SOME LEMMAS ON INTERPOLATING BLASCHKE PRODUCTS AND A CORRECTION

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1. Introduction. A Blaschke product on the unit disc,

$$B(z) = cz^k \prod_{n=1}^{\infty} \frac{\bar{z}_n}{z_n} \cdot \frac{z_n - z}{1 - \bar{z}_n z},$$

where $|c| = 1$ and k is a non-negative integer, is said to be *interpolating* if the condition

$$\text{(C)} \qquad \prod_{\substack{n=1;\\ n\neq m}}^{\infty} \left| \frac{z_n - z_m}{1 - \bar{z}_n z_m} \right| \geqq \delta > 0$$

is satisfied for a constant δ independent of m. A Blaschke product always belongs to the set I of inner functions; it has norm 1 and radial limits of modulus 1 almost everywhere. The most general inner function can be uniquely factored into a product BS, where B is a Blaschke product and

$$S(z) = \exp\left[-\frac{1}{2\pi} \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right]$$

for some positive singular measure $\mu(\theta)$ on the unit circle. The discussion will be carried out in terms of the hyperbolic geometry on the open unit disc D, its metric

$$\psi(z, \omega) = \left| \frac{z - \omega}{1 - \bar{z}\omega} \right|, \qquad z, \omega \in D,$$

and its neighbourhoods $N(x, \epsilon) = \{z' \in D\colon \psi(z, z') < \epsilon\}$.

Lemma 1 states that an interpolating Blaschke product B is bounded away from zero as long as we do not come hyperbolically close to one of its zeros. This is stated in (**2**) as Lemma 1, and is used extensively there, but the proof given there is not correct. I learned, verbally, the proof presented here from Kenneth Hoffman. In the next lemma and the examples that follow, we consider the infimum of $|B|$ over the points whose hyperbolic distance from the zero set of B is at least ϵ. This infimum will increase with ϵ, but depending on the particular function B, it may not approach 1 when ϵ tends to 1. Lemma 3 characterizes interpolating Blaschke products (or finite products of them) as those inner functions f for which $\{z\colon |f(z)| < \delta\}$ is contained in a union $\bigcup_n N(\alpha_n, \epsilon)$ of disjoint hyperbolic neighbourhoods for some $\delta > 0$.

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2. Interpolating Blaschke products. We give first the corrected lemma.

LEMMA 1. *If $B(z)$ is the Blaschke product of a sequence (z_n) satisfying the condition* (C) *for a given δ, and if $\epsilon > 0$ is given, there exists a constant δ_0 depending only on δ and ϵ such that $|B(z)| \geqq \delta_0$ whenever $\psi(z, z_n) \geqq \epsilon$ for all n.*

Proof. Let ϵ_0 be an arbitrarily small positive constant such that $\epsilon_0 \leqq \epsilon$, and $\epsilon_0 < \psi(\delta, \epsilon_0)$. The latter condition ensures that the sets $\Delta_n = N(z_n, \epsilon_0)$ are disjoint. For each positive integer n, the composite function f_n is defined by

$$f_n(z) = B\left(\frac{z + z_n}{1 + \bar{z}_n z}\right).$$

This function has the same range on the set $D_{\epsilon_0} = \{z \in D: |z| < \epsilon_0\}$ that B has on Δ_n. From the condition (C), we obtain $|f_n'(0)| = (1 - |z_n|^2)|B'(z_n)| \geqq \delta$. It can be shown from the Schwarz lemma that there corresponds to δ and ϵ_0 a constant δ_0 with the following property: Every f analytic on D, and satisfying $|f| < 1, f(0) = 0$, and $|f'(0)| \geqq \delta$ has an image on the set D_{ϵ_0} which covers the set $\{z \in D: |z| < \delta_0\}$. See the discussion on the Schwarz lemma in (**3**, pp. 165–171). This means that the Blaschke product B takes on each value λ, for $|\lambda| < \delta_0$, in each Δ_n.

To prove the lemma, we must show that $|B(z)| \geqq \delta_0$ for all z not in the union of the Δ_n. If $|B(z)| < \delta_0$ for such a z, then for some integer m, $|B_m(z)| < \delta_0$, where B_m is the partial product consisting of the first m factors of B. However, $|B_m| \geqq |B|$ holds in D, and hence B_m must take on the value $B_m(z)$ in Δ_i, $i = 1, 2, \ldots, m$. Moreover, B_m is a rational function, and it can take any given value only m times. This is a contradiction, and hence $|B(z)| \geqq \delta_0$ whenever $\psi(z, z_n) \geqq \epsilon \geqq \epsilon_0$ holds for all n.

We observe from the above proof that δ_0 could be chosen first, and the existence of a suitable ϵ follows, as long as δ_0 is not too large.

LEMMA 2. *Suppose that $B(z)$ is the Blaschke product of a sequence (z_n) satisfying* (C) *for a given δ. There is a positive constant δ_1 depending only on δ, and for each $\delta_0 < \delta_1$, there exists $\epsilon > 0$ such that $|B(z)| \geqq \delta_0$ whenever $\psi(z, z_n) \geqq \epsilon$ for all n.*

Proof. Take any δ_0 small enough so that the argument from Lemma 1, using the Schwarz lemma, will apply for some $\delta_0 < \psi(\delta, \epsilon_0)$. Then the conclusion holds with $\epsilon = \epsilon_0$. The constant δ_1 can be obtained as the supremum of all such δ_0.

The following example shows that the constant δ_1 of Lemma 2 can have a value less than 1. The function

$$A(z) = \exp\left(\frac{z + 1}{z - 1}\right)$$

is an inner function without zeros on D. As z approaches 1 on the real axis, $A(z)$ approaches zero. This means that $A(z)$ tends to zero as z approaches 1 between any two hypercycles. From this, one can see that $\{z: |A(z)| < \delta\}$ will contain arbitrarily large hyperbolic neighbourhoods for each positive value of δ. For $|\lambda| < 1$, the function $A_\lambda = (A - \lambda)/(1 - \bar{\lambda}A)$ is an interpolating Blaschke product with zeros on an oricycle touching $\{z: |z| = 1\}$ at $z = 1$. However, $\{z: |A_\lambda(z)| < \rho\}$, for any ρ with $|\lambda| < \rho$, must contain those large arbitrarily hyperbolic ϵ-neighbourhoods on which A is small.

For an example of an interpolating Blaschke product with $\delta_1 = 1$, it suffices to assume that

$$\lim_{m\to\infty} \prod_{\substack{n=1;\\ n\neq m}}^{\infty} \left| \frac{z_n - z_m}{1 - \bar{z}_n z_m} \right| = 1.$$

The proof depends on showing that, in the Schwarz lemma argument, as $\delta \to 1$ it is possible to take ϵ_0 and δ_0 arbitrarily close to 1.

The following is a partial converse of Lemma 2.

LEMMA 3. *Let F be any inner function. Suppose, for some $\delta < 1$, that $\{\bar{z}: |F(z)| < \delta\}$ is contained in the union of disjoint hyperbolic ϵ-neighbourhoods $N(\alpha_n, \epsilon)$, $\alpha_n \in D$ for $n = 1, 2, 3 \ldots$, of fixed radius. Then F is the product of a finite number of interpolating Blaschke products.*

Proof. Let $F = BS$, where B is a Blaschke product, and S a singular function. We show first that S is the identity, by showing that if either the discrete or the continuous part of $\mu(\theta)$ is non-trivial, then S must approach zero along some radial path. The argument of the first example just given then shows that $\{z: |S(z)| < \rho\}$ will contain arbitrarily large hyperbolic neighbourhoods for each $\rho > 0$. If the continuous part $dF(\theta)$ of the singular measure $d\mu(\theta)$ is non-trivial, we can choose θ_0 with $F'(\theta_0) = +\infty$. We then apply the argument of Fatou's lemma (**1**, p. 34) to obtain

$$\lim_{r\to 1} \log|S_1(re^{i\theta_0})| = -F'(\theta_0) = -\infty,$$

where S_1 is the inner function obtained from $dF(\theta)$ alone, since

$$\log|S_1(re^{i\theta})| = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)}\, dF(\phi).$$

We can also show that for any θ where the discrete part of $\mu(\theta)$ has non-zero measure k, $S(re^{i\theta})$ must have a factor of the form

$$\exp\left[k\left(\frac{z + e^{i\theta}}{z - e^{i\theta}}\right)\right], \qquad k > 0,$$

which approaches zero as $z \to e^{i\theta}$ radially.

Thus, $F = B$, and the zeros (z_m) of this Blaschke product must each lie in one of the sets $N(\alpha_n, \epsilon)$. The given conditions impose a finite maximum for the number of z_m's which can appear in any $N(\alpha_n, \epsilon)$. We decompose B into a finite product of Blaschke products, each having at most one zero in any given

$N(\alpha_n, \epsilon)$. If $\tilde{B}$ is any one of these, we know that $\tilde{B}$ also satisfies the conditions of the lemma, since $|\tilde{B}| \geqq |B|$. If we let $\tilde{B}_m$ be $\tilde{B}$ with the zero z_m from $N(\alpha_n, \epsilon)$ removed, then $|\tilde{B}_m| \geqq |\tilde{B}| \geqq \delta$ on the circular boundary of $N(\alpha_n, \epsilon)$. However, $\tilde{B}_m$ is zero-free inside, and hence $|\tilde{B}_m| \geqq \delta$ holds throughout. This means that $|\tilde{B}_m(z_m)| \geqq \delta$, which is precisely (C).

In particular, this shows that if B is a Blaschke product which is not the finite product of interpolating Blaschke products, then B will be arbitrarily small at points arbitrarily far (hyperbolically) from its zeros. We might also note that from Lemma 3 and Rouché's theorem, we can easily prove that the set of interpolating Blaschke products is open in I.

References

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