The Elements of Quaternions *(First Paper).*

By Dr William Peddie.

In this paper the laws of addition and subtraction of vectors were considered, and examples of their extreme usefulness in geometrical applications were given.

Adams's Hexagons and Circles.

By J. S. Mackay, M.A., LL.D.

Figure 24.

In triangle ABC, AD, BE, CF are concurrent at O; through O parallels are drawn to EF, FD, DE, meeting the sides of ABC in L, M, P, Q, S, T, and the sides of DEF in L′, M′, P′, Q′, S′, T′. The two hexagons LMPQST, L′M′P′Q′S′T′ thus formed have the following properties:

(1) The sides L′M′, P′Q′, S′T′ of the latter are parallel to the sides of ABC.

The complete quadrilateral AFOEBC has its diagonal AO cut harmonically by EF and BC;

therefore A, U, O, D is a harmonic range,

and E·A U O D is a harmonic pencil.

Now OP′EQ′ is a parallelogram;

therefore P′Q′ is bisected by EO;

therefore P′Q′ is parallel to that ray of the harmonic pencil which is conjugate to EO, namely EA.

In like manner S′T′ is parallel to AB, and L′M′ to BC.

(2) The sides QS, TL, MP of the former are parallel to the sides of DEF.

Since PEQ′P′ and QEP′Q′ are parallelograms,

therefore PE = QE.

Similarly TF = SF;

therefore PE : QE = TF : SF.

Now PT is parallel to EF;

therefore QS is parallel to EF.

In like manner TL is parallel to FD, and MP to DE.

https://doi.org/10.1017/S0013091500031278 Published online by Cambridge University Press

(3) The two hexagons are similar, and the first is four times the second.

This follows from the fact that the hexagons are made up of similar and similarly situated triangles, the ratio of whose homologous sides is that of 2 : 1.

(4) If D, E, F be the points of contact of the incircle with BC, CA, AB, the two hexagons obtained as before are inscriptible in circles; the radius of the greater circle is double the radius of the less; the centre of the greater circle is the centre of the incircle; and if Γ be the point of concurrency of AD, BE, CF the centre of the less circle is the mid point of ΓI.

FIGURE 25

Angle M L′Q′ = CDE = CED = EP′Q′;

therefore L′, M′, P′, Q′ are concyclic.

Similarly P′, Q′, S′, T′ are concyclic,

and S′, T′, L′, M′ are concyclic;

hence, by a theorem of Poncelet's, all the six points are concyclic.*

That the six points, L, M, P, Q, S, T are concyclic, follows from the preceding.

Since LM, PQ, ST, are chords of the greater circle, and they are bisected at D, E, F, therefore the centre of the greater circle is found by drawing perpendiculars to BC, CA, AB at D, E, F. But these perpendiculars are concurrent at the incentre.

Since Γ is the centre of similitude of the two hexagons, it is the external centre of similitude of the two circles circumscribed about them; and since I is the centre of the greater circle, the centre of the smaller circle must lie on ΓI. But because the radius of the greater circle is double the radius of the less, the centre of the less circle must be at I′ such that

$$I\Gamma : I'\Gamma = 2 : 1.$$

The preceding properties have been taken from C. Adams's *Die Lehre von den Transversalen*, pp. 77-80 (1843). The following considerations may be added:

(5) If, instead of the points of contact of the incircle D, E, F, there be taken the points of contact of the excircles D_1, E_1, F_1, or D_2, E_2, F_2, or D_3, E_3, F_3 analogous properties will be obtained.

Hence the existence of three other pairs of circles.

---

* Adams's proof of this is somewhat different.

(6) It is known that Γ is the symmedian point of the triangle DEF; hence the circle L′M′P′Q′S′T′ is the triplicate-ratio, or first Lemoine, circle of DEF.

Γ is frequently called the Gergonne point of triangle ABC.

(7) If the points of concurrency of the triads

$$\left.\begin{matrix} AD_1, BE_1, CF_1 \\ AD_2, BE_2, CF_2 \\ AD_3, BE_3, CF_3 \end{matrix}\right\} \text{ be } \left\{\begin{matrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{matrix}\right.$$

then $\Gamma_1 \quad \Gamma_2 \quad \Gamma_3$ are the symmedian points of the triangles $D_1E_1F_1$, $D_2E_2F_2$, $D_3E_3F_3$;
and three of the circles referred to in (5) will be Lemoine circles of these triangles.

Γ_1, Γ_2, Γ_3 are frequently called the adjoint Gergonne points of triangle ABC.

(8) Let the tangents to the circumcircle ABC at the points A, B, C meet each other at K_1, K_2, K_3; then, if triangle ABC be acute-angled, the circle ABC will be the incircle of triangle $K_1K_2K_3$, and if triangle ABC be obtuse-angled, the circle ABC will be an excircle of triangle $K_1K_2K_3$. Hence the relation in which triangle ABC stands to $K_1K_2K_3$ will, if ABC be acute-angled, be that in which triangle DEF stands to ABC, and so on. The two systems of points therefore

$$\begin{matrix} D, E, F, I, \Gamma, A, B, C, \ldots \\ A, B, C, O, K, K_1, K_2, K_3, \ldots \end{matrix}$$

correspond.

(9) As the straight lines through Γ parallel to the sides of DEF cut the sides of ABC in six concyclic points, so the straight lines through K parallel to the sides of ABC will cut the sides of $K_1K_2K_3$ in six concyclic points. Hence the existence of another circle connected with the triangle.

Since I, the circumcentre of DEF, is the centre of the circle LMPQST, therefore O, the circumcentre of ABC, will be the centre of this new circle.

(10) If in the second system of points referred to in (8) those points be taken which correspond to $\Gamma_1, \Gamma_2, \Gamma_3$ in the first system, three other sets of six concyclic points will be obtained.