THE HILBERT TRANSFORM ON REARRANGEMENT-INVARIANT SPACES

D. W. BOYD

Introduction. The purpose of this paper is to investigate conditions under which the Hilbert transform defines a bounded linear operator from a given function space into itself. The spaces with which we deal have the property of rearrangement-invariance which is defined in §1. This class of spaces includes the Lebesgue, Orlicz, and Lorentz spaces.

Let f be a locally integrable function on $R = (-\infty, \infty)$, and let H denote the Hilbert transform:

$$(Hf)(x) \overset{\text{a.e.}}{=} \lim_{\epsilon \to 0+} \frac{1}{\pi} \left(\int_{-\infty}^{x-\epsilon} + \int_{x+\epsilon}^{\infty} \right) \frac{f(t)}{x-t}\, dt$$

$$= \frac{1}{\pi} (P) \int_{-\infty}^{\infty} \frac{f(t)}{x-t}\, dt.$$

If X is a Banach space, let $[X]$ denote the space of bounded linear operators from X into itself. A classical result of M. Riesz states that $H \in [L^p]$ if and only if $1 < p < \infty$. Our main result generalizes this as follows.

THEOREM 3.7. *Let X be a rearrangement-invariant space. Define the operator E_s for $0 < s < \infty$ by $(E_s f)(x) = f(sx)$, $f \in X$. Denote the norm of E_s as a member of $[X]$ by $h(s; X)$. Then, $H \in [X]$ if and only if*

$$sh(s; X) \to 0 \text{ as } s \to 0+, \quad \text{and } h(s; X) \to 0 \text{ as } s \to \infty.$$

In §4, we apply this theorem to the spaces of Lorentz, by explicit calculation of $h(s; X)$. Using this result, we can give examples showing that reflexivity of X is both unnecessary and insufficient in order that $H \in [X]$. This may be somewhat surprising, since the condition for L^p to be reflexive (i.e. $1 < p < \infty$) is the same as the condition for $H \in [L^p]$. However, $1 < p < \infty$ also ensures that L^p is uniformly convex, so the following result is welcome:

Let X be the Lorentz space $\Lambda(\phi, p)$, $1 < p < \infty$. Then $H \in [X]$ if and only if X is uniformly convex.

This result is non-trivial since there are spaces of the form $\Lambda(\phi, p)$ which are reflexive but not uniformly convex.

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In §5, we apply the main theorem to the Orlicz spaces and obtain the following:

Let X be an Orlicz space. Then $H \in [X]$ if and only if X is reflexive.

Some generalizations of Theorem 3.7 to operators other than H are mentioned in §6. In fact, *if T is of weak type (p, p) for all $p \in (1, \infty)$ and if $H \in [X]$, then $T \in [X]$.*

1. Rearrangement-invariant Banach spaces. Suppose that $(\Sigma, \mathfrak{F}, \mu)$ is a totally σ-finite measure space. Let $\mathfrak{M}(\Sigma)$ denote the class of complex-valued measurable functions on Σ, and $\mathfrak{P}(\Sigma)$ denote the subclass of $\mathfrak{M}(\Sigma)$ consisting of non-negative functions.

Definition 1.1. A function ρ from $\mathfrak{P}(\Sigma)$ to $[0, \infty]$ is called a *length function* if it satisfies the following conditions for each $f, g \in \mathfrak{P}(\Sigma)$:

(i) $\rho(f) = 0 \Leftrightarrow f(x) = 0$ a.e. on Σ,
$\rho(f + g) \leqslant \rho(f) + \rho(g)$,
$\rho(af) = a\rho(f)$ for any constant $a \geqslant 0$.

(ii) If $f_n \in \mathfrak{P}(\Sigma)$ for $n = 1, 2, \ldots$, and $f_n \uparrow f$ a.e., then $\rho(f_n) \uparrow \rho(f)$.

(iii) If $E \in \mathfrak{F}$, $\mu(E) < \infty$, and χ_E is its characteristic function, then $\rho(\chi_E) < \infty$.

(iv) If $E \in \mathfrak{F}$, $\mu(E) < \infty$, then there exists a constant $A_E < \infty$, such that, for every $f \in \rho(\Sigma)$,

$$\int_E f\, d\mu \leqslant A_E\, \rho(f).$$

Given a length function ρ, the *associate* of ρ, denoted ρ', is defined for each $g \in \mathfrak{P}(\Sigma)$ by

$$\rho\ (g) = \sup_f \int_\Sigma fg\, d\mu \qquad (f \in \mathfrak{P}(\Sigma), \rho(f) \leqslant 1).$$

Let X denote the set of $f \in \mathfrak{M}(\Sigma)$ for which $\rho(|f|) < \infty$. Then, identifying functions which differ at most on a set of measure zero, X is a Banach space with norm $||f|| = \rho(|f|)$, for each $f \in X$. The space so defined by ρ' is called the associate of X and is denoted X'. Luxemburg, who gave the above definition of length function, has shown that $\rho'' = \rho$, so that $X'' = X$; see **(9**, p. 10**)**.

We wish to place a further restriction on the function ρ. To do this, we need the idea of the non-increasing rearrangement of a function in $\mathfrak{M}(\Sigma)$ onto the half-line $R^+ = [0, \infty)$. Given $f \in \mathfrak{M}(\Sigma)$, define a function f_* from R^+ to $[0, \infty]$, by

$$f_*(y) = \mu\{x: f|(x)| > y\}, \qquad y \geqslant 0.$$

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f_* is non-increasing. Denote the left-continuous inverse of f_* by f^*. Then, f^* is called the *non-increasing* rearrangement of f onto R^+; see **(2**, or **10)** for more details.

By definition, if m denotes Lebesgue measure on R^+,

$$m\{t: f^*(t) > y\} = \mu\{x: |f(x)| > y\}.$$

The following results are fundamental:

(1) $$\int_\Sigma |fg|\, d\mu \leqslant \int_0^\infty f^*g^*\, dm, \qquad f, g \in \mathfrak{M}(\Sigma).$$

(2) If F is non-negative, non-decreasing, and continuous on the left, then

$$\int_\Sigma F(|f|)\, d\mu = \int_0^\infty F(f^*)\, dm.$$

The integral mean of f^* is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds, \qquad t > 0,$$

and $f^{**}(0) = f^*(0+)$. If $f, g, h \in \mathfrak{M}(\Sigma)$, with $f = g + h$, then

(3) $$f^{**}(t) \leqslant g^{**}(t) + h^{**}(t).$$

Another definition of f^{**} is possible if Σ is non-atomic, namely

(4) $$tf^{**}(t) = \sup_E \int_E |f|\, d\mu \qquad (E \in \mathfrak{F}, \mu(E) \leqslant t).$$

Definition 1.2. Suppose that σ is a length function on $\mathfrak{P}(R^+)$, and that, for each $u \in \mathfrak{P}(R^+)$,

(v) $$\sigma(u) = \sigma(u^*).$$

Then σ will be called a *rearrangement-invariant length function* on $\mathfrak{P}(R^+)$, or an *r-function* on $\mathfrak{P}(R^+)$.

If σ is an r-function on $\mathfrak{P}(R^+)$, and if ρ is a length function on $\mathfrak{P}(\Sigma)$ defined by

(vi) $$\rho(f) = \sigma(f^*), \qquad \text{for each } f \in \mathfrak{P}(\Sigma),$$

then we call ρ an *r-function* on $\mathfrak{P}(\Sigma)$.

The space corresponding to ρ will be denoted by $X(\Sigma)$, and will be called a *rearrangement-invariant space.*

The proof of sections (a), (b), and (c) of the following lemma is essentially given in **(8)**. Sections (d) and (e) follow immediately in case the measure space $(\Sigma, \mathfrak{F}, \mu)$ is the real line R, with Lebesgue measure. For the general situation, the reader is referred to **(1)**.

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LEMMA 1.3. *Let σ be an r-function on $\mathfrak{P}(R^+)$, and ρ be defined as in* (vi). *Let σ' and ρ' denote the associates of σ and ρ respectively. Then,*

(a) *σ' is an r-function on $\mathfrak{P}(R^+)$.*

(b) *If $u \in \mathfrak{P}(R^+)$, then*

$$\sigma(u) = \sup_v \int_0^\infty u^* v^* \, dm \qquad (v \in \mathfrak{P}(R^+), \sigma'(v) \leqslant 1).$$

(c) *If $u, v \in \mathfrak{P}(R^+)$, and if $u^{**} \leqslant v^{**}$, then $\sigma(u) \leqslant \sigma(v)$.*

(d) *ρ is a length function on $\mathfrak{P}(\Sigma)$.*

(e) *For any $g \in \mathfrak{P}(\Sigma)$, $\rho'(g) = \sigma'(g^*)$.*

Remark. Because of section (e) of the above, we denote the associate of $X(\Sigma)$ by $X'(\Sigma)$, rather than $(X(\Sigma))'$, since the determination of the associate of $X(\Sigma)$ is independent of the nature of Σ. This is in contrast to the conjugate of $X(\Sigma)$ (the space of bounded linear functionals on $X(\Sigma)$), since there are many examples from the theory of Orlicz spaces for which $X'(R) \neq (X(R))^*$, while $X'([0, 1]) = (X([0, 1]))^*$.

2. The Hilbert transform and related operators. Suppose that $f \in \mathfrak{M}(R)$, and that f is locally integrable. The *Hilbert transform* of f is defined by the principal-value integral given in the Introduction, whenever the defining limit exists a.e. In this section, we shall obtain a relationship between H and two operators P and P' defined as follows:

If $u \in \mathfrak{P}(R^+)$, then, for $t > 0$,

$$(Pu)(t) = \frac{1}{t} \int_0^t u(s)\, ds \quad \text{and} \quad (P'u)(t) = \int_t^\infty u(s) \frac{ds}{s},$$

whenever the required integrals exist a.e.

The notation $[X, Y]$ will denote the space of bounded linear operators from X into Y. Also, if $f, g \in \mathfrak{P}(\Sigma)$, then we write

$$\langle f, g \rangle = \int_\Sigma fg \, d\mu.$$

THEOREM 2.1. *If X and Y are rearrangement-invariant spaces, then $H \in [X(R), Y(R)]$ if and only if both P and P' are in $[X(R^+), Y(R^+)]$.*

LEMMA 2.2. *Suppose that X and Y are as in the theorem, and that the operator S is defined by*

$$(Su)(t) = \int_0^\infty \frac{u(s)}{t + s} ds, \qquad u \in \mathfrak{M}(R^+), t \geqslant 0,$$

whenever this integral exists a.e. Then, $H \in [X(R), Y(R)]$ implies

$$S \in [X(R^+), Y(R^+)].$$

Proof. Let $X(R^+)$ have the r-function σ, and $Y(R^+)$ have the r-function τ.

Given $u \in X(R^+)$, with u non-increasing, define f on R by

$$(1) \qquad f(x) = \begin{cases} 0, & x > 0, \\ u(-x), & x \leqslant 0, \end{cases}$$

so that $f^*(t) = u(t)$, $t > 0$. Then, for any $x > 0$,

$$(2) \qquad \pi(Hf)(x) = (Su)(x).$$

Hence, if we define $g \in \mathfrak{M}(R)$ by

$$(3) \qquad g(x) = \begin{cases} \pi(Hf)(x), & x > 0 \\ 0, & x \leqslant 0, \end{cases}$$

it follows that $g^*(t) = (Su)(t)$, $t > 0$, and that

$$(4) \qquad |g(x)| \leqslant \pi|(Hf)(x)|, \qquad x \in R.$$

Thus,

$$(5) \qquad \begin{aligned} ||Su||_{Y(R^+)} &= \tau(Su) = \tau(g^*) = ||g||_{Y(R)} \leqslant \pi||Hf||_{Y(R)} \\ &\leqslant \pi c||f||_{X(R)} = \pi c||u||_{X(R^+)} \end{aligned}$$

where $c = ||H|| < \infty$.

If $u \in X(R^+)$, but is not necessarily non-increasing, then $(Su)(t) \leqslant (Su^*)(t)$, by (1) of §1. Hence, since $\sigma(u) = \sigma(u^*)$, (5) implies that $S \in [X(R^+), Y(R^+)]$.

LEMMA 2.3. **(10)**. *Suppose that $f \in \mathfrak{M}(R)$ and that*

$$\int_0^\infty f^*(t) \sinh^{-1}\frac{1}{t}\, dt < \infty.$$

Then $(Hf)(x)$ exists a.e., and for each $t > 0$,

$$(6) \qquad (Hf)^{**}(t) \leqslant \frac{4}{\pi} \int_0^\infty \frac{f^{**}(s)}{\sqrt{(t^2 + s^2)}}\, ds.$$

By an obvious inequality, (6) can be rewritten as

$$(7) \qquad (Hf)^{**}(t) \leqslant A \cdot [(P + P')f^{**}](t),$$

where $A = 4/\pi$.

Also, the condition $S \in [X(R^+), Y(R^+)]$, which is the conclusion of Lemma 2.2, is easily seen to be equivalent to the condition that both P and P' be in $[X(R^+), Y(R^+)]$. Finally, we observe that, for $f \in \mathfrak{D}(P) \cap \mathfrak{D}(P')$,

$$PP'f = (P + P')f = P'Pf.$$

($\mathfrak{D}(T)$ denotes the domain of the operator T).

LEMMA 2.4. *Suppose that $\mathfrak{D}(P') \supset X(R^+)$, and that $P \in [X, Y]$. Then, for all $f \in X(R)$,*

$$\int_0^\infty f^*(s) \sinh^{-1}\frac{1}{s}\, ds < \infty.$$

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Proof. Since $\mathfrak{D}(P') \supset X(R^+)$, if $f \in X(R)$, then $(P'f^*)(t) < \infty$ for almost all $t > 0$. But $(P'f^*)$ is non-increasing, so

$$(8) \qquad \int_1^\infty f^*(s)\frac{ds}{s} = (P'f^*)(1) < \infty.$$

Since $P \in [X, Y]$, and since $\mathfrak{D}(P) \supset Y(R^+)$, by condition (iv) on the length function, we have $\mathfrak{D}(P^2) \supset X(R^+)$. Hence,

$$(9) \qquad \int_0^1 \log\frac{1}{s}\cdot f^*(s)\,ds = (P^2f^*)(1) < \infty.$$

Combining (8) and (9), we obtain the lemma.

Proof of Theorem 2.1. Suppose $H \in [X(R), Y(R)]$. Then, by Lemma 2.2, and the remarks following Lemma 2.3, P and P' are in $[X(R^+), Y(R^+)]$.

Conversely, if P and P' are in $[X(R^+), Y(R^+)]$, then by Lemma 2.4, each $f \in X(R)$ satisfies the conditions of Lemma 2.3. Thus, $\mathfrak{D}(H) \supset X(R)$, and

$$(1) \qquad (Hf)^{**}(t) \leqslant A\cdot[(P + P')f^{**}](t), \qquad t > 0.$$

But, by remarks following Lemma 2.3,

$$(2) \qquad (P + P')f^{**} = (P + P')Pf^* = ((P + P')f^*)^{**}.$$

Hence, by Lemma 1.3(c), (1) and (2) imply that

$$(3) \qquad \|(Hf)^*\|_{Y(R^+)} \leqslant A\cdot\|(P + P')f^*\|_{Y(R^+)} \leqslant B\|f^*\|_{X(R^+)},$$

where $B = A(\|P\| + \|P'\|)$.

Since X and Y are rearrangement-invariant spaces, (3) shows that

$$H \in [X(R), Y(R)].$$

3. Positive integral operators associated with H. Our goal in this section is to prove Theorem 3.7, which was stated in the introduction. We begin by considering operators T, with $\mathfrak{D}(T) \subset \mathfrak{M}(R^+)$, of the form

$$(1) \qquad (Tf)(t) = \int_0^\infty a(s)f(st)\,ds, \qquad t > 0,$$

where

$$\mathfrak{D}(T) = \left\{f \in \mathfrak{M}(R^+) : \int_0^\infty |a(s)|\cdot|f(st)|\,ds < \infty \text{ a.e.}\right\}.$$

Note that P and P' are of this form, namely

$$(Pf)(t) = \int_0^1 f(st)ds \quad\text{and}\quad (P'f)(t) = \int_1^\infty f(st)\frac{ds}{s}.$$

Define the dilation operator E_s, with $\mathfrak{D}(E_s) = \mathfrak{M}(R^+)$, by

$$(E_sf)(t) = f(st), \qquad s \in (0, \infty).$$

Let $h(s; X, Y)$ denote the norm of E as a member of $[X(R^+), Y(R^+)]$.

From now on, whenever it is clear which Σ is meant, we shall write $[X, Y]$ for $[X(\Sigma), Y(\Sigma)]$.

THEOREM 3.1. *Suppose that X and Y are rearrangement-invariant spaces, and that T is defined by* (1). *Suppose that*

$$\int_0^\infty |a(s)|h(s; X, Y)\, ds = c < \infty.$$

Then, $T \in [X, Y]$, with $||T|| \leqslant c$.

Proof. (cf. **5**, p. 230).
Let $g \in Y'$, with $||g||_{Y'} \leqslant 1$. Then

$$(2) \qquad \int_0^\infty |g(t)|dt \int_0^\infty |a(s)|\cdot|f(st)|\, ds = \int_0^\infty |a(s)|\cdot\langle|E_s f|, |g|\rangle\, ds$$

$$\leqslant \int_0^\infty |a(s)|\cdot h(s; X, Y)||f||_X\, ||g||_{Y'} \leqslant c\, ||f||_X.$$

Taking the supremum in (2) over all $g \in Y'$, $||g||_{Y'} \leqslant 1$, noting that, for all such g, the left member of (2) is larger than $|\langle Tf, g\rangle|$, we obtain $||Tf||_Y \leqslant c||f||_X$.

LEMMA 3.2.
(a) *$sh(s; X, Y) = h(1/s; Y', X')$,*
(b) *$h(s; X, Y)$ is non-increasing in s, and $sh(s; X, Y)$ is non-decreasing in s,*
(c) *If $h(s) = h(s; X, Y)$, then*

$$h(1)\cdot\min(1/s, 1) \leqslant h(s) \leqslant h(1)\cdot\max(1/s, 1).$$

Proof. (a) follows from $s\langle E_s f, g\rangle = \langle f, E_{1/s}\, g\rangle$.
(b) Since $(E_s f)^* = E_s f^*$,

$$h(s) = \sup||E_s f|| = \sup||E_s f^*|| \qquad (||f|| \leqslant 1).$$

But, $E_s f^*$ is non-increasing in s, so that h is non-increasing. Using (a), it follows directly that $sh(s)$ is non-decreasing in s.
(c) follows directly from (b).

LEMMA 3.3.
(a) *Suppose that T is defined as in Lemma 3.1, with $a(s) \geqslant 0$ for $s \in [0, \infty)$ and that $T \in [X, Y]$. Define*

$$A(x) = \int_0^x a(s)ds.$$

Then,

$$h(s; X, Y) \leqslant ||T||/A(s) \qquad \textit{for } s > 0.$$

(b) *If $a(s) > 0$ on a set of positive measure and $T \in [X, Y]$, then $X \subseteq Y$.*

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Proof. (a) Since $T \in [X, Y]$, with norm c for every f, g non-negative and non-increasing, we have

$$\text{(1)} \qquad \langle Tf, g\rangle = \int_0^\infty g(t)\,dt \int_0^\infty a(s) f(st)\,ds$$
$$= \int_0^\infty a(s)\langle E_s f, g\rangle\,ds \leqslant c\|f\|_X\,\|g\|_{Y'}.$$

$\langle E_s f, g\rangle$ is non-increasing in s, so that, if $x \in (0, \infty)$,

$$\text{(2)} \qquad \langle E_x f, g\rangle \int_0^x a(s)ds \leqslant \int_0^x a(s)\langle E_s f, g\rangle ds$$
$$\leqslant \int_0^\infty a(s)\langle E_s f, g\rangle ds \leqslant c\|f\|_X\,\|g\|_{Y'}.$$

Taking the supremum in (2) over all $g \in Y'$, $\|g\|_{Y'} \leqslant 1$, then over all $f \in X$, $\|f\|_X \leqslant 1$, we obtain

$$\text{(3)} \qquad h(x; X, Y) \leqslant c/A(x).$$

(b) Since $a(s) > 0$ on a set of positive measure, $A(x_0) > 0$, for some $x_0 > 0$. Hence, by (a),

$$\|\mathrm{E}_{x_0} f\|_Y \leqslant c\|f\|_X/A(x_0).$$

Thus,

$$\|f\|_Y = \|E_{1/x_0} E_{x_0} f\|_Y \leqslant h(1/x_0; Y, Y)\|E_{x_0} f\|_Y$$
$$\leqslant \max(x_0, 1)\cdot c\|f\|_X/A(x_0) = c_1\cdot\|f\|_X.$$

By the way in which X and Y are defined, $X \subseteq Y$.

The next theorem, combined with Theorem 2.1, is practically Theorem 3.6. Note that we abbreviate $h(s; X, X)$ to $h(s; X)$. It is obvious that $h(1; X) = 1$.

THEOREM 3.4. *Suppose that X is a rearrangement-invariant space and that $\|P\|$, $\|P'\|$ denote the norms of P and P' respectively, as members of* $[X]$ (i.e. $\|P\| = \infty$, if $P \notin [X]$). *Then,*

$$\|P\| \leqslant \int_0^1 h(s; X)ds \leqslant 2\sqrt{2}\,\|P\|$$

and

$$\|P'\| \leqslant \int_1^\infty h(s; X)\frac{ds}{s} \leqslant 2\sqrt{2}\,\|P'\|.$$

Proof. We observe that we need only prove one of the statements. For, if $f, g \in \mathfrak{P}(R^+)$, we have $\langle Pf, g\rangle = \langle f, P'g\rangle$, so that $\|P\|_{[X]} = \|P'\|_{[X']}$. And, by Lemma 3.2,

$$\int_0^1 h(s; X)ds = \int_1^\infty h(s; X')\frac{ds}{s}.$$

We shall prove the second of the statements.

The left-hand inequality is Theorem 3.1. The right-hand inequality is trivial if $P' \notin [X]$. Thus, assuming $P' \in [X]$, we have $(P')^2 = Q \in [X]$, with $||Q|| \leqslant c^2$, if $c = ||P'||$. By a simple calculation,

$$(Qf)(t) = \int_1^\infty f(st)\cdot\log s\cdot\frac{ds}{s},$$

so that, by Lemma 3.3(a), we have

$$(1) \qquad h(s;X) \leqslant 2||Q||/(\log s)^2 \leqslant 2c^2/(\log s)^2, \qquad s > 1.$$

However, $h(s;X) \leqslant 1$, for $s \geqslant 1$, so that

$$(2) \qquad h(s;X) \leqslant \min\{1, 2c^2/(\log s)^2\}.$$

Thus, if $\log a = \sqrt{2}\, c$, then

$$(3) \qquad \int_1^\infty h(s;X)\frac{ds}{s} \leqslant \int_1^a \frac{ds}{s} + \int_a^\infty \frac{2c^2}{(\log s)^2}\frac{ds}{s}$$
$$= \log a + 2c^2/\log a = 2\sqrt{2}\, c.$$

This completes our proof.

One obvious fact about $h(s)$ we have not used is that

$$h(st) \leqslant h(s)h(t), \qquad s, t > 0.$$

This means that $\log h(s)$ is a subadditive function of $\log s$, so a theorem of Hille and Phillips (**6**, p. 244) may be applied to establish the following result.

LEMMA 3.5. *Let*

$$\alpha = \inf_{0<s<1} -\log h(s)/\log s, \qquad \beta = \sup_{1<s<\infty} -\log h(s)/\log s.$$

Then, $0 \leqslant \beta \leqslant \alpha \leqslant 1$ and

$$\alpha = \lim_{s\to 0+} -\log h(s)/\log s, \qquad \beta = \lim_{s\to\infty} -\log h(s)/\log s.$$

It is now easy to establish the following result:

LEMMA 3.6.
(a) *The following conditions on $h(s)$ are equivalent:*

(i) $\int_0^1 h(s)\, ds < \infty$.

(ii) $sh(s) \to 0$ *as* $s \to 0+$.
(iii) *There exists an* $s_0 \in (0, 1)$ *such that* $h(s_0) < 1/s_0$.
(iv) *There exist constants* $K > 0$ *and* $0 \leqslant \gamma < 1$ *such that* $h(s) \leqslant Ks^{-\gamma}$ *for* $0 < s \leqslant 1$.

(b) *The following are equivalent:*

(i) $\int_1^{\infty} h(s)\frac{ds}{s} < \infty.$

(ii) $h(s) \to 0$ *as* $s \to \infty$.

(iii) *There exists an* $s_0 \in (1, \infty)$ *such that* $h(s_0) < 1$.

(iv) *There exist constants* $K > 0$ *and* $0 < \gamma \leqslant 1$ *such that* $h(s) \leqslant Ks^{-\gamma}$ *for* $1 \leqslant s < \infty$.

Proof. (a) Let α be as in Lemma 3.5, and for each $\epsilon > 0$, let $\sigma(\epsilon)$ be such that $s \in (0, \sigma(\epsilon))$ implies

$$\alpha \leqslant -\log h(s)/\log s \leqslant \alpha + \Sigma,$$

so that

$$s^{-\alpha} \leqslant h(s) \leqslant s^{-\alpha-\Sigma} \qquad \text{for } s \in (0, \sigma(\epsilon)).$$

(i) implies (iii) trivially.

(iii) implies (iv); for if (iii) holds, then $\alpha < 1$, so for some $\epsilon > 0$, $\alpha + \epsilon < 1$, and (iv) holds with $\gamma = \alpha + \epsilon$ and $K = [\sigma(\epsilon)]^{-\alpha-\epsilon}$.

(iv) implies (ii) trivially.

(ii) implies (i); for if (ii) holds then $\alpha < 1$, and with ϵ as above,

$$\int_0^{\sigma(\epsilon)} h(s)\, ds \leqslant \int_0^{\sigma(\epsilon)} s^{-\alpha-\epsilon}\, ds < \infty,$$

so that (i) holds.

The proof of (b) is entirely similar.

THEOREM 3.7. *Let X be a rearrangement-invariant space. Then $H \in [X(R)]$ if and only if*

(i) $sh(s; X) \to 0$ *as* $s \to 0+$ *and*

(ii) $h(s; X) \to 0$ *as* $s \to \infty$.

Proof. By Theorem 2.1, $H \in [X(R)]$ if and only if P and P' are both in $[X(R^+)]$. By Theorem 3.4, this is equivalent to

$$\int_0^1 h(s)ds < \infty \quad \text{and} \quad \int_1^{\infty} h(s)\frac{ds}{s} < \infty.$$

By Lemma 3.6, this last is equivalent to $sh(s) \to 0$, as $s \to 0+$, and $h(s) \to 0$, as $s \to \infty$.

Remark. The conditions (i) and (ii) on $h(s; X)$ may be replaced by any pair of the equivalent conditions given in Lemma 3.6. The form of the theorem we shall most often use replaces (i) and (ii) by the condition that there exist $s_0 > 1$ such that $h(s_0; X) < 1$ and $h(s_0; X') < 1$.

4. The Lorentz spaces. The spaces to be discussed here were defined in **(7)**. Suppose that $\phi \in \mathfrak{P}(R^+)$ is non-increasing, and that

$$\Phi(t) = \int_0^t \phi(s)\, ds < \infty \qquad \text{for } t < \infty .$$

The space $\Lambda(\phi, p)$ is defined by the length function

$$\sigma_p(u) = \left\{ \int_0^\infty \phi(t)[u^*(t)]^p\, dt \right\}^{1/p}, \qquad p \geqslant 1.$$

The associate of $\Lambda(\phi, 1) = \Lambda(\phi)$, is denoted $M(\phi)$, and has the length function

$$\sigma'(u) = \sup_{s>0} \left\{ \int_0^s u^*(t)dt \Big/ \Phi(s) \right\}.$$

Note that if ϕ is identically 1, then $\Lambda(\phi, p) = L^p$.

Before calculating $h(s)$ for these spaces, we note that it is easily proved that

$$h(s;\ \Lambda(\phi, p)) = [h(s;\ \Lambda(\phi))]^{1/p}, \qquad p \geqslant 1.$$

Also, $M(\phi)$ is the associate of $\Lambda(\phi)$ so that

$$h(s;\ \Lambda(\phi)) = h(1/s;\ M(\phi))/s;$$

so we need only derive $h(s)$ for the space $M(\phi)$.

THEOREM 4.1. *Define $N(s)$ by*

$$N(s) = \sup_{t>0} \Phi(t)/\Phi(st), \qquad s > 0.$$

Then

$$h(s;\ \Lambda(\phi, p)) = N(s)^{1/p}, \qquad h(s;\ M(\phi)) = (1/s)N(1/s).$$

Proof. Let $f \in M(\phi)$, with $||f|| \leqslant 1$, so that

$$\int_0^t f^*(x)dx \leqslant \Phi(t). \tag{1}$$

Hence,

$$\int_0^t (E_s f)^*(x)\, dx = \int_0^t f^*(sx)\, dx \tag{2}$$

$$= \frac{1}{s} \int_0^{st} f^*(x)\, dx \leqslant \frac{1}{s}\, \Phi(st),$$

from (1). However, by definition of N, $\Phi(st) \leqslant \Phi(t)\cdot N(1/s)$. Substituting this in (2) and using the definition of the norm in $M(\phi)$, we obtain

$$||E_s f|| \leqslant (1/s)N(1/s).$$

However, if we let $f = \phi$, which has norm 1, equality holds throughout in (2), and in fact,

$$||E_s\, \phi|| = (1/s)N(1/s).$$

Hence, $h(s; M(\phi)) = (1/s)\cdot N(1/s)$, and the remainder of the theorem follows as in the preceding remarks.

Examples. It is shown in **(4)** that, for $p > 1$, $\Lambda(\phi, p)(R)$ is reflexive if and only if $\phi \notin L^1(R^+)$, and that $\Lambda(\phi)$ and $M(\phi)$ are never reflexive. In fact, $M(\phi)$ is non-separable.

If we take $\phi(t) = t^{\alpha-1}$, for $0 < \alpha < 1$, then $N(s) = s^{-\alpha}$. Hence $H \in [\Lambda(\phi)]$ and $H \in [M(\phi)]$, so that neither reflexivity nor separability of X is necessary for $H \in [X]$.

Taking $\phi(t) = (1 + t)^{-1}$, $N(s) = \max(1/s, 1)$, so that $H \notin [\Lambda(\phi, p)]$; but $\Lambda(\phi, p)$ is reflexive if $p > 1$. Hence, reflexivity is insufficient for $H \in [X]$.

Because of the last example, the following result would seem to be welcome.

THEOREM 4.2. *If* $1 < p < \infty$ *then* $H \in \Lambda(\phi, p)$ *if and only if* $\Lambda(\phi, p)$ *is uniformly convex.*

LEMMA 4.3 (Halperin). *The space* $\Lambda(\phi, p)(\Sigma)$ *where* $\mu(\Sigma) = \infty$, $1 < p < \infty$, *is uniformly convex if and only if*

$$\sup_{t>0} \Phi(t)/\Phi(2t) < 1.$$

Proof. See **(3)**.

LEMMA 4.4. *If* $N(s)$ *is defined as in Theorem* 4.1, *then either* $N(s) < 1$ *for all* $s > 1$, *or else* $N(s) = 1$ *for all* $s > 1$.

Proof. Suppose that there is an $s_0 > 1$ such that $N(s_0) = 1$. Let

$$s_1 = \sup\{s\colon N(s) = 1\} > 1.$$

We may assume $s_1 < \infty$. Define $s = (2s_1 + 1)/3$ so that $s \in (1, s_1)$ and $N(s) = 1$. There exists a sequence of values of t for which

$$1 = N(s) = \lim \Phi(t)/\Phi(st), \tag{1}$$

and thus

$$\lim[\Phi(st) - \Phi(t)]/\Phi(t) = 0. \tag{2}$$

Since ϕ is non-increasing, if we let $s' = 2s - 1$, we have

$$\Phi(s't) - \Phi(st) = \int_{st}^{(2s-1)t} \phi(u)\,du \leqslant (s-1)t\cdot\phi(st) \leqslant \int_t^{st} \phi(u)\,du = \Phi(st) - \Phi(t).$$

Thus

$$0 \leqslant [\Phi(s't) - \Phi(st)]/\Phi(t) \leqslant [\Phi(st) - \Phi(t)]/\Phi(t) \to 0.$$

Hence $N(s') = 1$ because

$$\lim \Phi(s't)/\Phi(t) = \lim \Phi(st)/\Phi(t) = 1.$$

However, $s' = 2s - 1 > s_1$, which contradicts the choice of s_1.

Proof of Theorem 4.2. We have $H \in [\Lambda(\phi, p)]$ if and only if P and P' are in $[\Lambda(\phi, p)]$. For any ϕ, we have $P \in [\Lambda(\phi, p)]$ for $p > 1$. For

$$N(s) = h(s; \Lambda(\phi)) \leqslant \max(1/s, 1),$$

by Lemma 3.2. Thus, $h(s;\, \Lambda(\phi, p)) \leqslant \max(1/s^{1/p}, 1)$, so that

$$\int_0^1 h(s)\, ds < \infty,$$

so that $P \in [\Lambda(\phi, p)]$, by Theorem 3.4.

Suppose that $\Lambda(\phi, p)$ is uniformly convex. Then by Lemma 4.3, $N(2) < 1$, so by Lemma 3.7 and Theorem 3.4, $P' \in [\Lambda(\phi, p)]$.

Conversely, suppose that $P' \in [\Lambda(\phi, p)]$. Then, by the above-mentioned results, there exists an $s_0 > 1$ such that

$$N(s_0) = h(s_0;\, \Lambda(\phi, p))^p < 1.$$

By Lemma 4.4, this implies that $N(2) < 1$, and so by Lemma 4.3, $\Lambda(\phi, p)$ is uniformly convex.

5. The Orlicz spaces. In this section, we calculate $h(s; X)$ whenever X is an Orlicz space. The definition of Orlicz space which is used here is that given in **(9)**. This definition includes the spaces L^1 and L^∞ as well as L^p, for $1 < p < \infty$.

Definition 5.1. Let ϕ be a non-decreasing function on $[0, \infty)$, continuous on the right, and for which $\phi(0) = 0$. Denote its left-continuous inverse by ψ and write

$$\Phi(u) = \int_0^u \phi(t)dt, \qquad \Psi(u) = \int_0^u \psi(t)dt, \qquad u \geqslant 0.$$

Then Φ and Ψ are called *complementary Young's functions.*

It is possible that Ψ is infinite for all $v > b$, where b is some finite number. By definition, Φ and Ψ are convex wherever finite.

Define the functional M_Ψ on $\mathfrak{M}(\Sigma)$ by

$$(1) \qquad M_\Psi(f) = \int_\Sigma \Psi(|f(x)|)\mu(dx) = \int_0^\infty \Psi(f^*(t))\, dt.$$

If $f \in \mathfrak{P}(\Sigma)$, let

$$\rho_\Sigma(f) = \inf\{c\colon M_\Psi(f/c) \leqslant 1\}.$$

By **(9)**, ρ_Σ is a length function. Since $\rho_\Sigma(f) = \rho_{R^+}(f^*)$, ρ_Σ is also an r-function.

The space corresponding to ρ_Σ is denoted $L_{M\Psi}$, and its dual is called L_Φ. Similarly, $L_{M\Phi}$ and L_Ψ are defined. The following is proved in **(9)**:

$$||f||_{M\Psi} \leqslant ||f||_\Psi \leqslant 2||f||_{M\Psi}.$$

Because of this, if H is a bounded operator from any one of the four Orlicz spaces to itself, then this is true for all the spaces.

We need the following definition before deriving an expression for $h(s; X)$.

Definition 5.2. Suppose that Φ is a function from $[0, \infty)$ to $[0, \infty]$, and that there are numbers $a \in [0, \infty)$, $b \in (0, \infty]$, with $a \leqslant b$, such that

$$\Phi(t) = \begin{cases} 0, & \text{if } t \in [0, a], \\ \infty, & \text{if } t \in [b, \infty), \end{cases}$$

and Φ is non-decreasing in $[a, b]$. Define a function K from $[0, \infty)$ to $[0, \infty]$, by

$$K(s) = \begin{cases} \sup\limits_{t \in (a,b)} \Phi(st)/\Phi(t) & \text{if } a \neq b \text{ and } s \in [0, 1], \\ 0 & \text{if } 0 < a = b < \infty, \\ \sup\limits_{t \in (0,\infty)} \Phi(st)/\Phi(t) & \text{if } a = 0, b = \infty, \text{and } s \in (1, \infty), \\ \infty & \text{if } a \neq 0 \text{ or } b \neq \infty. \end{cases}$$

We call K the *upper quotient function of* Φ.

Remarks. If $a \neq b$, $s \in [0, 1]$, and $t \in (0, \infty)$, then

$$\Phi(st) \leqslant K(s)\Phi(t).$$

Also, if Φ is continuous whenever finite, then K is continuous on the left wherever finite.

Note that even when $a = 0$ and $b = \infty$, $K(s)$ is either finite for all $s \in (1, \infty)$ or infinite for all $s \in (1, \infty)$.

The next theorem seems to have intrinsic interest apart from its use here. If all the inverses appearing were strict, the result would be almost obvious.

THEOREM 5.3. *Suppose that Φ is as described in Definition 5.2 and further that Φ is strictly increasing and continuous in $[a, b]$. Let Φ^{-1} be the right continuous inverse of Φ, and define*

$$G(s) = \inf_{t>0} \Phi^{-1}(st)/\Phi^{-1}(t).$$

Let K be as in Definition 5.2 and let K^{-1} be the right-continuous inverse of K. Then, for all $s \in [0, \infty)$,

$$G(s) = K^{-1}(s).$$

Proof. Note that G is right-continuous; let G^{-1} be its left-continuous inverse. The following are easily verified:

$$(K \circ K^{-1})(s) \leqslant s \leqslant (K^{-1} \circ K)(s), \tag{1}$$

$$(G^{-1} \circ G)(s) \leqslant s \leqslant (G \circ G^{-1})(s). \tag{2}$$

We shall give the proof when $a = 0$, $b = \infty$ so that Φ maps $[0, \infty)$ one-one onto itself. We assume further that $s \in [0, 1]$. The remaining cases are handled in the same way, requiring slightly more care.

Let $I = [0, 1]$. Then each of K, K^{-1}, G, G^{-1} maps I into I.

From

$$\Phi(st) \leqslant K(s)\Phi(t), \qquad s \in I, t > 0,$$

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it follows that

$$\Phi(K^{-1}(s)t) \leqslant K(K^{-1}(s))\Phi(t) \leqslant s\Phi(t), \qquad \text{by (1).}$$

Hence, since Φ^{-1} is a strict inverse and is increasing,

$$K^{-1}(s)t \leqslant \Phi^{-1}(s\Phi(t)). \tag{3}$$

By the definition of G, given $s \in I$, and $\epsilon > 0$, there exists

$$t_0 = t_0(s, \epsilon) \in (0, \infty)$$

such that

$$\Phi^{-1}(st_0) \leqslant (G(s) + \epsilon)\Phi^{-1}(t_0).$$

The range of Φ contains t_0 so that $t_0 = \Phi(t_1)$ for some $t_1 \in (0, \infty)$ and hence

$$\Phi^{-1}(s\Phi(t_1)) \leqslant (G(s) + \epsilon)\Phi^{-1}(\Phi(t_1)). \tag{4}$$

Combining (3) and (4), we obtain

$$K^{-1}(s)t_1 \leqslant \Phi^{-1}(s\Phi(t_1)) \leqslant (G(s) + \epsilon)t_1;$$

so, since $t_1 > 0$, and $\epsilon > 0$ is arbitrary,

$$K^{-1}(s) \leqslant G(s), \qquad s \in I. \tag{5}$$

Now, we apply analogous steps starting with G, to obtain (in place of (3)),

$$\Phi(s\Phi^{-1}(t)) \leqslant G^{-1}(s)t, \qquad s \in I, t > 0. \tag{6}$$

By definition of K, given $s \in I$, $\epsilon > 0$, there exists $t_1 = t_1(s, \epsilon)$ such that (in place of (4)),

$$(K(s) - \epsilon)t_1 \leqslant \Phi(s\Phi^{-1}(t_1)). \tag{7}$$

Combining (6) and (7), we obtain

$$K(s) \leqslant G^{-1}(s), \qquad s \in I \tag{8}$$

and hence

$$K^{-1}(s) \geqslant G(s), \qquad s \in I. \tag{9}$$

Together, (5) and (9) show that $G = K^{-1}$ for $s \in I$.

COROLLARY 5.4. *If G is as in Theorem 5.3, then*

$$\Phi(G(s)t) \leqslant s\Phi(t) \qquad \textit{for all } s, t \geqslant 0.$$

The proof follows directly from Theorem 5.3, and equation (1) of its proof, once it is noticed that, if $a > 0$, or $b < \infty$ in the definition of Φ, then $G(s) \leqslant 1$ for all s.

THEOREM 5.5. *The function $h(\cdot\,; X)$ for the Orlicz spaces is given by:*

$$h(s; L_{M\Phi}) = 1/K^{-1}(s)$$

where K is as in Theorem 5.3.

Also

$$h(s; L_\Phi) = (1/s)\cdot h(1/s; L_{M\Psi}).$$

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Proof. Suppose that $f \in L_\Phi$ with $||f|| = 1$. Then

$$(1) \qquad M_\Phi(f) = \int_0^\infty \Phi(f^*(t))dt = 1$$

so that

$$(2) \qquad M_\Phi(G(s)\cdot E_s f) = \int_0^\infty \Phi(G(s)f^*(st))\,dt$$

$$= \int_0^\infty \Phi(G(s)f^*(t))\,\frac{dt}{s} \qquad (st \to t)$$

$$\leqslant \int_0^\infty s\Phi(f^*(t))\,\frac{dt}{s}, \qquad \text{by Corollary 5.4,}$$

$$= M_\Phi(f) = 1.$$

Hence, by definition of the norm,

$$(3) \qquad ||E_s f|| \leqslant 1/G(s),$$

so that $h(s; L_{M\Phi}) \leqslant 1/G(s)$.

However, if we take f to be the characteristic function of the interval $[0, t]$, we have $||f|| = 1/\Phi^{-1}(1/t)$ so that

$$(4) \qquad \sup_{t>0} \frac{||E_s f||}{||f||} = \sup_{t>0} \frac{\Phi^{-1}(1/t)}{\Phi^{-1}(s/t)} = \sup_{t>0} \frac{\Phi^{-1}(t)}{\Phi^{-1}(st)} = \frac{1}{G(s)}.$$

Hence, equality holds in (3), proving the required result upon application of Theorem 5.3 and Lemma 3.2(a).

The conditions under which $L_\Phi(\Sigma)$ (hence $L_{M\Phi}$, L_Ψ, $L_{M\Psi}$) is reflexive have been given in **(9)**. The criterion differs depending on the nature of Σ, and is simpler if $\mu(\Sigma) < \infty$ or if Σ is purely atomic.

Definition 5.6. Φ is said to satisfy the (δ_2, Δ_2) condition, if and only if there exists a constant $m > 0$ such that

$$\Phi(2v) \leqslant m\Phi(v) \qquad \text{for all } v > 0.$$

LEMMA 5.7 **(9)**. *Let L_Φ denote $L_\Phi(\Sigma)$ where $\mu(\Sigma) = \infty$ and Σ has no atoms. Then $L_\Phi^* = L_{M\Psi}$ if and only if Φ satisfies (δ_2, Δ_2). L_Φ is reflexive if and only if both Φ and Ψ satisfy (δ_2, Δ_2).*

THEOREM 5.8. *$H \in [L_\Phi(R)]$ if and only if $L_\Phi(R)$ is reflexive.*

LEMMA 5.9. *Φ satisfies (δ_2, Δ_2) if and only if there exists an $s_0 > 1$ such that $h(s_0; L_{M\Phi}) < 1$.*

Proof. Let a and b be as in Theorem 5.3.

If $a > 0$, or $b < \infty$, then it is easily checked that Φ cannot have property (δ_2, Δ_2). In this case, $K(s) = \infty$, for $s > 1$ so that

$$h(s; L_{M\Phi}) = 1/K^{-1}(s) = 1 \qquad \text{for } s > 1.$$

Conversely, if $h(s; L_{M\Phi}) = 1$, for all $s > 1$, then $K(2) = \infty$, proving that Φ does not have property (δ_2, Δ_2).

If $a = 0$, $b = \infty$ and if Φ has property (δ_2, Δ_2), then $K(2) \leqslant m$, by Definition 5.6 so that $K(2^n) \leqslant m^n < \infty$ for any integer n so that $K(s) < \infty$ for all $s > 1$. Thus there exists $s_0 > 1$ with $K^{-1}(s_0) > 1$ and hence such that $h(s_0; L_{M\Phi}) < 1$.

Conversely, if $h(s_0; L_{M\Phi}) < 1$, then $K(s) < \infty$ for $s > 1$, and in particular $K(2) < \infty$, so that Φ satisfies (δ_2, Δ_2).

Proof of Theorem 5.8. We have $H \in [L_\Phi(R)]$ if and only if both P and P' are in $[L_\Phi(R^+)]$. Also $P \in [L_\Phi]$ if and only if $P' \in [L_{M\Psi}]$ since $L_{M\Psi}$ and L_Φ are associates. And $P' \in [L_\Phi]$ if and only if $P' \in [L_{M\Phi}]$, since $L_{M\Phi}$ and L_Φ are equivalent spaces.

However $P' \in [L_{M\Phi}]$ if and only if $h(s_0; L_{M\Phi}) < 1$ for some $s_0 > 1$, by Lemma 3.7 and Theorem 3.4. And by Lemma 5.9, this is true if and only if Φ satisfies the condition (δ_2, Δ_2). Similarly, $P' \in [L_{M\Psi}]$ if and only if Ψ satisfies (δ_2, Δ_2).

Combining these results we obtain that $H \in [L_\Phi(R)]$ if and only if both Φ and Ψ satisfy (δ_2, Δ_2) which is true if and only if $L_\Phi(R)$ is reflexive, by Lemma 5.7. This completes our proof.

By remarks made earlier, Theorem 5.8 is true if L_Φ is replaced by any of the other Orlicz spaces.

A result analogous to Theorem 5.8 has been obtained by Ryan **(11)** for the "conjugate function" operator. The method used was quite different.

6. Further results. By using an inequality of Calderón, it is possible to prove the following theorem by methods similar to those used in §3. This is, in essence, an interpolation theorem. The connection between some of the ideas developed in this paper and the theory of interpolation spaces may be found in **(1)**.

THEOREM 6.1. *Let Σ be a totally σ-finite measure space. Suppose that T is a linear operator with $\mathfrak{D}(T) \supset L^p(\Sigma)$ for $1 < p < \infty$ and that $T \in [L^p(\Sigma)]$ for all p satisfying $1 < p < \infty$.*

Let X be a rearrangement-invariant space for which

(i) *$sh(s; X) \to 0$ as $s \to 0+$,*

(ii) *$h(s; X) \to 0$ as $s \to \infty$.*

Then $\mathfrak{D}(T) \supset X(\Sigma)$ and $T \in [X(\Sigma)]$.

The statement of Theorem 6.1 can be strengthened. It is possible to allow T to be "quasilinear," and to require only that T be of "weak type" (p, p), for $1 < p < \infty$.

The conditions on $h(s; X)$ are necessary in general, since they are necessary for $H \in [X(R)]$, and H satisfies the other conditions of the theorem.

Theorem 6.1 is, in particular, applicable to the singular integral operators of Calderón and Zygmund.

It is possible to characterize the functions which may appear in the form $h(s; X)$ in the following sense:

THEOREM 6.2. *Suppose that g is any function on* $(0, \infty)$ *which satisfies the following conditions:*

(a) $g(s) > 0$ *for* $0 < s < \infty$,

(b) $g(st) \leqslant g(s)g(t)$,

(c) *g is non-increasing and* $sg(s)$ *is non-decreasing,*

(d) $g(1) = 1$.

Then there is a rearrangement-invariant space X for which

$$g(s) = h(s; X).$$

Proof. Consider the following function on $\mathfrak{P}(R^+)$:

$$\rho(f) = \sup_{t>0} g(1/t) \cdot f^{**}(t).$$

Because of the subadditive property of the ** mapping, it is easy to verify that ρ is a length function which is obviously rearrangement-invariant. Let X be the space defined by this length function; then $h(s; X) = g(s)$. For if $f \in X$, then, using (b),

$$\rho(E_s f) = \sup_{t>0} g(1/t) f^{**}(st) = \sup_{t>0} g(s/t) f^{**}(t)$$
$$\leqslant g(s) \cdot \sup_{t>0} g(1/t) f^{**}(t) = g(s) \cdot \rho(f).$$

But if f is the characteristic function of the interval $[0, t]$, then it is easy to verify, using (c), that

$$\rho(f) = g(1/t).$$

Hence, $\rho(E_s f)/\rho(f) = g(s/t)/g(1/t)$ which is equal to $g(s)$ for $t = 1$.

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University of Toronto