 : x^1 : \cdots : x^n : t]\} \tag{20}$$

can be identified with $E = \ker d_o t$: indeed, $\ker d_o(tu - g) = \ker(d_o t - d_o g) = \ker d_o t = E$, because $d_o g = 0$.

Lemma 3.1. *The stabilizing subgroups corresponding to the origins $o^{(k)}$, for $k = 0, 1, 2, 3$, are:*

$$\begin{aligned} H &= \mathbb{R}^{n+1} \rtimes \mathsf{GL}(n+1)\,, \\ H^{(1)} &= \mathbb{R}^{n+1} \rtimes ((\mathbb{R}^n \rtimes \mathsf{GL}(n)) \times \mathbb{R}^\times)\,, \\ H^{(2)} &= (\mathbb{R}^{n+1} \rtimes (\mathbb{R}^n \rtimes \mathsf{O}(d, n-d)) \times \mathbb{R}^\times)\,, \\ H^{(3)} &= \mathbb{R}^{n+1} \rtimes (\mathsf{O}(d, n-d) \times \mathbb{R}^\times)\,. \end{aligned}$$

Proof. An element of G stabilizing the line generated by p is a $(n+2) \times (n+2)$ matrix with determinant one, displaying all zeros in the first column, save for the first entry, that has to be equal to the inverse of the determinant of the rightmost lower $(n+1) \times (n+1)$ block; in other words,

$$H = G_{[p]} = \mathsf{Aff}(E \oplus \mathbb{R}q) = \mathbb{R}^{n+1} \rtimes \mathsf{GL}(n+1)\,. \tag{21}$$

The same can be seen on the infinitesimal level; passing to the Lie algebra $\mathfrak{g}$ of G, we consider the decomposition

$$\mathfrak{g} = \mathfrak{sl}(\mathcal{W}) = \mathfrak{so}(\mathcal{W}) \oplus S^2_0(\mathcal{W})\,, \tag{22}$$

where $\mathfrak{so}(\mathcal{W}) = \mathfrak{so}(d+1, n+1-d)$ is identified with the space of skew-symmetric forms $\Lambda^2\mathcal{W}$ and $S^2_0(\mathcal{W})$ denotes the space of trace-free symmetric forms with respect to $g_\mathcal{W}$, cf. (19); therefore, since $\mathcal{W}$ splits into the sum

$$\mathcal{W} = \mathbb{R}p \oplus (E \oplus \mathbb{R}q)$$

of the $(n+1)$-dimensional space $\mathbb{R}q \oplus E$ and the one-dimensional subspace $\mathbb{R}p$, we obtain the decompositions

$$\mathfrak{so}(\mathcal{W}) = (\mathbb{R}p \wedge (E \oplus \mathbb{R}q)) \oplus \mathfrak{so}(E \oplus \mathbb{R}q)\,, \tag{23}$$

$$S^2_0(\mathcal{W}) = \mathbb{R}p \odot (E \oplus \mathbb{R}p) \oplus \mathbb{R}q \odot (E \oplus \mathbb{R}q) \oplus S^2_0 E \oplus \mathbb{R}(p \odot q - e_0 \otimes e_0), \tag{24}$$

where $e_0 \in E$ is a suitable vector. It is now easy to see that the Lie algebra $\mathfrak{g}_0 = \mathfrak{h}$ of the stabilizer H is given by

$$\mathfrak{h} = \mathfrak{gl}(E \oplus \mathbb{R}q) \quad \oplus \quad (\mathbb{R}p \otimes (E \oplus \mathbb{R}q)^*) = \mathfrak{gl}(E \oplus \mathbb{R}q) \quad \oplus \quad ((\mathbb{R}p)^* \otimes (E \oplus \mathbb{R}q))^*\,. \tag{25}$$

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Since

$$T_oM = T_{[p]}\mathbb{P}\mathcal{W} = (\mathbb{R}p)^* \otimes \frac{\mathcal{W}}{\mathbb{R}p} \simeq (\mathbb{R}p)^* \otimes (E \oplus \mathbb{R}q)\,,$$

we obtain

$$\mathfrak{h} \simeq T_oM \oplus \mathfrak{gl}(T_oM)\,.$$

The last identification allows to rewrite (21) as follows:

$$H = T_oM \rtimes \mathsf{GL}(T_oM)\,,$$

where the factor $\mathsf{GL}(T_oM)$ (respectively, T_oM) is the image (respectively, kernel) of the isotropy representation

$$j : H \longrightarrow \mathsf{GL}(T_oM)\,. \tag{26}$$

In light of what we have found, it is easy to pass from (25) to the Lie algebra $\mathfrak{h}^{(1)}$ of $H^{(1)}$

$$\mathfrak{h}^{(1)} = \mathfrak{h}_{o^{(1)}} = (\mathbb{R}p \otimes (E \oplus \mathbb{R}q)^*) \oplus \mathfrak{gl}(E \oplus \mathbb{R}q)_E\,, \tag{27}$$

where

$$\mathfrak{gl}(E \oplus \mathbb{R}q)_E = \mathfrak{gl}(E) \oplus E \oplus \mathbb{R}q\,, \tag{28}$$

is the subalgebra preserving E. On the level of Lie groups, this means that

$$H^{(1)} = (E \oplus \mathbb{R}q)^* \rtimes ((E \rtimes \mathsf{GL}(E)) \times (\mathbb{R}q)^\times) \simeq \mathbb{R}^{n+1} \rtimes ((\mathbb{R}^n \rtimes \mathsf{GL}(n)) \times \mathbb{R}^\times)\,,$$

or, more intrinsically,

$$H^{(1)} = T_oM \rtimes (\mathsf{Aff}(T_o\mathcal{S}_g) \times \mathbb{R}^\times)\,. \tag{29}$$

We shall show now that the subgroup of (29) that stabilizes $o^{(2)}$ is precisely

$$H^{(2)} = T_oM \rtimes (\mathsf{E}(T_o\mathcal{S}_g) \times \mathbb{R}^\times)\,, \tag{30}$$

where

$$\mathsf{E}(T_o\mathcal{S}_g) = E \rtimes \mathsf{O}(E) = \mathbb{R}^n \rtimes \mathsf{O}(d, n-d)$$

is the group of rigid motions of $E \simeq T_o\mathcal{S}_g$. Note that the isotropy representation (26) tells us that the factor T_oM of $H^{(1)}$ survives in $H^{(2)}$; it is also easy to see that the 'conformal factor' $\mathbb{R}^\times$, since it scales the dependent variable, does not affect the second jet at zero of

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the quadric $\mathcal{S}_g$, see (18); indeed, if we identify second-order jets with quadratic forms (see § 2.3.3), then the second jet at zero of the quadric $\mathcal{S}_g$ is g itself. Similarly, a transformation coming from the $\mathsf{GL}(T_o\mathcal{S}_g)$ component of the group $\mathsf{E}(T_o\mathcal{S}_g)$ preserves $o^{(2)}$ if and only if it preserves g; therefore, it must be an element of $\mathsf{O}(T_o\mathcal{S}_g)$.

In order to finish the proof of (30), it remains to show that the 'translational' component $T_o\mathcal{S}_g$ of $\mathsf{E}(T_o\mathcal{S}_g)$ does not move $o^{(2)}$; we postpone this to the proof of the analogous property in § 4 (see Remark 4.1), together with the proof that the aforementioned component does move $o^{(3)}$, eventually showing that

$$H^{(3)} = T_oM \rtimes (\mathsf{O}(T_o\mathcal{S}_g) \times \mathbb{R}^\times)\,,$$

thus concluding the whole proof. □

Remark 3.1. The structure of $H^{(1)}$ is that of

$$H^{(1)} = \mathcal{H} \rtimes \mathrm{Aut}(\mathcal{H})\,,$$

where

$$\mathrm{Lie}(\mathcal{H}) = E \oplus E^* \oplus \mathbb{R}p\,,$$

with $\mathcal{H}$ being the $(2n+1)$-dimensional Heisenberg group. Indeed, from (27) and (28), it follows that

$$\mathfrak{h}^{(1)} = \mathfrak{h}_{o^{(1)}} = (\mathfrak{gl}(E) \oplus E \oplus \mathbb{R}q) \oplus (\mathbb{R}p \otimes E) \oplus \mathbb{R}p\,,$$

that is,

$$\mathfrak{h}^{(1)} = \mathrm{Lie}(\mathcal{H}) \oplus \mathfrak{gl}(E) \oplus \mathbb{R}q\,. \tag{31}$$

In terms of traceless $(n+2) \times (n+2)$ matrices, an element

$$(a, b^t, \alpha, A, \beta)$$

of the algebra (31) corresponds to the matrix

$$\begin{pmatrix} -\beta - tr(A) & 0 & \alpha \\ 0 & A & b \\ 0 & a^t & \beta \end{pmatrix}.$$

Remark 3.2 The hypersurface $\mathcal{S}_g$ is a homogeneous manifold, namely,

$$\mathcal{S}_g = \mathsf{SO}(\mathcal{W})/(\mathsf{SO}(\mathcal{W}) \cap H^{(1)})\,.$$

It is indeed convenient, before passing to the application of Theorem 2.1, to prove the analogous result for the affine case; after that, the two cases will go on in parallel, due to the fact that the structure of the model fibre of J^3 over J^2 does not feel the topology of the underlying manifold, that has changed from $\mathbb{P}^{n+1}$ to $\mathbb{A}^{n+1}$.

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4. Stabilizers of the $\mathsf{Aff}(n+1)$-action on $J^\ell(n, \mathbb{A}^{n+1})$

By the symbol $\mathbb{A}^{n+1}$, we denote the linear space $\mathbb{R}^{n+1}$, regarded as an affine space. The affine space $\mathbb{A}^{n+1}$ is a manifold with the action of the affine group

$$G = \mathsf{Aff}(n+1) = \mathbb{R}^{n+1} \rtimes \mathsf{GL}(n+1)\,,$$

such that the vector normal subgroup $\mathbb{R}^{n+1}$ acts simply transitively. We fix the standard basis

$$\{e_0, e_1, \ldots, e_n\}$$

of $\mathbb{R}^{n+1} = \mathbb{R}e_0 \oplus E$, and we let

$$(u, \boldsymbol{x}) := (u, x^1, \ldots, x^n)$$

be the corresponding coordinates. We have then the same n-dimensional space E as before, with the same coordinates, but now the $(n+1)$-dimensional underlying manifold is

$$M := \mathbb{A}^{n+1} = \mathbb{R}^{n+1} = \mathbb{R}e_0 \oplus E\,.$$

Since $\mathbb{A}^{n+1}$ still possesses the zero, we set $o := 0 \in \mathbb{A}^{n+1}$. In analogy to (18), we let $\mathcal{S}_g^{\mathrm{aff}}$ be the *quadric*

$$\mathcal{S}_g^{\mathrm{aff}} = \{u = g(\boldsymbol{x}, \boldsymbol{x})\}\,, \tag{32}$$

where g is the same scalar product on E as before, see (17). In § 4, we shall prove that $\mathcal{S}_g^{\mathrm{aff}}$ is a fiducial hypersurface, see Proposition 5.1.

As before, we let $o^{(k)} := [\mathcal{S}_g^{\mathrm{aff}}]_o^k$, for $k \geq 0$, so that the point $o^{(1)} = [\mathcal{S}_g^{\mathrm{aff}}]_o^1$ will be again the tangent space $T_o\mathcal{S}_g^{\mathrm{aff}} \in J^1$, that is the hyperplane $E = \mathbb{R}^n$ of $\mathbb{R}^{n+1}$.

Lemma 4.1. *The stabilizing subgroups of the origins $o^{(k)}$, for $k = 0, 1, 2, 3$, are:*

$$H = \mathsf{GL}(n+1)\,, \tag{33}$$

$$H^{(1)} = (\mathbb{R}^n \rtimes \mathsf{GL}(n)) \times \mathbb{R}^\times\,, \tag{34}$$

$$\begin{aligned} H^{(2)} &= (\mathbb{R}^n \rtimes \mathsf{O}(d, n-d)) \times \mathbb{R}^\times\,, \\ H^{(3)} &= \mathsf{O}(d, n-d) \times \mathbb{R}^\times\,. \end{aligned} \tag{35}$$

Proof. Formula (33) is well known; an affine transformation preserves the zero, that is the origin $o^{(0)} = o = 0$ of $\mathbb{A}^{n+1}$, if and only if it is linear, i.e., an element of $\mathsf{GL}(n+1)$.

Concerning (34), let us note that a $(n+1)\times(n+1)$ non-singular matrix preserves the hyperplane $\mathbb{R}^n$, that is the origin $o^{(1)}$, if and only if it has the form

$$\begin{pmatrix} A & \boldsymbol{w} \\ 0 & \mu \end{pmatrix},$$

where $A \in \mathsf{GL}(n)$, $\boldsymbol{w} \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^\times$. Identity

$$\begin{pmatrix} A & \boldsymbol{w} \\ 0 & \mu \end{pmatrix} = \mu \cdot \begin{pmatrix} \mu^{-1}A & \mu^{-1}\boldsymbol{w} \\ 0 & 1 \end{pmatrix}$$

shows that $\mathrm{Stab}_H(o^{(1)})$ is obtained from the subgroup $\mathbb{R}^n \rtimes \mathsf{GL}(n)$ of matrices of the form

$$\begin{pmatrix} A & \boldsymbol{w} \\ 0 & 1 \end{pmatrix}, \tag{36}$$

by multiplying it by the group $\mathbb{R}^\times$.

To deal with (35), it is convenient to introduce, by a slight abuse of notation, two special elements of $H^{(1)}$, namely

$$M_{A,\mu} := \begin{pmatrix} A & 0 \\ 0 & \mu \end{pmatrix}, \quad \boldsymbol{w} := \begin{pmatrix} I_n & \boldsymbol{w} \\ 0 & 1 \end{pmatrix}.$$

It is worth observing that $M_{A,\mu}$ acts naturally by A on the hyperplane $\mathbb{R}^n$ while rescaling by μ the elements of the complementary line $\mathbb{R}e_0$, whereas the vector $\boldsymbol{w}$ acts on the affine hyperplane $u=1$ by translation; in particular, it 'tilts' the line $\mathbb{R}e_0$ into the line $\mathbb{R}(e_0+\boldsymbol{w})$.

Since

$$\begin{pmatrix} A & \boldsymbol{w} \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} I_n & A^{-1}\boldsymbol{w} \\ 0 & 1 \end{pmatrix},$$

any element of $H^{(1)}$ can be expressed as product of the above special elements:

$$H^{(1)} = \{M_{A,\mu} \cdot \boldsymbol{w} \mid A \in \mathsf{GL}(n)\,, \mu \in \mathbb{R}^\times\,, \boldsymbol{w} \in \mathbb{R}^n\}. \tag{37}$$

Let us pass to the first claim of (35), i.e., to the computation of the stabilizer $\mathrm{Stab}_{H^{(1)}}(o^{(2)})$ of the second-order jet at $0 \in \mathbb{R}^n$ of the quadric hypersurface $\mathcal{S}_g^{\mathrm{aff}} = \{(Q(\boldsymbol{x}), \boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}^n\}$, where

$$Q(\boldsymbol{x}) := g(\boldsymbol{x}, \boldsymbol{x})$$

is the quadratic form associated to the scalar product (17).

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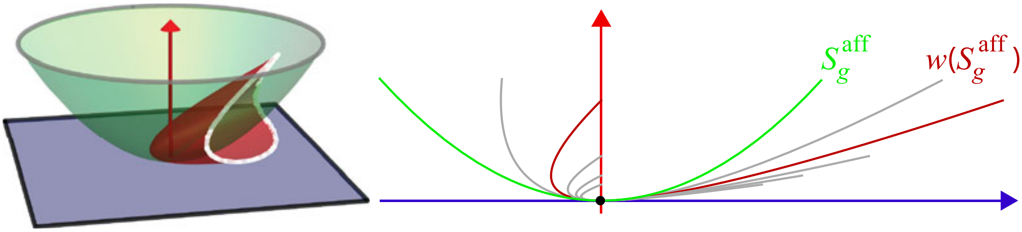

To begin with, we act by a transformation of type $\boldsymbol{w}$ on the hypersurface $\mathcal{S}_g^{\mathrm{aff}}$; it turns out that, even if the resulting hypersurface $\boldsymbol{w}(\mathcal{S}_g^{\mathrm{aff}})$ looks like a 'slanted paraboloid' (see the picture below), the second-order jet at zero of $\boldsymbol{w}(\mathcal{S}_g^{\mathrm{aff}})$ is the same as the original hypersurface $\mathcal{S}_g^{\mathrm{aff}}$.

In order to see this, let us observe that

$$\boldsymbol{w}(\mathcal{S}_g^{\mathrm{aff}}) = \{\boldsymbol{w}\cdot(Q(\boldsymbol{x}),\boldsymbol{x}) \mid \boldsymbol{x}\in\mathbb{R}^n\} = \{Q(\boldsymbol{x}),(\boldsymbol{x}+Q(\boldsymbol{x})\boldsymbol{w}) \mid \boldsymbol{x}\in\mathbb{R}^n\}\,,$$

where the function $\boldsymbol{t}(\boldsymbol{x}) := \boldsymbol{x}+Q(\boldsymbol{x})\boldsymbol{w}$ is a small deformation of the identity in a sufficiently small neighbourhood of zero. As such, $\boldsymbol{t}(\boldsymbol{x})$ will admit a (local) inverse. We claim that

$$\boldsymbol{x}(\boldsymbol{t}) := \boldsymbol{t} - Q(\boldsymbol{t})\boldsymbol{w} \tag{38}$$

approximates the inverse of $\boldsymbol{t}(\boldsymbol{x})$ up to third-order terms. Indeed,

$$\begin{aligned}\boldsymbol{t}(\boldsymbol{x}(\boldsymbol{t})) &= \boldsymbol{t} - Q(\boldsymbol{t})\boldsymbol{w} + Q(\boldsymbol{t}-Q(\boldsymbol{t})\boldsymbol{w})\boldsymbol{w}\\ &= \boldsymbol{t} - Q(\boldsymbol{t})\boldsymbol{w} + (Q(\boldsymbol{t}) - 2Q(\boldsymbol{t})\langle\boldsymbol{w},\boldsymbol{t}\rangle + Q^2(\boldsymbol{t})Q(\boldsymbol{w}))\boldsymbol{w}\\ &= \boldsymbol{t} - (2Q(\boldsymbol{t})\langle\boldsymbol{w},\boldsymbol{t}\rangle + Q^2(\boldsymbol{t})Q(\boldsymbol{w}))\boldsymbol{w}\\ &= \boldsymbol{t} + O(\|\boldsymbol{t}\|^3)\,.\end{aligned}$$

This will allow us to work with the graph of the function $f(\boldsymbol{t}) := Q(\boldsymbol{x}(\boldsymbol{t}))$ instead of the hypersurface $\boldsymbol{w}(\mathcal{S}_g^{\mathrm{aff}})$, as long as only jets from zero up to second order are concerned. In particular,

$$\nabla f(0) = \nabla Q(0)\cdot\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{t}}(0) = \nabla Q(0)\,,$$

since the Jacobian $\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{t}}$ at zero is the identity. We have then proved that $[f]_0^1 = o^{(1)}$. Analogously,

$$\begin{aligned}\frac{\partial^2 f}{\partial t^i\partial t^j}(0) = \frac{\partial}{\partial t^j}\left(\frac{\partial Q}{\partial x^k}\frac{\partial x^k}{\partial t^i}\right)(0) &= \frac{\partial^2 Q}{\partial x^h\partial x^k}\frac{\partial x^h}{\partial t^j}\frac{\partial x^k}{\partial t^i}(0) + \frac{\partial Q}{\partial x^k}\frac{\partial^2 x^k}{\partial t^j\partial t^i}(0)\\ &= \frac{\partial^2 Q}{\partial x^h\partial x^k}(0)\delta_j^h\delta_i^k = \frac{\partial^2 Q}{\partial x^i\partial x^j}(0)\,,\end{aligned}$$

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since the first derivatives of Q vanish at the origin. Then, we also have that $[f]^2_0 = o^{(2)}$, i.e., $\boldsymbol{w} \cdot o^{(2)} = o^{(2)}$.

Therefore, in view of (37),

$$H^{(2)} = \{M_{A,\mu} \in H^{(1)} \mid M_{A,\mu} \cdot o^{(2)} = o^{(2)}\}, \tag{39}$$

so that it remains to compute the second-order jet at zero of the hypersurface

$$M_{A,\mu}(\mathcal{S}^{\mathrm{aff}}_g) = \{(\mu Q(\boldsymbol{x}), A \cdot \boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}^n\} = \{(\mu A^{-1\,*}(Q)(\boldsymbol{x}), \boldsymbol{x}), \mid \boldsymbol{x} \in \mathbb{R}^n\}\,,$$

and impose that it be equal to $o^{(2)}$.

Since both Q and $\mu A^{-1\,*}(Q)$ are quadratic forms, their second-order jets at zero coincides if and only if

$$\mu A^{-1\,*}(Q) = Q \Leftrightarrow A^*(Q) = \mu Q\,,$$

i.e., A is a conformal transformation of Q, and μ is the corresponding conformal factor, uniquely determined by A. In other words, $A \in \mathsf{O}(d, n-d) \cdot \mathbb{R}^\times$ and $\mu^n = \det(A)^2$, which concludes the proof that $H^{(2)} = (\mathbb{R}^n \rtimes \mathsf{O}(d, n-d)) \times \mathbb{R}^\times$.

The last case, i.e., the second claim of (35), will be dealt with in a similar fashion; to compute $H^{(3)}$, we first rewrite $H^{(2)}$ as

$$H^{(2)} = \{M_{A,\mu} \cdot \boldsymbol{w} \mid A \in \mathsf{O}(d, n-d) \times \mathbb{R}^\times\,,\ \mu^n = \det(A)^2\,,\ \boldsymbol{w} \in \mathbb{R}^n\},$$

in analogy to (37).

Since both Q and $\mu A^{-1\,*}(Q)$ have vanishing third-order jets at zero, in order to compute $H^{(3)}$ it suffices to impose that transformations of type $\boldsymbol{w}$ preserve $o^{(3)}$, i.e.,

$$H^{(3)} = \{M_{A,\mu} \cdot \boldsymbol{w} \in H^{(2)} \mid \boldsymbol{w} \cdot o^{(3)} = o^{(3)}\},$$

in analogy to the previous case (39). In this last case, however, the third-order jet at zero of $\boldsymbol{w}(\mathcal{S}^{\mathrm{aff}}_g)$ will not be the same as $o^{(3)}$, unless $\boldsymbol{w} = 0$. We have already observed that $f(\boldsymbol{t})$ and $Q(\boldsymbol{x})$ have the same derivatives at 0 up to order 2.

To study the third-order jet at zero of $\boldsymbol{w}(\mathcal{S}^{\mathrm{aff}}_g)$, we need to compute the third derivatives of f, where now $f(\boldsymbol{t}) = Q(\boldsymbol{x}(\boldsymbol{t}))$, with $\boldsymbol{x}(\boldsymbol{t})$ being the *true inverse* of $\boldsymbol{t}(\boldsymbol{x})$, and not the approximated one, i.e., (38). The reason why we use the same symbol for both the exact and the approximated (local) inverse, beside an evident notation simplification, is that the final result will depend only on the approximated one.

$$\begin{aligned}\frac{\partial^3 f}{\partial t^i \partial t^j \partial t^l} &= \frac{\partial}{\partial t^j}\left(\frac{\partial^2 Q}{\partial x^h \partial x^k}\frac{\partial x^h}{\partial t^j}\frac{\partial x^k}{\partial t^i} + \frac{\partial Q}{\partial x^k}\frac{\partial^2 x^k}{\partial t^j \partial t^i}\right)\\ &= \frac{\partial^3 Q}{\partial x^h \partial x^k \partial x^s}\frac{\partial x^s}{\partial t^l}\frac{\partial x^h}{\partial t^j}\frac{\partial x^k}{\partial t^i} + \frac{\partial^2 Q}{\partial x^h \partial x^k}\frac{\partial^2 x^h}{\partial t^j \partial t^l}\frac{\partial x^k}{\partial t^i} + \frac{\partial^2 Q}{\partial x^h \partial x^k}\frac{\partial^2 x^k}{\partial t^i \partial t^l}\frac{\partial x^h}{\partial t^j}\\ &\quad + \frac{\partial^2 Q}{\partial x^k \partial x^s}\frac{\partial^2 x^k}{\partial t^j \partial t^i}\frac{\partial x^s}{\partial t^l} + \frac{\partial Q}{\partial x^k}\frac{\partial^3 x^k}{\partial t^i \partial t^j \partial t^l}.\end{aligned}$$

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Evaluating the last expression at 0, we obtain

$$\begin{aligned}\frac{\partial^3 f}{\partial t^i \partial t^j \partial t^l}(0) &= 2Q_{hk}\frac{\partial^2 x^h}{\partial t^j \partial t^l}(0)\delta_i^k + 2Q_{hk}\frac{\partial^2 x^k}{\partial t^i \partial t^l}(0)\delta_j^h + 2Q_{ks}\frac{\partial^2 x^k}{\partial t^j \partial t^i}(0)\delta_l^s \\ &= 2Q_{hi}\frac{\partial^2 x^h}{\partial t^j \partial t^l}(0) + 2Q_{jk}\frac{\partial^2 x^k}{\partial t^i \partial t^l}(0) + 2Q_{kl}\frac{\partial^2 x^k}{\partial t^j \partial t^i}(0)\,. \end{aligned} \tag{40}$$

Now, for the purpose of computing the second derivatives of $\boldsymbol{x}$ at 0 in (40), we can use the approximated inverse that is (38):

$$\frac{\partial^2 \boldsymbol{x}(\boldsymbol{t})}{\partial t^i \partial t^j}(0) = \frac{\partial^2 (\boldsymbol{t} - Q(\boldsymbol{t})\boldsymbol{w})}{\partial t^i \partial t^j}(0) = -2Q_{ij}\boldsymbol{w}\,.$$

Indeed, the discrepancy between the true and the approximated inverse, being of third order in $\boldsymbol{x}$, will still vanish in 0, even after a double differentiation.

Therefore, the third-order term of the Taylor expansion of f around 0 (where, it is worth stressing, f is the one computed via the true inverse of $\boldsymbol{t}(\boldsymbol{x})$) is precisely

$$\begin{aligned}\frac{1}{3!}\frac{\partial^3 f}{\partial t^i \partial t^j \partial t^l}(0)t^i t^j t^l &= -\frac{1}{6}\left(2Q_{hi}2Q_{jl}w^h + 2Q_{jk}2Q_{il}w^k + 2Q_{kl}2Q_{ji}w^k\right)t^i t^j t^l \\ &= -2Q(\boldsymbol{t})\langle \boldsymbol{t}, \boldsymbol{w}\rangle\,. \end{aligned} \tag{41}$$

Since we have already observed that $\boldsymbol{w}(\mathcal{S}_g^{\mathrm{aff}})$ and $\mathcal{S}_g^{\mathrm{aff}}$ have the same jets at 0 up to order 2, and the third-order derivatives of Q are zero, formula (41) shows that $[\mathcal{S}_g^{\mathrm{aff}}]_0^3 = o^{(3)}$ if and only if $\boldsymbol{w} = 0$.

This shows that $\mathrm{Stab}_{H(2)}(o^{(3)}) = \mathsf{O}(d, n-d) \times \mathbb{R}^\times$, thus concluding the entire proof. $\square$

Remark 4.1. As we have anticipated, the proof of Lemma 4.1 also provides the missing steps in the proof of Lemma 4.1; observe also that the residual action of the group G on the fibre $J^3_{o^2}$ is exactly the same, that is, that of $\mathsf{E}(\mathbb{R}^n) \times \mathbb{R}^\times$. It is then reasonable to continue analysing the two cases in parallel.

5. $\mathsf{PGL}(n)$- and $\mathsf{Aff}(n)$-invariant PDEs on hypersurfaces of $\mathbb{P}^{n+1}$ and $\mathbb{A}^{n+1}$

Proposition 5.1. *The projective hyperquadric $\mathcal{S}_g$ defined by (18) (respectively, the quadric hypersurface $\mathcal{S}_g^{\mathrm{aff}}$ defined by (32)) is a fiducial hypersurface of order both 2 and 3 with respect to the action of the affine group $\mathsf{Aff}(n+1)$ on the affine space $\mathbb{A}^{n+1}$ (respectively, of the projective group $\mathsf{SL}(n+2)$ on the projective space $\mathbb{P}^{n+1}$), in the sense of Definition 2.5.*

Proof. For the order $k=2$, the proof is analogous to the Euclidean case, see [1, Proposition 4.1]. Indeed, J^1 is the same as $\mathbb{P}T^*\mathbb{R}^{n+1}$ or, equivalently, the flag space $F_{0,n}$, on which the linear group $\mathsf{GL}(n+1)$ already acts transitively, let alone $\mathsf{Aff}(n+1)$. So, $o^{(1)}$ is the flag $(0, \mathbb{R}^n)$ and the action of H on $J^1_0 = \mathbb{P}(\mathbb{R}^{n+1\,*})$ is transitive. Therefore, since

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the $\mathsf{Aff}(n+1)$-orbit of o is the entire M, the $\mathsf{Aff}(n+1)$-orbit of $o^{(1)}$ is the entire space J^1, viz.

$$J^1(n, \mathbb{A}^{n+1}) = \mathsf{Aff}(n+1)/H^{(1)}\,.$$

To deal with the case $k=3$, we shall study the orbit $H^{(1)} \cdot o^{(2)}$ in $J^2_{o^{(1)}}$, bearing in mind the identification

$$J^2_{o^{(1)}} \equiv S^2 T^*_o \mathcal{S}^{\text{aff}}_g \otimes N_o \mathcal{S}^{\text{aff}}_g = S^2 \mathbb{R}^{n\,*} \otimes \langle \partial_u \rangle\ , \tag{42}$$

cf. (15), and the description (34) of $H^{(1)}$. Since the quadratic form Q associated to the scalar product is non-degenerate, its $\mathsf{GL}(n)$-orbit will be open. Incidentally, we see the appearance of a $\mathsf{Aff}(n+1)$-invariant *second-order* PDE, namely the Monge–Ampère equation $\mathcal{E} \subset J^2$ given by $\det(u_{ij}) = 0$.

Summing up,

$$\check{J}^2 = \mathsf{Aff}(n+1) \cdot o^{(2)} = \mathsf{Aff}(n+1)/H^{(2)}$$

is an open subset of $J^2(n, \mathbb{A}^{n+1})$ (which is contained in the complement $J^2(n, \mathbb{A}^{n+1}) \smallsetminus \mathcal{E}$ of the Monge–Ampère equation $\mathcal{E}$). Therefore, the assumption (A1) of Definition 1.5 is met for the order $k=3$.

It remains to check assumption (A2) of Definition 2.5; we begin by showing that the orbit $H^{(2)} \cdot o^{(3)}$ is a proper affine sub-space of $J^3_{o^{(2)}}$. To this end, we shall need the identification ,

$$J^3_{o^{(2)}} \equiv S^3 \mathbb{R}^{n\,*} \otimes \langle \partial_u \rangle\ , \tag{43}$$

that is analogous to (42). Indeed, from the proof of Lemma 4.1, it is clear that the $H^{(2)}$-orbit of $o^{(3)}$ is made of the elements

$$[\boldsymbol{w}(\mathcal{S}^{\text{aff}}_g)]^3_o\,,$$

with $\boldsymbol{w} \in \mathbb{R}^n$. Therefore, from formula (41), it follows immediately that

$$[\boldsymbol{w}(\mathcal{S}^{\text{aff}}_g)]^3_o - o^{(3)} = [-2Q(\boldsymbol{x})\langle \boldsymbol{x}, \boldsymbol{w}\rangle]^3_0$$

and then (43) allows to identify the difference $[\boldsymbol{w}(\mathcal{S}^{\text{aff}}_g)]^3_o - o^{(3)}$ with the element

$$-2\boldsymbol{w}^{\#} \odot g \tag{44}$$

of the vector space $S^3 \mathbb{R}^{n\,*} \otimes \langle \partial_u \rangle$, where $\boldsymbol{w}^{\#}$ is the dual covector to $\boldsymbol{w}$ by means of the scalar product (17). In other words, as $\boldsymbol{w}$ ranges in $\mathbb{R}^n$, (44) describes the linear subspace

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$$\mathbb{R}^{n\,*}\odot\langle g\rangle\subset S^3\mathbb{R}^{n\,*}\,.$$

By construction, this is the linear space modelling the fibre $H^{(2)}\cdot o^{(3)}$. Since the same is true for any fibre, assumption (A2) of Definition 2.5 is met; indeed, as we pointed out in § 2.2.2, assumption (A2), in the case when (10) holds, is the same as having a (proper) affine subbundle and (10) immediately follows from (35).

The projective case can be dealt with analogously. □

5.1. The main result

Theorem 5.1. *Fix a scalar product g of signature $(d, n-d)$ as in (17) and let $\mathcal{S}_g^{\mathrm{aff}}\subset\mathbb{A}^{n+1}$ (respectively, $\mathcal{S}_g\subset\mathbb{P}^{n+1}$) be the corresponding fiducial (quadratic) hypersurface. Let*

$$S_0^3\mathbb{R}^{n\,*}:=\frac{S^3\mathbb{R}^{n\,*}}{\mathbb{R}^{n\,*}\odot\langle g\rangle}\tag{45}$$

denote the space of trace-free cubic forms on $\mathbb{R}^n$. Then, for any $\mathsf{CO}(d, n-d)$-invariant hypersurface

$$\Sigma\subset S_0^3\mathbb{R}^{n\,*}\,,$$

we obtain an $\mathsf{Aff}(n+1)$-invariant third-order PDE $\mathcal{E}_\Sigma\subset J^3(n,\mathbb{A}^{n+1})$ (respectively, an $\mathsf{SL}(n+2)$-invariant third-order PDE $\mathcal{E}_\Sigma\subset J^3(n,\mathbb{P}^{n+1})$).

Proof. Let us begin with the affine case. The first step consists in proving that $\tau_{\mathcal{R}}(H^{(2)})$-invariant hypersurfaces in

$$\frac{S^3T_o^*\mathcal{S}_g^{\mathrm{aff}}\otimes N_o\mathcal{S}_g^{\mathrm{aff}}}{R_{o^{(2)}}}$$

are the same as $\mathsf{CO}(p, n-p)$-invariant hypersurfaces in $S_0^3\mathbb{R}^{n\,*}$. To this end, recall the structure of $H^{(2)}$, studied in Lemma 4.1 (see, in particular, formula (35)) and observe that the factor $\mathbb{R}^\times$ acts by multiplication by $\mu\in\mathbb{R}^\times$ on $N_o\mathcal{S}_g^{\mathrm{aff}}$. The factor $\mathsf{O}(p, n-p)$ acts naturally on $S^3T_o^*\mathcal{S}_g^{\mathrm{aff}}$, which can be identified with $S^3\mathbb{R}^{n\,*}$. According to Proposition 5.1, an element $\boldsymbol{w}$ in the factor $\mathbb{R}^n$ acts by shifting along $R_{o^{(2)}}=\mathbb{R}^{n\,*}\odot\langle g\rangle$ by $-2\boldsymbol{w}^\#\odot g$, see also (44), and hence its action on the quotient is trivial.

The claim then follows from Theorem 2.1, recalling that, up to a covering, $\mathsf{CO}(d, n-d)=\mathsf{O}(d, n-d)\times\mathbb{R}^\times$.

Since the projective case can be dealt with analogously, we omit the proof. □

5.2. Coordinate description

Since the problem is, by its nature, a local one, we shall not consider the projective case, since the affine space $\mathbb{A}^{n+1}$ can be considered as an affine neighbourhood embedded

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in $\mathbb{P}^{n+1}$. Again, we extend the global coordinate system $\{u, x^1, \ldots, x^n\}$ of $\mathbb{A}^{n+1}$ to a (local) coordinate system of $J^3(n, \mathbb{A}^{n+1})$; see also § 2.3.1.

Lemma 5.1. *Let $\mathcal{E}_\Sigma$ be the $\mathsf{Aff}(n+1)$-invariant equation associated to the $\mathsf{CO}(d, n-d)$-invariant hypersurface Σ, as in Theorem 4.1. Then, in the aforementioned coordinate system on J^3, the equation $\mathcal{E}_\Sigma$ can be described as $\{f = 0\}$, where the function $f = f(u_{ij}, u_{ijk})$, that does not depend on $u, x^1, \ldots, x^n, u_1, \ldots, u_n$, is the same function describing the hypersurface $\Sigma_{o^{(1)}}$ of $J^3_{o^{(1)}}$.*

Proof. It is a consequence of Lemma 2.4, where the bundle is

$$J^1 \times J^3_{o^{(1)}} \subset J^3(n, \mathbb{A}^{n+1})$$

and the subgroup $T \subset G = \mathsf{Aff}(n+1)$ will be the $(2n+1)$-dimensional group

$$T = \mathbb{R}^{n+1} \rtimes \left\{ \begin{pmatrix} I_n & 0 \\ \boldsymbol{w} & 1 \end{pmatrix} \mid \boldsymbol{w} \in \mathbb{R}^n \right\}.$$

The first factor of T acts by translations on $\mathbb{R}^{n+1}$ and the lifted translations fix the u_i's and, similarly, the u_{ij}'s and the u_{ijk}'s. Therefore, it is enough the first factor of T to fulfill the hypothesis of Lemma 2.4.

Let us consider now

$$\phi = \begin{pmatrix} I_n & 0 \\ \boldsymbol{w} & 1 \end{pmatrix}.$$

Easy computations show that $\phi^{(1)\,*}(u_i) = u_i + w_i$, whereas $\phi^{(2)\,*}(u_{ij}) = u_{ij}$ and $\phi^{(3)\,*}(u_{ijk}) = u_{ijk}$. The first fact shows that T acts transitively on J^1 (since the translations act transitively on J^0 and the ϕ's act transitively on the fibres of $J^1 \to J^0$). The second fact shows that T acts trivially on the fibre $J^3_{o^{(1)}}$. Thus, the result follows from Lemma 2.4 applied to the group T. □

Example 5.1. For $n = 2$, a straightforward computation based on the proof of Lemma 5.1 (see [9, § 6] for more details) shows that the subset $\mathcal{E} := \{f = 0\}$ of J^3, where

$$\begin{aligned} f &= 6u_{xx}u_{xxx}u_{xy}u_{yy}u_{yyy} - 6u_{xx}u_{xxx}u_{xyy}u_{yy}^2 - 18u_{xx}u_{xxy}u_{xy}u_{xyy}u_{yy} \\ &\quad + 12u_{xx}u_{xxy}u_{xy}^2u_{yyy} - 6u_{xx}^2u_{xxy}u_{yy}u_{yyy} + 9u_{xx}u_{xxy}^2u_{yy}^2 - 6u_{xx}^2u_{xy}u_{xyy}u_{yyy} \\ &\quad + 9u_{xx}^2u_{xyy}^2u_{yy} + u_{xx}^3u_{yyy}^2 - 6u_{xxx}u_{xxy}u_{xy}u_{yy}^2 + 12u_{xxx}u_{xy}^2u_{xyy}u_{yy} - 8u_{xxx}u_{xy}^3u_{yyy} \\ &\quad + u_{xxx}^2u_{yy}^3, \end{aligned} \tag{46}$$

is invariant with respect to the group $\mathsf{Aff}(3)$. In [9], it is also shown that the same subset $\mathcal{E}$, in the real case, shows two different characters, depending on whether it projects over the open subset $\det u_{ij} > 0$, or $\det u_{ij} < 0$: the former corresponds to the invariant

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PDE associated with $\mathsf{CO}(2) = \mathsf{CO}(0,2)$, the latter to the invariant PDE associated with $\mathsf{CO}(1,1)$; see also § 5.4. In the first case, the invariant PDE is actually a *system* of two PDEs; this corresponds to (46) being the sum of two positive quantities; in the second case, the invariant subset $\mathcal{E}$ turns out to be the *union* of two scalar PDEs.

5.3. Complex CO_n-invariant hypersurfaces in $S^3_0(\mathbb{C}^n)$, with $n = 3, 4$

The departing point of the main Theorem 5.1 is a $\mathsf{CO}(d, n-d)$-invariant hypersurface Σ in the trace-free third symmetric power $S^3_0\mathbb{R}^{n\,*}$ of the n-dimensional real vector space $\mathbb{R}^{n\,*}$. While a general classification in the real case is still unattainable, much can be said in the case of small values of n, if we work over the field of complex number.

Therefore, only in this section, $V = \mathbb{C}^n$ is going to be a complex vector space, with $n = 3, 4$; having set $W := S^3_0(V)$, we shall study complex $\mathsf{CO}(V)$-invariant hypersurfaces Σ in W; in particular, there will be no signature, so that we consider the complex conformal group $\mathsf{CO}(V) = \mathsf{CO}_n(\mathbb{C})$, rather that its split real counterparts $\mathsf{CO}_{d,n-d}(\mathbb{R})$.

More accurately, we will derive a description of complex invariant hypersurfaces Σ in the irreducible $\mathsf{CO}(V)$-module $W = S^3_0(V)$ of traceless symmetric three-forms of the standard module $V = \mathbb{C}^n$ for $n=3$ and, partially, for $n=4$ from the known results of invariants' theory, see [11, 13]; afterwards, one can reduce the description of the real hypersurfaces that are invariant with respect to the corresponding normal real forms $\mathsf{CO}_{1,2}(\mathbb{R})$ and $\mathsf{SO}_{2,2}(\mathbb{R})$, as well as with respect to the compact real forms $\mathsf{CO}_3(\mathbb{R})$ and $\mathsf{CO}_4(\mathbb{R})$, to the description of the real forms of the above-obtained complex hypersurfaces.

By employing the same notation of [10], we will denote by $R(k\pi_1)$ the irreducible representation of the simple Lie algebra $\mathfrak{so}_n(\mathbb{C})$, whose highest weight is $k\pi_1$, always assuming that $n \geq 3$ and denoting by π_1 the first fundamental weight of $\mathfrak{so}_n(\mathbb{C})$; in particular, $R(\pi_1)$ is the tautological representation in the space $V = \mathbb{C}^n$ and $R(3\pi_1)$ is the highest irreducible component $W = S^3_0 V$ in the symmetric cube $S^3 V$.

5.3.1. The complex case with n = *3*

Recall that the Lie algebra $\mathfrak{so}_3(\mathbb{C})$ is isomorphic to the Lie algebra $\mathfrak{sl}(U) = \mathfrak{sl}_2(\mathbb{C})$ and that all irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules are exhausted by the symmetric power $S^k U$ of the tautological module $U = \mathbb{C}^2$. The tensor product $S^k U \otimes S^\ell U'$ is decomposed into irreducible submodules by the Klebsh–Gordon formula

$$S^k U \otimes S^\ell U' = \sum_{i=0}^{\infty} S^{k+\ell-2i} U.$$

The tautological representation of $\mathfrak{so}_3(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$ is the adjoint representation $V = S^2 U$ and the representation

$$R(3\pi_1) = S^3_0(V) = S^3_0(S^2 V) = S^6(U).$$

This is the $\mathfrak{sl}_2(\mathbb{C})$-module of binary forms of order 6. The full algebra of (polynomial) invariants $\mathbb{C}[S^6(U)]^{\mathfrak{sl}_2(\mathbb{C})}$ is known, see [11]. It is generated by five invariants $f_2, f_4, f_6, f_{10}, f_{15}$, of degrees 2, 4, 6, 10, 15, where the last invariant

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Table 1. Invariant hypersurfaces for $\mathrm{n}=3$ *and* $d \leq 10$.

Degree d	Polynomial f
2	$f = f_2$
4	$f = af_2^2 + bf_4$
6	$f = af_2^2 + bf_2f_4 + cf_6$
8	$f = af_2^4 + bf_2^2f_4 + cf_2f_6 + df_4^2$
10	$f = af_2^5 + bf_2^3f_4 + cf_2^2f_6 + df_4f_6 + ef_{10}$

$f_{15} \in A := \mathbb{C}[f_2, f_4, f_6, f_{10}]$ and the algebra A is the algebra of polynomials in four (independent) variables f_i.

Theorem 5.2. *Any complex* $\mathsf{SO}(V)$*-invariant hypersurface in* S_0^3V*, with* $V = \mathbb{C}^3$ *has the form* $\Sigma_f^c = \{f = c\}$ *where* $f \in \mathbb{C}[f_2, f_4, f_6, f_{10}f_{15}]$ *and* $c \in \mathbb{C}$ *is a constant. Any* $\mathsf{CO}(V)$*-invariant hypersurface has the form* $\Sigma_f^0 = \{f = 0\}$ *where* $f = f(f_2, f_4, f_6, f_{10}, f_{15})$ *is a homogeneous polynomial of* f_i*,* $\deg(f_i) = i$*.*

Moreover, any homogeneous invariant hypersurface of degree ≤ 210 has the form $f=0$ where the polynomial f is given in Table 1; the explicit form of the generators can be found in [13].

5.3.2. The complex case with $\mathrm{n}=4$

Consider now the case $n=4$. Then, $\mathfrak{so}(V) = \mathfrak{so}_4(\mathbb{C}) = \mathfrak{so}(U) + \mathfrak{so}(U')$, $U = U' = \mathbb{C}^2$ and the tautological module is $V = U \otimes U'$. Then $S_0^2V = S^2U \otimes S^2U'$ and

$$V \otimes S_0^2(V) = SU \otimes S^2U \otimes U' \otimes S^2U' = (S^3U + U) \otimes (S^3U' + U').$$

Then, $S_0^3V = S^3U \otimes S^3U'$.

It is known that the algebra of invariants of the $\mathfrak{sl}_2(\mathbb{C})$-module S^3U of ternary forms is generated by the discriminant δ, see [13], where for

$$p(x,y) = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3,$$

the discriminant is

$$\delta(p) = a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3.$$

Hence, the algebra of invariants $\mathbb{C}[S_0^3V]^{\mathfrak{so}_3(\mathbb{C})}$ contains δ and δ'.

Theorem 5.3. *Any polynomial* $f = f(\delta, \delta')$ *defines an invariant hypersurface* $f=c$*, where* $c \in \mathbb{C}$ *is constant. Any homogeneous polynomial* $f = f(\delta, \delta')$ *defines an* $\mathsf{CO}(V)$ *invariant hypersurface* $f=0$*.*

We stress that not all invariants are polynomials of δ and δ', that is, there may be other invariants.

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5.3.3. A glimpse into the real case

A standard method to cook out real invariants, having at one's disposal the complex ones, is by means of the anti-involution σ in $\mathfrak{so}(\mathbb{C}^n)$, i.e., the complex conjugation in $\mathbb{C}^n$; the anti-involution σ determines the real form $\mathfrak{so}(k,\ell)$ and then the real and the imaginary parts of the complex generators of the algebra $A=\mathbb{C}[\mathbb{C}^n]^{\mathfrak{so}(\mathbb{C}^n)}$ turn out to be real invariants that generate the whole real algebra of invariants. But there is a catch: the so-obtained real generators might be dependent.

Even though, in general, the description of a minimal system of generators of A is a very complicated problem, in practice it is possible to describe the invariants in small degrees k.

The so-called symbolic method for constructing invariants boils down to obtaining scalar invariants by contracting tensor products $y_1\otimes y_2\otimes\cdots\otimes y_k$ of cubic forms with the inverse metric g^{ij}. For example, for $k=2$, one can construct the invariant $I=y_{ijk}z_{i'j'k'}g^{ii'}g^{jj'}g^{kk'}$; in the case of binary form, one has to use also the determinant $\det y_{ij}$.

This way, one can get a description of the invariants in the small degree.

5.4. The $\mathsf{Aff}(3)$ case

Going back to the real-differentiable setting, if we set $n=2$, then it is is easy to use the results contained into Theorem 5.1 and Lemma 5.1 to write down explicitly the unique $\mathsf{Aff}(3)$-invariant scalar third-order PDE $\mathcal{E}$ imposed on hypersurfaces of $\mathbb{A}^3$. To clarify what we mean by 'unique', it should be stressed from the outset that, in general, the $\mathsf{Aff}(n+1)$-invariant PDE $\mathcal{E}\subset J^3$ constructed according to Theorem 5.1 projects onto an open subset $\check{J}^2$ of J^2; this is a direct consequence of the assumption (A1) on the action of G, see § 2.2.2. In turn, there are as many open subsets $\check{J}^2$, as the $\mathsf{GL}(n)$-equivalence classes of fiducial hypersurfaces (32); if we denote by d_n the ceiling of $n/2$, then these classes are labelled by the signatures

$$(n,0)\,,(n-1,1)\,,\dots\,,(n-d_n,d_n)\,,$$

i.e., there is d_n+1 of them. The union of all the subsets $\check{J}^2$ is dense in J^2, and its boundary is the unique second-order $\mathsf{Aff}(n+1)$-invariant PDE, that is the Monge–Ampère equation $\det hess(u)=0$; see also the proof of the assumption (A1) of Proposition 5.1.

In the case $n=2$, we have only two open subsets of J^2, corresponding to the Riemanian $(+,+)$ and to the Lorentzian $(+,-)$ signature of the Hessian of the surface in $\mathbb{A}^3$, denoted, respectively, by $\check{J}^2_+$ and $\check{J}^2_-$. In view of the important link between the $\mathsf{Aff}(3)$-invariant PDEs and the geometry of affine surfaces, we sketch the relation between such PDEs and the Fubini-Pick invariant.

Let $u=f(x^1,\dots,x^n)$ describe a hypersurface S of $\mathbb{A}^{n+1}$ which is the graph of the function f. Let us consider the basis

$$\left(\partial_u\,,\,D^{(1)}_1\,,\dots,D^{(1)}_n\right)=\left(\partial_u\,,\,\partial_{x^1}+u_1\partial_u\,,\dots\,,\partial_{x^n}+u_n\partial_u\right).$$

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The above basis is *unimodular* as $\det(\partial_u\,, D^{(1)}_1\,, \dots, D^{(1)}_n) = 1$. The components of the Blaschke metric G are

$$G_{ij} = \rho u_{ij}, \tag{47}$$

where

$$\rho = \left[\det\left(u_{ij}\right)\right]^{-\frac{1}{n+2}}\,,$$

whereas the components of the Fubini-Pick cubic form C are

$$C_{ijk} = -\frac{1}{2}\left(\rho u_{ijk} + f_{ij}D_k(\rho) + f_{jk}D_i(\rho) + f_{ik}D_j(\rho)\right)\,,$$

where D_h are the total derivatives, see also (14). The Fubini-Pick invariant is the function defined as

$$G^{i_1i_2}G^{j_1j_2}G^{h_1h_2}C_{i_1j_1h_1}C_{i_2j_2h_2}, \tag{48}$$

which, in the case $n = 2$ and up to a non-zero factor, is equal to the right-hand side term of (46). Then, the $\mathsf{Aff}(3)$-invariant PDE is $\mathcal{E} := \{f = 0\}$, with f given by (46), see [9] for more details. Another approach, based on the study of the singularities of the group action that has been used in [8], lead to the very same equation (46).

We stress that the equation $\mathcal{E}$ projects onto the whole of J^2, because (46) is defined on the whole J^3; however, if we take the intersections

$$\mathcal{E}\cap \check{J}^2_+\,, \quad \mathcal{E}\cap \check{J}^2_-\,,$$

we obtain precisely the two equations, say, $\mathcal{E}_{\Sigma_+}$ and $\mathcal{E}_{\Sigma_-}$, that come from Theorem 5.1; they correspond to the $\mathsf{CO}(2)$-invariant subset $\Sigma_+ := \{0\}$ and to the $\mathsf{CO}(1,1)$-invariant subset Σ_- made of two invariant lines, respectively. In other words,

$$\mathcal{E} = \overline{\mathcal{E}_{\Sigma_+}\cup\mathcal{E}_{\Sigma_-}}\,,$$

whence the adjective ‘unique’.

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